Cancellation Theorem

Vladimir Voevodsky

Received: December 8, 2008
Revised: May 15, 2010

Abstract. In this paper we give a direct proof of the fact that for any schemes of finite type X, Y over a Noetherian scheme S the natural map of presheaves with transfers

$$\underline{Hom}(\mathbf{Z}_{tr}(X), \mathbf{Z}_{tr}(Y)) \to \underline{Hom}(\mathbf{Z}_{tr}(X) \otimes_{tr} \mathbf{G}_m, \mathbf{Z}_{tr}(Y) \otimes_{tr} \mathbf{G}_m)$$

is a (weak) $\mathbf{A}^1$-homotopy equivalence. As a corollary we deduce that the Tate motive is quasi-invertible in the triangulated categories of motives over perfect fields.

2010 Mathematics Subject Classification: 14F42, 19E15

Contents

1 Introduction

Let $SmCor(k)$ be the category of finite correspondences between smooth schemes over a field k. Denote by $\mathbf{G}_m$ the scheme $\mathbf{A}^1 - \{0\}$. One defines the sheaf with transfers S^1_t by the condition that $\mathbf{Z}_{tr}(\mathbf{G}_m) = S^1_t \oplus \mathbf{Z}$ where $\mathbf{Z}$ is split off by the projection to the point and the point 1. For any scheme Y consider the sheaf with transfers $F_Y = \underline{Hom}(S^1_t, S^1_t \otimes \mathbf{Z}_{tr}(Y))$ which maps a smooth scheme X to $Hom(S^1_t \otimes \mathbf{Z}_{tr}(X), S^1_t \otimes \mathbf{Z}_{tr}(Y))$. The main result of this paper is Corollary 4.9 which asserts that for any Y the obvious map $\mathbf{Z}_{tr}(Y) \to F_Y$ defines a quasi-isomorphism of singular simplicial complexes

$$C_*(\mathbf{Z}_{tr}(Y)) \to C_*(F_Y)$$

as complexes of presheaves i.e. for any X the map of complexes of abelain groups

$$C_*(\mathbf{Z}_{tr}(Y))(X) \to C_*(F_Y)(X)$$

is a quasi-isomorphism. We then deduce from this result the "Cancellation Theorem" for triangulated motives which asserts that if k is a perfect field then for any K, L in $DM_-^{eff}(k)$ the map

$$Hom(K, K') \to Hom(K(1), K'(1))$$

is bijective.

This result was previously known in two particular situations. For varieties over a field k with resolution of singularities it was proved in [4]. For K' being the motivic complex $\mathbf{Z}(n)[m]$ and any field k it was proved in [5]. Both proofs are very long.

The main part of our argument does not use the assumption that we work with smooth schemes over a field and we give it for separated schemes of finite type over a noetherian base. To be able to do it we define in the first section the category of finite correspondences for separated schemes of finite type over a base. The definition is a straightforward generalization of the definition for schemes over a field based on the constructions of [2] and can be skipped. In the second section we define intersection of relative cycles with Cartier divisors and prove the properties of this construction which we need. In the third we prove our main theorem 4.6 and deduce from it the cancellation theorem over perfect fileds 4.10.

In this paper we say "a relative cycle" instead of "an equidimensional relative cycle". All schemes are separated. The letter S is typically reserved for the base scheme which is assumed to be noetherian. All the standard schemes $\mathbf{P}^1$, $\mathbf{A}^1$ etc. are over S. When no confusion is possible we write XY instead of $X \times_S Y$.

I would like to thank Pierre Deligne who explained to me how to compute the length function.

2 Finite correspondences

For a scheme X of finite type over a noetherian scheme S we denote by $c(X/S)$ the group of finite relative cycles on X over S. In [2] this group was denoted by $c_{equi}(X/S, 0)$. If S is regular or if S is normal and the characteristic of X is zero, $c(X/S)$ is the free abelian group generated by closed irreducible subsets of X which are finite over S and surjective over a connected component of S. For the general definition see [2, after Lemma 3.3.9]. A morphism $f : S' \to S$ defines the pull-back homomorphism $c(X/S) \to c(XS'/S')$ which we denote by $cycl(f)$.

For two schemes X, Y of finite type over S we define the group $c(X, Y)$ of *finite correspondences* from X to Y as $c(XY/X)$.

Let us recall the following construction from [2, §3.7]. Let $X' \to X \to S$ be morphisms of finite type, $\mathcal{W}$ a relative cycle on X' over X and $\mathcal{Z}$ a relative cycle on X over S. Then one defines a cycle $Cor(\mathcal{W},\mathcal{Z})$ on X' as follows. Let Z_i be the components of the support of $\mathcal{Z}$ present with multiplicites n_i and $e_i : Z_i \to X$ the corresponding closed embeddings. Let $e'_i : Z_i \times_X X' \to X'$ denote the projections. We set

$$Cor(\mathcal{W},\mathcal{Z}) = \sum_i n_i (e'_i)_* cycl(e_i)(\mathcal{W})$$

where $(e'_i)_*$ is the (proper) push-forward on cycles.
Let X, Y be schemes of finite type over S and

$$f \in c(X,Y) = c(XY/X)$$

$$g \in c(Y,Z) = c(YZ/Y)$$

finite correspondences. Let

$$p_X : XY \to Y$$

$$p_Y : XYZ \to XZ$$

be the projections. We define the composition $g \circ f$ by the formula:

$$g \circ f = (p_Y)_* Cor(cycl(p_X)(g), f) \tag{2.1}$$

This operation is linear in both arguments and thus defines a homomorphism of abelian groups

$$c(X,Y) \otimes c(Y,Z) \to c(X,Z)$$

The lemma below follows immediately from the definition of $Cor(-,-)$ and the fact that the (proper) push-forward commutes with the $cycl(-)$ homomorphisms ([2, Prop. 3.6.2]).

