Rationally Isotropic Quadratic Spaces Are Locally Isotropic: II

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Abstract. The results of the present article extend the results of [Pa]. The main result of the article is Theorem 1.1 below. The proof is based on a moving lemma from [LM], a recent improvement due to O. Gabber of de Jong's alteration theorem, and the main theorem of [PR]. A purity theorem for quadratic spaces is proved as well in the same generality as Theorem 1.1, provided that R is local. It generalizes the main purity result from [OP] and it is used to prove the main result in [ChP].

1 Introduction

Let A be a commutative ring and P be a finitely generated projective A-module. An element $v \in P$ is called unimodular if the A-submodule vA of P splits off as a direct summand. If $P = A^n$ and $v = (a_1, a_2, \dots, a_n)$ then v is unimodular if and only if $a_1A + a_2A + \cdots + a_nA = A$.
Let $\frac{1}{2} \in A$. A quadratic space over A is a pair (P, α) consisting of a finitely generated projective A-module P and an A-isomorphism $\alpha : P \to P^*$ satisfying $\alpha = \alpha^*$, where $P^* = \mathrm{Hom}_R(P, R)$. Two spaces (P, α) and (Q, β) are *isomorphic* if there exists an A-isomorphism $\varphi : P \to Q$ such that $\alpha = \varphi^* \circ \beta \circ \varphi$.
Let (P, φ) be a quadratic space over A. One says that it is *isotropic* over A, if there exists a unimodular $v \in P$ with $\varphi(v) = 0$.

Theorem 1.1. *Let R be a semi-local regular integral domain containing a field. Assume that all the residue fields of R are infinite and $\frac{1}{2} \in R$. Let K be the fraction field of R and (V, φ) a quadratic space over R. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R.*

This Theorem is a consequence of the following result.

Theorem 1.2. *Let k be an infinite perfect field of characteristic different from 2, B a k-smooth algebra. Let $p_1, p_2, \dots, p_n$ be prime ideals of B, $S = B - \cup_{j=1}^{n} p_j$ and $R := B_S$ be the localization of B with respect to S (note that B_S is a semi-local ring). Let K be the ring of fractions of R with respect to all non-zero divisors and (V, φ) be a quadratic space over R. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R.*

For arbitrary discrete valuation rings, Theorem 1.1 holds trivially. It also holds for arbitrary regular local two-dimensional rings in which 2 is invertible, as proved by M. Ojanguren in [O].

To conclude the Introduction let us add a historical remark which might help the general reader. Let R be a regular local ring, G/R a reductive group scheme. The question whether a principal homogeneous space over R which admits a rational section actually admits a section goes back to the foundations of étale cohomology. It was raised by J.-P. Serre and A. Grothendieck (séminaire Chevalley "Anneaux de Chow"). In the geometric case, this question has essentially been solved, provided that G/R comes from a ground field k. Namely, J.-L. Colliot-Thélène and M. Ojanguren in [CT-O] deal with the case where the ground field k is infinite and perfect. There were later papers [Ra1] and [Ra2] by M.S. Raghunathan, which handled the case k infinite but not necessarily perfect. O. Gabber later announced a proof in the general case. One may then raise the question whether a similar result holds for homogeneous spaces. A specific instance is that of projective homogeneous spaces. An even more specific instance is that of smooth projective quadrics (question raised in [C-T], Montpellier 1977). This last case is handled in the present paper. Remark 3.5 deals with the semi-local case.

The key point of the proof of Theorem 1.2 is the combination of the moving lemma in [LM] and Gabber's improvement of the alteration theorem due to de Jong with the generalization of Springer's result in [PR]. Theorem 1.1 is deduced from Theorem 1.2 using D. Popescu's theorem.

2 Auxiliary results

Let k be a field. To prove Theorem 1 we need auxiliary results. We start recalling the notion of transversality as it is defined in [LM, Def.1.1.1].