Lemma 2.1 *Let $Y \to X \to S$ be a sequence of morphisms of finite type, $p : Y \to Y'$ a morphism over X, $\mathcal{Y} \in Cycl(Y/X, r) \otimes \mathbf{Q}$ and $\mathcal{X} \in Cycl(X/S, s) \otimes \mathbf{Q}$. Assume that p is proper on the support of $\mathcal{Y}$. Then*

$$p_* Cor(\mathcal{Y},\mathcal{X}) = Cor(p_*(\mathcal{Y},\mathcal{X})).$$

Lemma 2.2 *For any $f \in c(X,Y)$, $g \in c(Y,Z)$, $h \in c(Z,T)$ one has*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

PROOF: Consider the following diagram

$$
\begin{array}{ccccccccc}
XT & \xleftarrow{4} & XYT & \longrightarrow & YT & & & & \\
{\scriptstyle 7}\uparrow & & {\scriptstyle 8}\uparrow & & {\scriptstyle 2}\uparrow & & & & \\
XZT & \xleftarrow{9} & XYZT & \longrightarrow & YZT & \longrightarrow & ZT & \longrightarrow & T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
XZ & \xleftarrow{5} & XYZ & \xrightarrow{9} & YZ & \xrightarrow{1} & Z & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & XY & \xrightarrow{3} & Y & & & & \\
 & & \downarrow & & & & & & \\
 & & X & & & & & &
\end{array}
$$

where the morphisms are the obvious projections. Note that all the squares are cartesian. We will also use the projection $6 : XZ \to Z$.
We have $f \in c(XY/X)$, $g \in c(YZ/Y)$ and $h \in c(ZT/Z)$. The compositions are given by:

$$g \circ f = 5_* Cor(cycl(3)(g), f)$$

$$h \circ g = 2_* Cor(cycl(1)(h), g)$$

$$(h \circ g) \circ f = 4_* Cor(cycl(3)(h \circ g), f) = 4_* Cor(cycl(3)(2_* Cor(cycl(1)(h), g)), f)$$

$$h \circ (g \circ f) = 7_* Cor(cycl(6)(h), g \circ f) = 7_* Cor(cycl(6)(h), 5_* Cor(cycl(3)(g), f))$$

We have:

$$4_* Cor(cycl(3)(2_* Cor(cycl(1)(h), g)), f) =$$
$$= 4_* Cor(8_* cycl(3) Cor(cycl(1)(h), g), f) =$$
$$= 4_* 8_* Cor(cycl(3) Cor(cycl(1)(h), g), f) =$$
$$= 4_* 8_* Cor(Cor(cycl(1 \circ 9)(h), cycl(3)(g)), f)$$

where the first equality holds by [2, Prop. 3.6.2], the second by Lemma 2.1 and the third by [2, Th. 3.7.3]. We also have:

$$7_* Cor(cycl(6)(h), 5_* Cor(cycl(3)(g), f)) =$$
$$= 7_* 9_* Cor(cycl(6 \circ 5)(h), Cor(cycl(3)(g), f))$$

by [2, Lemma 3.7.1]. We conclude that $(h \circ g) \circ f = h \circ (g \circ f)$ by [2, Prop. 3.7.7].

We denote by $Cor(S)$ the category of finite correspondences whose objects are schemes of finite type over S, morphisms are finite correspondences and the composition of morphisms is defined by (2.1).

For a morphism of schemes $f : X \to Y$ let Γ_f be its graph considered as an element of $c(XY/X)$. One verifies easily that $\Gamma_{gf} = \Gamma_g \circ \Gamma_f$ and we get a functor $Sch/S \to Cor(S)$. Below we use the same symbol for a morphism of schemes and its graph considered as a finite correspondence.
The external product of cycles defines pairings

$$c(X, Y) \otimes c(X', Y') \to c(XX', YY')$$

and one verifies easily using the results of [2] that this pairing extends to a tensor structure on $Cor(S)$ with $X \otimes Y := XY$.

3 Intersecting relative cycles with divisors

Let X be a noetherian scheme and D a Cartier divisor on X i.e. a global section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$. One defines the cycle $cycl(D)$ associated with D as follows. Let U_i be an open covering of X such that D_{U_i} is of the form $f_{i,+}/f_{i,-} \in \mathcal{M}^*(U_i)$. Then $cycl(D)$ is determined by the property that

$$cycl(D)_{|U_i} = cycl(f_{i,+}^{-1}(0)) - cycl(f_{i,-}^{-1}(0))$$

where on the right hand side one considers the cycles associated with closed subschemes ([2,]). One defines the support of D as the closed subset $supp(D) := supp(cycl(D))$.
We say that a cycle $\mathcal{Z} = \sum n_i z_i$ on X intersects D properly if the points z_i do not belong to $supp(D)$. Let Z_i be the closure of z_i considered as a reduced closed subscheme and $e_i : Z_i \to X$ the closed embedding. If $\mathcal{Z}$ and D intersect properly we define their intersection $(\mathcal{Z}, D)$ as the cycle

$$(\mathcal{Z}, D) := \sum n_i (e_i)_*(cycl(e_i^*(D)))$$

If $p : X \to S$ is a morphism of finite type and $\mathcal{Z}$ is a relative cycle of relative dimension d over S, we say that D intersects $\mathcal{Z}$ properly relative to p (or properly over S) if the dimension of fibers of $supp(D) \cap supp(\mathcal{Z})$ over S is $\le d - 1$. This clearly implies that $\mathcal{Z}$ intersects D properly and $(\mathcal{Z}, D)$ is defined.

Proposition 3.1 *Let $p : X \to S$ be a morphism of finite type, $\mathcal{Z}$ a relative equidimensional cycle of relative dimension d on X over S and D a Cartier divisor on X which intersects $\mathcal{Z}$ properly over S. Then:*

1. *$(\mathcal{Z}, D)$ is a relative cycle of relative dimension $d-1$ over S,*
2. *let $f : S' \to S$ be a morphism, $X' = (X \times_S S')_{red}$ and let $q_{red} : X' \to X$ be the restriction of the projection to X'. If $q_{red}^*(D)$ is well defined then*

$$f^*(cycl(Z), D) = (f^*(cycl(Z)), q_{red}^*(D)). \tag{3.1}$$

where f^* refers to the pull-back of relative cycles as defined in [2].

Proof: Let $\mathcal{Z} = \sum_i n_i z_i$ where z_i are points on the generic fibers of p and $n_i \neq 0$. As usually we denote by $[z_i]$ the reduced closed subschemes with generic points z_i.
Since our problem is local in the Zariski topology on X and additive in D we may assume that $D = D(f)$ where $f \in \mathcal{O}(X)$ is a function on X which is not zero divisor. The condition that D intersects $\mathcal{Z}$ properly over S is equivalent to the condition that for each i and each point y of S the restriction of f to $([z_i] \times_S Spec(k_y))_{red}$ is not a zero divisor. Localizing around $[z_i]$ we may assume that the restriction of f to $(X \times_S Spec(k_y))_{red}$ is not a zero divisor for any y. Under these assumptions $q^*_{red}(D)$ is well defined for any $f : S' \to S$. The proposition follows now from Lemma 3.2.

Lemma 3.2 *Let Z be an integral scheme, S a reduced scheme, $p : Z \to S$ an equidimensional morphism and $Spec(k) \stackrel{s_0}{\to} Spec(R) \stackrel{s_1}{\to} S$ a fat point over a point $s : Spec(k) \to S$ of S (see [2, p.23]). Let $Z_s = Z \times_S Spec(k)$ and let $q : Z_s \to Z$ be the projection. Let $f \in \mathcal{O}(Z)$ be a function such that the image of f in $\mathcal{O}(Z_s)_{red}$ is not a zero divisor. Then*

$$(s_0, s_1)^*(D(f)) = ((s_0, s_1)^*(\eta), f \circ q_{red}) \tag{3.2}$$

where η is the generic point of Z considered as a cycle on Z and $q_{red} : Z_{s,red} \to Z$ is the restriction of q to the maximal reduced subscheme of Z_s.

Proof: Observe first the cycles on both sides of (3.2) are supported in points of codimension 1 of Z_s. Let z be such a point. We want to show that the multiplicities of the left and right hand sides of (3.2) in z coincide.
To compute $(s_0, s_1)^*(\eta)$ one considers the surjection $\psi : \mathcal{O}_{Z_R} \to H$ such that $ker(\psi)$ is supported in the closed fiber of $Z_R \to Spec(R)$ and H is flat over R. Let p_j be the minimal prime ideals of $\mathcal{O}_{Z_s}$ and $A_i = \mathcal{O}_{Z_s}/p_i$. Then by definition (see [2, Lemma 3.1.2]),

$$(s_0, s_1)^*(\eta) = \sum_j length_{A_j}(q_0^*(H) \otimes A_j) p_j$$

Therefore, for a point z of codimension 1 on Z_s we have

$$mlt_z(((s_0, s_1)^*(\eta), f \circ q_{red})) =$$

$$= \sum_j length_{A_j}(q_0^*(H) \otimes A_j) length_{\mathcal{O}_{Z_s,z}}((A_j/f_j) \otimes \mathcal{O}_{Z_s,z})$$

where f_j is the restriction of $f \circ q_{red}$ to $[p_j]$.
Let $F = \mathcal{O}_Z / f\mathcal{O}_Z$. We have $D(f) = \sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i}) y_i$ where y_i are the generic points of the scheme $Y = f^{-1}(0)$. Let $F_i = F \otimes \mathcal{O}_{[y_i]}$. By definition, we have

$$(s_0, s_1)^*(D(f)) = \sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, Cycl(q_0^*(G_i)).$$

where G_i is a quotient of $q_1^*(F_i)$ which is flat over R and such that the kernel of the projection $\phi_i : q_1^*(F_i) \to G_i$ is supported in the closed fiber of $Z_R \to Spec(R)$. Our conditions imply that this cycle is supported in points of codimension 1 of Z_s and for such a point z the multiplicity of $(s_0, s_1)^*(D(f))$ in z equals

$$mlt_z((s_0, s_1)^*(D(f))) = \sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, length_{\mathcal{O}_{Z_s,z}}(q_0^*(G_i) \otimes \mathcal{O}_{Z_s,z}) \tag{3.3}$$