Definition 2.1. *Let $f : X \to Z$, $g : Y \to Z$ be morphisms of k-smooth schemes. We say that f and g are transverse if*

1. *$Tor_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ for all $q > 0$.*
2. *The fibre product $X \times_Z Y$ is a k-smooth scheme.*

Lemma 2.2. *Let $f : X \to Z$ and $g : Y \to Z$ be transverse, and $pr_Y : Y \times_Z X \to Y$ and $h : T \to Y$ be transverse, then f and $g \circ h$ are transverse.*

This is just Lemma 1 from [Pa].

Since this moment and till Remark 2.6 (including that Remark) let k be an infinite perfect field of characteristic different from 2. Let U be a smooth irreducible quasi-projective variety over k and let $j : u \to U$ be a closed point of U. In particular, the field extension $k(u)/k$ is finite. It is also separable since k is perfect. Thus $u = Spec(k(u))$ is a k-smooth variety.

Lemma 2.3. *Let U be as above. Let Y be a k-smooth irreducible variety of the same dimension as U. Let $v = \{v_1, v_2, \dots, v_s\} \subset U$ be a finite set of closed points. Let $q : Y \to U$ be a projective morphism such that $q^{-1}(v) \neq \emptyset$. Assume $q : Y \to U$ and $j_v : v \hookrightarrow U$ are transverse. Then q is finite étale over an affine neighborhood of the set $v \subset U$.*

Proof. There is a $v_i \in v$ such that $q^{-1}(v_i) \neq \emptyset$. By [Pa, Lemma 2] q is finite étale over a neighborhood V_i of the point $v_i \in U$. This implies that $V_i \subset q(Y)$. It follows that $q(Y) = U$, since q is projective and U is irreducible. Whence for each $i = 1, 2, \dots, s$ one has $q^{-1}(v_i) \neq \emptyset$. By [Pa, Lemma 2] for each $m = 1, 2, \dots, s$ the morphism q is finite étale over a neighborhood V_m of the point $v_m \in U$. Since U is quasi-projective, q is finite étale over an affine neighborhood V of the set $v \subset U$.

□

Let U be as above. Let $p : \mathcal{X} \to U$ be a smooth projective k-morphism. Let $X = p^{-1}(u)$ be the fibre of p over u. Since p is smooth the $k(u)$-scheme X is smooth. Since $k(u)/k$ is separable X is smooth as a k-scheme. Thus for a morphism $f : Y \to \mathcal{X}$ of a k-smooth scheme Y it makes sense to say that f and the embedding $i : X \hookrightarrow \mathcal{X}$ are transverse. So one can state the following

Lemma 2.4. *Let $p : \mathcal{X} \to U$ be as above, let $j_v : v \hookrightarrow U$ be as in Lemma 2.3 and let $X = p^{-1}(v)$ be as above. Let Y be a k-smooth irreducible variety with $dim(Y) = dim(U)$. Let $f : Y \to \mathcal{X}$ be a projective morphism such that $f^{-1}(X) \neq \emptyset$. Suppose that f and the closed embedding $i : X \hookrightarrow \mathcal{X}$ are transverse. Then the morphism $q = p \circ f : Y \to U$ is finite étale over an affine neighborhood of the set v.*

Proof. For each $i = 1, 2, \dots, s$ the extension $k(u)/k$ is finite. Since k is perfect, the scheme v is k-smooth. The morphism $p : \mathcal{X} \to U$ is smooth. Thus the morphism j_v and the morphism p are transverse. Morphisms j_v and $q = p \circ f$ are transverse by Lemma 2.2, since j_v and f are transverse. One has $q^{-1}(v) = f^{-1}(X) \neq \emptyset$. Now Lemma 2.3 completes the proof of the Lemma.