Let $K_0^\vee(Z_s)$ be the Grothendieck group of the bounded derived category of complexes of coherent sheaves Z_s whose cohomology are supported in codimension ≥ 1. Then the formula

$$l_{Z_s,z}(M) = length_{\mathcal{O}_{Z_s,z}}(M \otimes \mathcal{O}_{Z_s,z})$$

defines an additive functional on this group and we need to show that

$$l_{Z_s,z}(\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, q_0^*(G_i)) = = l_{Z_s,z}(\sum_j length_{A_j}(q_0^*(H) \otimes A_j) A_j/f_j)$$

Let f_s be the image of f in $\mathcal{O}_{Z_s}$ and let $K_s = cone(\mathcal{O}_{Z_s} \stackrel{\cdot f_s}{\to} \mathcal{O}_{Z_s})$. Since f_j are not zero divisors, we have $A_j/f_j = A_j \otimes_{\mathbf{L}} K$ and the additivity of length implies that $l_{Z_s,z}(M \otimes_{\mathbf{L}} K_s)$ is zero on any M which is supported in codimension ≥ 1. Since this condition holds for the difference $q_0^*(H) - (\sum_j length_{A_j}(q_0^*(H) \otimes A_j)A_j)$ we conclude that

$$l_{Z_s,z}(\sum_j length_{A_j}(q_0^*(H) \otimes A_j)A_j/f_j) = l_{Z_s,z}(q_0^*(H) \otimes_{\mathbf{L}} K_s) =$$

$$= l_{Z_s,z}(\mathbf{L}q_0^*(cone(H \stackrel{f}{\to} H)) = l_{Z_s,z}(cone(q_0^*(H) \stackrel{f}{\to} q_0^*(H))) \tag{3.4}$$

Let u be a generator of the maximal ideal of R. Then $ker(\phi_i)$ and $ker(\psi)$ are just the u-torsion elements in $q_1^*(F_i)$ and $\mathcal{O}_{Z_R}$ respectively. In particular, G_i are H-modules i.e. $G_i = G_i \otimes H$. Therefore, both (3.3) and (3.4) are zero if z does not belong to $W_s = Spec(q_0^*(H)) \subset Z_s$ and for $z \in W_s$ we have

$$mlt_z((s_0, s_1)^*(D(f))) = l_{W_s,z}(\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, \mathbf{L}q_W^*(h^*(G_i)))$$

and

$$mlt_z(((s_0, s_1)^*(\eta), f \circ q_{red})) = l_{W_s,z}(\mathbf{L}q_W^*(cone(H \stackrel{f}{\to} H)))$$

where $q_W : W_s \to Spec(H)$ and $h : Spec(H) \to Spec(Z_R)$ are the obvious morphisms. We claim that the difference

$$M = cone(H \stackrel{f}{\to} H) - (\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, h^*(G_i))$$

as an element of K_0 of H-modules is supported in points of $Spec(H)$ of codimension at ≥ 2 and therefore

$$l_{W_s,z}(\mathbf{L}q_W^*(M)) = 0$$

by Lemma 3.4. Indeed, both sides are zero in the generic points of the generic and of the closed fiber. The restriction of f to the generic fiber Z_K of Z_R is not a zero divisor since the map $q_K : Z_K \to Z$ is flat (because S is reduced) and since Z is integral f is not a zero divisor in $\mathcal{O}_Z$. Therefore, the generic fiber of $cone(H \stackrel{f}{\to} H)$ coincides with $q_K^*(F)$ which, as an element of K_0, coincides with $\sum_i length_{\mathcal{O}_{Z,y_i}}(F \otimes \mathcal{O}_{Z,y_i})\, q_K^*(F_i)$ up to codimension ≥ 2.

Lemma 3.3 *Let $p : W \to Spec(R)$ be a flat morphism such that R is a discrete valuation ring, let $s : Spec(k) \to Spec(R)$ be a morphism whose image is the closed point of $Spec(R)$, $W_s = W \times_{Spec(R)} Spec(k)$ and let $q_W : W_s \to W$ be the projection. Let further M be a coherent sheaf on W supported in the closed fiber of p. Then*