□

For a k-smooth variety W let $CH_d(W)$ be the group of dimension d algebraic cycles modulo rational equivalence on W (see [Fu]). The next lemma is a variant of the proposition [LM, Prop. 3.3.1] for the Chow groups $Ch_d := CH_d/2CH_d$ of algebraic cycles modulo rational equivalence with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Lemma 2.5 (A moving lemma). *Suppose that k is an infinite perfect field (the characteristic of k is different from 2 as above). Let W be a k-smooth scheme* *and let $i : X \hookrightarrow W$ be a k-smooth closed subscheme. Then $Cd_d(W)$ is generated by the elements of the form $f_*([Y])$ where Y is an irreducible k-smooth variety of dimension d, $[Y] \in Cd_d(Y)$ is the fundamental class of Y, $f : Y \to W$ is a projective morphism such that f and i are transverse and $f_* : Ch_d(Y) \to Ch_d(W)$ is the push-forward.*

Proof. The group $Ch_d(W)$ is generated by cycles of the form $[Z]$, where $Z \subset W$ is a closed irreducible subvariety of dimension d. Since k is perfect of characteristic different from 2, applying a recent result due to Gabber [I, Thm. 1.3], one can find a k-smooth irreducible quasi-projective variety Z' and a proper morphism $\pi : Z' \to Z$ with k-smooth quasi-projective variety Z' and such that the degree $[k(Z') : k(Z)]$ is odd. The morphism p is necessary projective, since the k-variety Z' is quasi-projective and p is a proper morphism (see [Ha, Ch.II, Cor.4.8.e]). Write π' for the composition $Z' \to Z \hookrightarrow W$. Clearly, $\pi'_*([Z']) = [Z] \in Cd_d(W)$. The lemma is not proved yet, since π' and i are not transverse.
However to complete the proof it remains to repeat literally the proof of proposition [LM, Prop. 3.3.1]. The proof of that proposition does not use the resolution of singularities. Whence the lemma.

□

Remark 2.6. *Note that at the end of the previous proof we actually used a Chow version of [LM, Prop. 3.3.1] instead of Prop. 3.3.1 itself.*

The following theorem proved in [PR] is a generalization of a theorem of Springer. See [La, Chap.VII, Thm.2.3] for the original theorem by Springer.

Theorem 2.7. *Let R be a local Noetherian domain which has an infinite residue field of characteristic different from 2. Let $R \subset S$ be a finite R-algebra which is étale over R. Let (V, φ) be a quadratic space over R such that the space $(V, \varphi) \otimes_R S$ contains an isotropic unimodular vector. If the degree $[S : R]$ is odd then the space (V, φ) already contains a unimodular isotropic vector.*

Remark 2.8. *Theorem 2.7 is equivalent to the main result of [PR], since the R-algebra S from Theorem 2.7 one always has the form $R[T]/(F(T))$, where $F(T)$ is a separable polynomial of degree $[S : R]$ (see [AK, Chap.VI, Defn.6.11, Thm.6.12]).*

Repeating verbatim the proof of Theorem 2.7 given in [PR] we get the following result.

Theorem 2.9. *Let R be a semi-local Noetherian integral domain SUCH THAT ALL ITS residue fields ARE INFINITE of characteristic different from 2. Let $R \subset S$ be a finite R-algebra which is étale over R. Let (V, φ) be a quadratic space over R such that the space $(V, \varphi) \otimes_R S$ contains an isotropic unimodular vector. If the degree $[S : R]$ is odd then the space (V, φ) already contains a unimodular isotropic vector.*

3 Proofs of Theorems 1.2 and 1.1

Proof of Theorem 1.2. Let k be an infinite perfect field of characteristic different from 2. Let $p_1, p_2, \dots, p_n$ be prime ideals of B, $S = B - \cup_{j=1}^{n} p_j$ and $R = B_S$ be the localization of B with respect to S.
Clearly, it is sufficient to prove the theorem in the case when B is an integral domain. So, in the rest of the proof we will assume that B is an integral domain. We first reduce the proof to the localization at a set of maximal ideals. To do that we follow the arguments from [CT-O, page 101]. Clearly, there exist $f \in S$ and a quadratic space (W, ψ) over B_f such that $(W, \psi) \otimes_{B_f} B_S = (V, \varphi)$. For each index j let m_j be a maximal ideal of B containing p_j and such that $f \notin m_j$. Let $T = B - \cup_{j=1}^{n} m_j$. Now B_T is a localization of B_f and one has $B_f \subset B_T \subset B_S = R$. Replace R by B_T.
From now on and until the end of the proof of Theorem 1.2 we assume that $R = \mathcal{O}_{U,\{u_1,u_2,\dots,u_n\}}$ is the semi-local ring of a finite set of closed points $u = \{u_1, u_2, \dots, u_n\}$ on a k- smooth d-dimensional irreducible affine variety U.
Let $\mathcal{X} \subset \mathbf{P}_R(V)$ be a projective quadric given by the equation $\varphi = 0$ in the projective space $\mathbf{P}_R(V) = Proj(S^*(V^\vee))$. Let $X = p^{-1}(u)$ be the scheme-theoretic pre-image of u under the projection $p : \mathcal{X} \to Spec(R)$. Shrinking U we may assume that u is still in U and the quadratic space (V, φ) is defined over U. We still write $\mathcal{X}$ for the projective quadric in $\mathbf{P}_U(V)$ given by the equation $\varphi = 0$ and still write $p : \mathcal{X} \to U$ for the projection. Let $\eta : Spec(K) \to U$ be the generic point of U and let $\mathcal{X}_\eta$ be the generic fibre of $p : \mathcal{X} \to U$. Since the equation $\varphi = 0$ has a solution over K there exists a K-rational point y of $\mathcal{X}_\eta$. Let $Y \subset \mathcal{X}$ be its closure in $\mathcal{X}$ and let $[Y] \in Ch_d(\mathcal{X})$ be the class of Y in the Chow groups with $\mathbb{Z}/2\mathbb{Z}$-coefficients.
Since p is smooth the scheme X is $k(u)$-smooth. Since $k(u)/k$ is a finite étale algebra X is smooth as a k-scheme. By Lemma 2.5 there exist a finite family of integers $n_r \in \mathbb{Z}$ and a finite family of projective morphisms $f_r : Y_r \to \mathcal{X}$ (with k-smooth irreducible Y_r's of dimension $dim(U)$) which are transverse to the closed embedding $i : X \hookrightarrow \mathcal{X}$ and such that $\sum n_r f_{r,*}([Y_r]) = [Y]$ in $Ch_d(\mathcal{X})$. Shrinking U we may assume that for each index r one has $f_r^{-1}(X) \neq \emptyset$. By Lemma 2.4 for any index r the morphism $q_r = p \circ f_r : Y_r \to U$ is finite étale over an affine neighborhood U' of the set u. Shrinking U we may assume that $U' = U$. Let $deg : Ch_0(\mathcal{X}_\eta) \to \mathbb{Z}/2\mathbb{Z}$ be the degree map. Since $deg(y) = 1$ and $\sum n_r f_{r,*}[Y_r] = [Y] \in Ch_d(\mathcal{X})$ there exists an index r such that the degree of the finite étale morphism $q_r : Y_r \to U$ is odd. Without loss of generality we may assume that the degree of q_1 is odd. The existence of the Y_1-point $f_1 : Y_1 \to \mathcal{X}$ of $\mathcal{X}$ shows that we are under the hypotheses of Theorem 2.9. Hence shrinking U once more we see that there exists a section $s : U \to \mathcal{X}$ of the projection $\mathcal{X} \to U$. Theorem 1.2 is proven.