$$\mathbf{L}q_W^*(M) \cong q_W^*(M) \oplus q_W^*(M)[1]$$

Proof: Let $s = is'$ be the factorization of s where $i : Spec(R/m) \to Spec(R)$ is the closed embedding and $s' : Spec(k) \to Spec(R/m)$ a flat morphism and let $q_W = q_i' q'$ be the corresponding factorization of q_W. Then it is sufficient to show that $\mathbf{L}q_i^*(M) \cong q_i^*(M) \oplus q_i^*(M)[1]$. Since $(q_i)_*$ is an exact full embedding it is further sufficient to show that $(q_i)_* \mathbf{L}q_i^*(M) \cong (q_i)_* q_i^*(M) \oplus (q_i)_* q_i^*(M)[1]$. The functor $(q_i)_* q_i^*$ is isomorphic to the functor $(-)\otimes B$ where $B = \mathcal{O}_W/p^*(m)$. Therefore, $(q_i)_* \mathbf{L}q_i^*$ is isomorphic to the functor $(-)\otimes_{\mathbf{F}} B$. Since R is a discrete valuation ring m is a principal ideal. Let u be a generator of m. Since p is flat the image of u in $\mathcal{O}_W$ is not a zero divisor. Therefore

$$(-)\otimes_{\mathbf{L}} B = cone((-) \stackrel{u}{\to} (-))$$

If M is supported in the closed fiber of p then $M \otimes B = M$ and the multiplication by u on M equals zero.

Lemma 3.4 *Under the assumptions of Lemma 3.3 let M be a coherent sheaf on W supported in codimension ≥ 2 and let w be a point of codimension 1 on W_s. Then*

$$length_{\mathcal{O}_{W_s,w}}(\mathbf{L}q_W^*(M) \otimes \mathcal{O}_{W_s,w}) = 0 \tag{3.5}$$

Proof: It is sufficient to show that (3.5) holds for $M = \mathcal{O}_W/p$ where p is a prime ideal of codimension ≥ 2. There are two types of prime ideals satisfying this condition - the ideals lying over the generic point and the ideals lying over the closed point. If p lies over the generic point and has codimension ≥ 2 then the closed fiber of the corresponding closed subscheme has codimension at least 2 and $\mathbf{L}q_W^*(M) \otimes \mathcal{O}_{W_s,w} = 0$ since w is of codimension 1.
If p lies in the closed fiber an has codimension ≥ 1 there then $q_W^*(M)$ has finite length in w and (3.5) follows by additivity of length from Lemma 3.3.

Corollary 3.5 *Let $X' \xrightarrow{f} X \to S$ be morphisms of finite type, $\mathcal{Z}$ a relative cycle on X over S and $\mathcal{W}$ a relative cycle on X' over X of dimension 0. Let further D be a Cartier divisor on X' which intersects $\mathcal{W}$ properly over X. Then D intersects $Cor(\mathcal{W},\mathcal{Z})$ properly over S and one has:*

$$(Cor(\mathcal{W},\mathcal{Z}),D)=Cor((\mathcal{W},D),\mathcal{Z}) \tag{3.6}$$

Proof: It is a straightforward corollary of the definition of $Cor(-,-)$ and (3.1).

Lemma 3.6 *Let $f: X'\to X$ be a morphism of schemes of finite type over S, $\mathcal{Z}$ a relative cycle on X' such that f is proper on $supp(\mathcal{Z})$ and D a Cartier divisor on X. Assume that $f^*(D)$ is defined and $\mathcal{Z}$ intersects $f^*(D)$ properly over S. Then $f_*(\mathcal{Z})$ intersects D properly over S and one has:*

$$f_*(\mathcal{Z},f^*(D))=(f_*(\mathcal{Z}),D) \tag{3.7}$$

Proof: Let d be the relative dimension of $\mathcal{Z}$ over S. To see that $f_*(\mathcal{Z})$ intersects D properly over S we need to check that the dimension of the fibers of $supp(D)\cap supp(f_*(\mathcal{Z}))$ over S is $\leq d-1$. This follows from our assumption and the inclusion

$$supp(D)\cap supp(f_*(\mathcal{Z}))\subset supp(D)\cap f(supp(\mathcal{Z}))=$$

$$=f(f^{-1}(supp(D))\cap supp(\mathcal{Z}))=f(supp(f^*(D))\cap supp(\mathcal{Z}))$$

To verify (3.7) it is sufficient to consider the situation locally around the generic points of $f(supp(f^*(D))\cap supp(\mathcal{Z}))$. Therefore we may assume that $D=D(g)$ is the divisor of a regular function g and $\mathcal{Z}=z$ is just one point with the closure Z. Replacing X' by Z and X by $f(Z)$ we may assume that X, X' are integral, f is surjective and X is local of dimension 1. Let $A=\mathcal{O}(X)$, $B=\mathcal{O}(X')$. Consider the function $l_g: M\mapsto l_A(M\otimes{}^L A/g)$ on $K_0(A-mod)$. This function vanishes on modules with the support in the closed point which implies that

$$l_g(B)=deg(f)l_g(A)=deg(f)l_A(A/g)$$

On the other hand $l_g(A)=l_A(B/(f^*(g)))$. Let x'_i be the closed points of X', k'_i their residue fields and k the residue field of the closed point of X. Let further M_i be the part of $B/(f^*(g))$ supported in x'_i. One can easily see that $l_A(B/(f^*(g)))=\sum_i[k'_i:k]l_B(M_i)$. Combining our equalities we get:

$$deg(f)l_A(A/g)=\sum_i[k'_i:k]l_B(M_i) \tag{3.8}$$

which is equivalent to (3.7).

4 Cancellation theorem

Consider a finite correspondence

$$\mathcal{Z} \in c(\mathbf{G}_m X, \mathbf{G}_m Y) = c(\mathbf{G}_m X \mathbf{G}_m Y / \mathbf{G}_m X).$$

Let f_1, f_2 be the projections to the first and the second copy of $\mathbf{G}_m$ respectively and let g_n denote the rational function $(f_1^{n+1}-1)/(f_1^{n+1}-f_2)$ on $\mathbf{G}_m X \mathbf{G}_m Y$.

Lemma 4.1 *For any $\mathcal{Z}$ there exists N such that for all $n \geq N$ the divisor of g_n intersects $\mathcal{Z}$ properly over X and the cycle $(\mathcal{Z}, D(g_n))$ is finite over X.*

Proof: Let $\bar{f}_1 \times \bar{q} : \bar{C} \to \mathbf{P}^1 X$ be a finite morphism which extends the projection $supp(\mathcal{Z}) \to \mathbf{G}_m X$. Let N be an integer such that the rational function $\bar{f}_1^N / f_2$ is regular in a neighborhood of $\bar{f}_1^{-1}(0)$ and the rational function $f_2/\bar{f}_1^N$ is regular in a neighborhood of $\bar{f}_1^{-1}(\infty)$. Then for any $n \geq N$ one has:

1. the restriction of $g_n f_2$ to $supp(\mathcal{Z})$ is regular on a neighborhood of $\bar{f}_1^{-1}(0)$ and equals 1 on $\bar{f}_1^{-1}(0)$
2. the restriction of g_n to $supp(\mathcal{Z})$ is regular a neighborhood of $\bar{f}_1^{-1}(\infty)$ and equals 1 on $\bar{f}_1^{-1}(\infty)$

Conditions (1),(2) imply that the divisor of g_n intersects $\mathcal{Z}$ properly over X and that the relative cycle $(\mathcal{Z}, D(g_n))$ is finite over X.

If $(\mathcal{Z}, D(g_n))$ is defined as a finite relative cycle we let $\rho_n(\mathcal{Z}) \in c(X, Y)$ denote the projection of $(\mathcal{Z}, D(g_n))$ to XY.

Remark 4.2 Note that we can define a finite correspondence $\rho_g(\mathcal{Z}) : X \to Y$ for any function g satisfying the conditions (1),(2) in the same way as we defined $\rho_n = \rho_{g_n}$. In particular, if n and m are large enough then the function $tg_n + (1-t)g_m$ defines a finite correspondence $h = h_{n,m} : X\mathbf{A}^1 \to Y$ such that $h_{|X \times \{0\}} = \rho_m(\mathcal{Z})$ and $h_{|X \times \{1\}} = \rho_n(\mathcal{Z})$, i.e. we get a canonical $\mathbf{A}^1$-homotopy from $\rho_m(\mathcal{Z})$ to $\rho_n(\mathcal{Z})$.

Lemma 4.3 *(i) For a finite correspondence $\mathcal{W} : X \to Y$ and any $n \geq 1$ one has $\rho_n(Id_{\mathbf{G}_m} \otimes \mathcal{W}) = \mathcal{W}$*

(ii) Let e_X be the composition $\mathbf{G}_m X \xrightarrow{pr} X \xrightarrow{\{1\} \times Id} \mathbf{G}_m X$. Then $\rho_n(e_X) = 0$ for any $n \geq 0$.

Proof: The cycle on $\mathbf{G}_m X \mathbf{G}_m Y$ over $\mathbf{G}_m X$ which represents $Id_{\mathbf{G}_m} \otimes \mathcal{W}$ is $\Delta_*(\mathbf{G}_m \times \mathcal{W})$ where Δ is the diagonal embedding $\mathbf{G}_m XY \to \mathbf{G}_m X \mathbf{G}_m Y$. The cycle $(\Delta_*(\mathbf{G}_m \times \mathcal{W}), g_n)$ is $\Delta_*(D \otimes \mathcal{W})$ where D is the divisor of the function $(t^{n+1}-1)/(t^{n+1}-t)$ on $\mathbf{G}_m$. The push-forward of $\Delta_*(D \otimes \mathcal{W})$ to XY is the cycle $deg(D)\mathcal{W}$. Since $deg(D) = 1$ we get the first statement of the lemma.
The cycle $\mathcal{Z}$ on $\mathbf{G}_m X \mathbf{G}_m X$ representing e_X is the image of the embedding $\mathbf{G}_m X \to \mathbf{G}_m X \mathbf{G}_m X$ which is diagonal on X and of the form $t \mapsto (t, 1)$ on $\mathbf{G}_m$. This shows that the restriction of g_n to $supp(\mathcal{Z})$ equals 1 and $(\mathcal{Z}, D(g_n)) = 0$.