□

Proof of Theorem 1.1. Let R be a regular semi-local integral domain containing a field. Let k be the prime field of R. By Popescu's theorem $R = \varinjlim B_\alpha$, where the B_α's are smooth k-algebras (see [P] or [Sw]). Let $can_\alpha : B_\alpha \to R$ be the canonical k-algebra homomorphism. We first observe that we may replace the direct system of the B_α's by a system of essentially smooth semi-local k-algebras which are integral domains. In fact, if m_j is a maximal ideal of R, we can take $p_{\alpha,j} := can_\alpha^{-1}(m_j)$, $S_\alpha := B_\alpha - \cup_{j=1}^n p_{\alpha,j}$ and replace each B_α by $(B_\alpha)_{S_\alpha}$, Note that in this case the canonical morphisms $can_\alpha : B_\alpha \to R$ take maximal ideals to maximal ones and every B_α is a regular semi-local k-algebra.

We claim that B_α is an integral domain. In fact, since B_α is a regular semi-local k-algebra it is a product $\prod_{i=1}^s B_{\alpha,i}$ of regular semi-local integral domains $B_{\alpha,i}$. The ideal $q_\alpha := can_\alpha^{-1}(0) \subset B_\alpha$ is prime and is contained in each of the maximal ideals $can_\alpha^{-1}(m_j)$ of the ring B_α. The latter ideal runs over all the maximal ideals of B_α. Thus the prime ideal q_α is contained in all maximal ideals of $B_\alpha = \prod_{i=1}^s B_{\alpha,i}$. Since q_α is prime after reordering the indices it must be of the form $q_1 \times \prod_{i=2}^s B_{\alpha,i}$. If $s \geq 2$ then the latter ideal is not contained in a maximal ideal of the form $\prod_{i=1}^{s-1} B_{\alpha,i} \times m$ for a maximal ideal m of $B_{\alpha,s}$. Whence $s = 1$ and B_α is indeed an integral domain.

There exists an index α and a quadratic space φ_α over B_α such that $\varphi_\alpha \otimes_{B_\alpha} R \cong \varphi$. For each index $\beta \geq \alpha$ we will write φ_β for the B_β-space $\varphi_\alpha \otimes_{B_\alpha} B_\beta$. Clearly, $\varphi_\beta \otimes_{B_\beta} R \cong \varphi$. The space φ_K is isotropic. Thus there exists an element $f \in R$ such that the space (V_f, φ_f) is isotropic. There exists an index $\beta \geq \alpha$ and a non-zero element $f_\beta \in B_\beta$ such that $can_\beta(f_\beta) = f$ and the space φ_β localized at f_β is isotropic over the ring $(B_\beta)_{f_\beta}$.

If $char(k) = 0$ or if $char(k) = p > 0$ and the field k is infinite perfect, then by Theorem 1.2 the space φ_β is isotropic. Whence the space φ is isotropic too.

If $char(k) = p > 0$ and the field k is finite, then choose a prime number l different from 2 and from p and take the field k_l which is the composite of all l-primary finite extensions k' of k in a fixed algebraic closure $\bar{k}$ of k. Note that for each field k'' which is between k and k_l and is finite over k the degree $[k'' : k]$ is a power of l. In particular, it is odd. Note as well that k_l is a perfect infinite field. Take the k_l-algebra $k_l \otimes_k B_\beta$. *It is a semi-local essentially k_l-smooth algebra, which is not an integral domain in general.* The element $1 \otimes f_\beta$ is not a zero divisor. In fact, k_l is a flat k-algebra and the element f is not a zero divisor in B_β.

The quadratic space $k_l \otimes_k \varphi_\beta$ localized at $1 \otimes f_\beta$ is isotropic over $(k_l \otimes_k B_\beta)_{1 \otimes f_\beta} = k_l \otimes_k (B_\beta)_{f_\beta}$ and $1 \otimes f_\beta$ is not a zero divisor in $k_l \otimes_k B_\beta$. By Theorem 1.2 the space $k_l \otimes_k \varphi_\beta$ is isotropic over $k_l \otimes_k B_\beta$. Whence there exists a finite extension $k \subset k' \subset k_l$ of k such that the space $k' \otimes_k \varphi_\beta$ is isotropic over $k' \otimes_k B_\beta$. Thus the space $k' \otimes_k \varphi$ is isotropic over $k' \otimes_k R$. Now $k' \otimes_k R$ is a finite étale extension of R of odd degree. All residue fields of R are infinite. By Theorem 2.9 the space φ is isotropic over R.