Lemma 4.4 *Let $\mathcal{Z} : \mathbf{G}_m X \to \mathbf{G}_m Y$ be a finite correspondence such that $\rho_n(\mathcal{Z})$ is defined. Then for any finite correspondence $\mathcal{W} : X' \to X$, $\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$ is defined and one has*

$$\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W})) = \rho_n(\mathcal{Z}) \circ \mathcal{W} \tag{4.1}$$

Proof: Let us show that (4.1) holds. In the process it will become clear that the left hand side is defined. We can write $\rho_n(\mathcal{Z}) \circ \mathcal{W}$ as the composition

$$X' \xrightarrow{\mathcal{W}} X \xrightarrow{(\mathcal{Z}, D(g_n))} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} Y$$

and $\rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$ as the composition

$$X' \xrightarrow{\mathcal{Y}} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} Y$$

where $\mathcal{Y} = (\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}), D(g_n))$. Consider the diagram

$$\begin{array}{ccccc}
\mathbf{G}_m X' \mathbf{G}_m Y & \xleftarrow{p_1} & \mathbf{G}_m X' X \mathbf{G}_m Y & \longrightarrow & \mathbf{G}_m X \mathbf{G}_m Y \\
 & & \downarrow & & \downarrow \\
 & & X'X & \xrightarrow{p_2} & X \\
 & & \downarrow & & \\
 & & X' & &
\end{array}$$

where the arrows are the obvious projections. If we consider $\mathcal{Z}$ as a cycle of dimension 1 over X then the cycle $\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W})$, considered as a cycle over X', is $(p_1)_* Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W})$ and we have

$$((p_1)_* Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W}), D(g_n)) =$$

$$= (p_1)_* (Cor(cycl(p_2)(\mathcal{Z}), \mathcal{W}), D(g_n)) = (p_1)_* Cor((cycl(p_2)(\mathcal{Z}), D(g_n)), \mathcal{W}) =$$

$$= (p_1)_* Cor(cycl(p_2)(\mathcal{Z}, D(g_n)), \mathcal{W})$$

where the first equality holds by (3.7), the second by (3.6) and the third by (3.1).
The last expression represents the composition $\mathcal{W} \circ (\mathcal{Z}, D(g_n))$ and we conclude that

$$\rho_n(\mathcal{Z}) \circ \mathcal{W} = \rho_n(\mathcal{Z} \circ (Id_{\mathbf{G}_m} \otimes \mathcal{W}))$$

Lemma 4.5 *Let $\mathcal{Z} : \mathbf{G}_m X \to \mathbf{G}_m Y$ be a finite correspondence such that $\rho_n(\mathcal{Z})$ is defined. Then for any morphism of schemes $f : X' \to Y'$, $\rho_n(\mathcal{Z} \otimes f)$ is defined and one has*

$$\rho_n(\mathcal{Z} \otimes f) = \rho_n(\mathcal{Z}) \otimes f \tag{4.2}$$

Proof: Consider the diagram

$$\begin{array}{ccccc} \mathbf{G}_m X X' \mathbf{G}_m Y Y' & \xleftarrow{p_1} & \mathbf{G}_m X X' \mathbf{G}_m Y & \longrightarrow & \mathbf{G}_m X \mathbf{G}_m Y \\ & & \downarrow & & \downarrow \\ & & XX' & \xrightarrow{p_2} & X \end{array}$$

where p_1 is defined by the embedding $X' \xrightarrow{f \times Id} X'Y'$ and the rest of the morphisms are the obvious projections. Consider $\mathcal{Z}$ as a cycle over X. Then $\rho_n(\mathcal{Z} \otimes f)$ is given by the composition

$$\mathbf{G}_m X X' \xrightarrow{\mathcal{Y}_1} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} YY'$$

where $\mathcal{Y}_1 = ((p_1)_* cycl(p_2)(\mathcal{Z}), g_n)$ and $\rho_n(\mathcal{Z}) \otimes f$ by the composition

$$\mathbf{G}_m X X' \xrightarrow{\mathcal{Y}_2} \mathbf{G}_m \mathbf{G}_m Y \xrightarrow{pr} YY'$$

where $\mathcal{Y}_2 = (p_1)_*(cycl(p_2)((\mathcal{Z}, g_n)))$. The equality $\mathcal{Y}_1 = \mathcal{Y}_2$ follows from (3.7) and (3.1).

For our next result we need to use presheaves with transfers. A presheaf with transfers on Sch/S is an additive contravariant functor from $Cor(S)$ to the category of abelian groups. For X in Sch/S we let $\mathbf{Z}_{tr}(X)$ denote the functor represented by X on $Cor(S)$. One defines tensor product of presheaves with transfers in the usual way such that $\mathbf{Z}_{tr}(X) \otimes \mathbf{Z}_{tr}(Y) = \mathbf{Z}_{tr}(X \times Y)$. To simplify notations we will write X instead of $\mathbf{Z}_{tr}(X)$ and identify morphisms $\mathbf{Z}_{tr}(X) \to \mathbf{Z}_{tr}(Y)$ with finite correspondences $X \to Y$. Note in particular that $\mathbf{G}_m$ denotes the presheaf with transfers $\mathbf{Z}_{tr}(\mathbf{G}_m)$ not the presheaf with transfers represented by $\mathbf{G}_m$ as a scheme. To preserve compatibility with the notation XY for the product of X and Y we write FG for the tensor product of presheaves with transfers F and G.
Let S^1_t denote the presheaf with transfers $ker(\mathbf{G}_m \to S)$. We consider it as a direct summand of $\mathbf{G}_m$ with respect to the projection $Id - e$ where e is defined by the composition $\mathbf{G}_m \to S \xrightarrow{1} \mathbf{G}_m$. In the following theorem we let $f \cong g$ denote that the morphisms f and g are $\mathbf{A}^1$-homotopic.