□

To state the first corollary of Theorem 1.1 we need to recall the notion of unramified spaces. Let R be a Noetherian integral domain and K be its fraction field. Recall that a quadratic space (W, ψ) over K is *unramified* if for every height one prime ideal $\wp$ of R there exists a quadratic space $(V_{\mathfrak{p}}, \varphi_{\wp})$ over $R_{\wp}$ such that the spaces $(V_{\wp}, \varphi_{\wp}) \otimes_{R_{\wp}} K$ and (W, ψ) are isomorphic.

Corollary 3.1 (A purity theorem). *Let R be a regular local ring containing a field of characteristic different from 2 and such that the residue field of R is infinite. Let K be the field of fractions of R. Let (W, ψ) be a quadratic space over K which is unramified over R. Then there exists a quadratic space (V, φ) over R extending the space (W, ψ), that is the spaces $(V, \varphi) \otimes_R K$ and (W, ψ) are isomorphic.*

Proof. By the purity theorem [OP, Theorem A] there exists a quadratic space (V, φ) over R and an integer $n \geq 0$ such that $(V, \varphi) \otimes_R K \cong (W, \psi) \perp \mathbb{H}_K^n$, where $\mathbb{H}_K$ is a hyperbolic plane. If $n > 0$ then the space $(V, \varphi) \otimes_R K$ is isotropic. By Theorem 1.1 the space (V, φ) is isotropic too. Thus $(V, \varphi) \cong (V', \varphi') \perp \mathbb{H}_R$ for a quadratic space (V', φ') over R. Now Witt's Cancellation theorem over a field [La, Chap.I, Thm.4.2] shows that $(V', \varphi') \otimes_R K \cong (W, \psi) \perp \mathbb{H}_K^{n-1}$. Repeating this procedure several times we may assume that $n = 0$, which means that $(V, \varphi) \otimes_R K \cong (W, \psi)$.

□

Remark 3.2. *Corollary 3.1 is used in the proof of the main result in [ChP]. The main result in [ChP] holds now in the case of a local regular ring R containing a field provided that the residue field of R is infinite and $\frac{1}{2} \in R$.*

Corollary 3.3. *Let R be a semi-local regular integral domain containing a field. Assume that all the residue fields of R are infinite and $\frac{1}{2} \in R$. Let K be the fraction field of R. Let (V, φ) be a quadratic space over R and let $u \in R^{\times}$ be a unit. Suppose the equation $\varphi = u$ has a solution over K then it has a solution over R, that is there exists a vector $v \in V$ with $\varphi(v) = u$ (clearly the vector v is unimodular).*

Proof. It is very standard. However for the completeness of the exposition let us recall the arguments from [C-T, Proof of Prop.1.2]. Let $(R, -u)$ be the rank one quadratic space over R corresponding to the unit $-u$. The space $(V, \varphi)_K \perp (K, -u)$ is isotropic thus the space $(V, \varphi) \perp (R, -u)$ is isotropic by Theorem 1.1. By the lemma below there exists a vector $v \in V$ with $\varphi(v) = u$. Clearly v is unimodular.

□

Lemma 3.4. *Let (V, φ) be as above. Let $(W, \psi) = (V, \varphi) \perp (R, -u)$. The space (W, ψ) is isotropic if and only if there exists a vector $v \in V$ with $\varphi(v) = u$.*

Proof. It is standard. See [C-T, the proof of Proposition 1.2.].

□

Remark 3.5. *It would be nice to extend the result of Corollary 3.1 to the semi-local case. The difficulty is to extend the purity theorem [OP, Theorem A] to that semi-local case.*

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