Theorem 4.6 *Let F be a presheaf with transfers such that there is an epimorphism $X \to F$ for a scheme X. Let $\phi : S^1_t \otimes F \to S^1_t Y$ be a morphism. Then there exists a unique up to an $\mathbf{A}^1$-homotopy morphism $\rho(\phi) : F \to Y$ such that $Id_{S^1_t} \otimes \rho(\phi) \cong \phi$.*

Proof: Let us fix an epimorphism $p : X \to F$. Then the morphism ϕ defines a finite correspondence $\mathcal{Z} : \mathbf{G}_m X \to \mathbf{G}_m Y$ and for n sufficiently large we may consider $\rho_n(\mathcal{Z}) : X \to Y$. Lemma 4.4 implies immediately that $\rho_n(\mathcal{Z})$ vanishes on $ker(p)$ and therefore it defines a morphism $\rho_n(\phi) : F \to X$.

Consider a morphism ϕ of the form $Id_{S^1_t} \otimes \psi$. Then $\mathcal{Z}$ is of the form $(Id_{\mathbf{G}_m} - e) \otimes \mathcal{W}$ where $\mathcal{W} : X \to Y$ corresponds to ψ. By Lemma 4.3 we have $\rho_n(\mathcal{Z}) = \mathcal{W}$ and therefore $\rho_n(Id_{S^1_t} \otimes \psi) = \psi$ for any $n \geq 1$. If ρ, ρ' are two morphims such that $Id_{S^1_t} \otimes \rho \cong \phi$ and $Id_{S^1_t} \otimes \rho' \cong \phi$ then for a sufficiently large n we have

$$\rho = \rho_n(Id_{S^1_t} \otimes \rho) \cong \rho_n(Id_{S^1_t} \otimes \rho') = \rho'$$

This implies the uniqueness part of the theorem.
To prove the existence let us show that for a sufficiently large n one has $Id_{S^1_t} \otimes \rho_n(\phi) \cong \phi$. Let $\widetilde{\phi}$ be the morphism $\mathbf{G}_m F \to \mathbf{G}_m Y$ defined by ϕ and let

$$\widetilde{\phi}^* : F\mathbf{G}_m \to Y\mathbf{G}_m$$

be the morphism obtained from $\widetilde{\phi}$ by the obvious permutation.

Lemma 4.7 *The morphisms $\widetilde{\phi} \otimes (Id_{\mathbf{G}_m} - e)$ and $(Id_{\mathbf{G}_m} - e) \otimes \widetilde{\phi^*}$ are $\mathbf{A}^1$-homotopic.*

Proof: One can easily see that these two morphisms are obtained from the morphisms

$$\phi \otimes Id_{S^1_t}, Id_{S^1_t} \otimes \phi^* : S^1_t F S^1_t \to S^1_t Y S^1_t$$

by using the standard direct sum decomposition. One can see further that $\phi \otimes Id_{S^1_t} = \sigma_Y (Id_{S^1_t} \otimes \phi^*) \sigma_F$ where σ_F and σ_Y are the permutations of the two copies of S^1_t in $S^1_t F S^1_t$ and $S^1_t Y S^1_t$ respectively. Lemma 4.8 below implies now that $\phi \otimes Id_{S^1_t} \cong Id_{S^1_t} \otimes \phi^*$.

Lemma 4.8 *The permutation on $S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to $\{-1\} Id \otimes Id$ where $\{-1\} : S^1_t \to S^1_t$ is defined by the morphism $\mathbf{G}_m \xrightarrow{x \mapsto x^{-1}} \mathbf{G}_m$.*

Proof: The same arguments as the ones used in [1, p.142] show that for any scheme X and any pair of invertible functions f, g on X the morphism $X \xrightarrow{f \otimes g} S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to the morphism $g \otimes f^{-1}$. This implies immediately that the permutation on $S^1_t S^1_t$ is $\mathbf{A}^1$-homotopic to the morphism $Id \otimes (\{-1\} Id)$ where $\{-1\} Id : S^1_t \to S^1_t$ is the morphism defined by the map $\mathbf{G}_m \xrightarrow{x \mapsto x^{-1}} \mathbf{G}_m$.

For a sufficiently large n we have

$$\rho_n(\phi \otimes (Id_{\mathbf{G}_m} - e)) = \rho_n(\phi) \otimes (Id_{\mathbf{G}_m} - e)$$

by Lemma 4.5. On the other hand

$$\rho_n((Id_{\mathbf{G}_m} - e) \otimes \phi^*) = \phi^*$$

by Lemma 4.3. By Lemma 4.7 we conclude that

$$\phi^* \cong \rho_n(\phi) \otimes (Id_{\mathbf{G}_m} - e)$$

which is equivalent to $Id_{S^1_t} \otimes \rho_n(\phi) \cong \phi$. Theorem 4.6 is proved.

COROLLARY 4.9 *Denote by F_Y the presheaf*

$$X \mapsto Hom(S^1_t X, S^1_t Y)$$

and consider the obvious map $Y \to F_Y$. Then for any X the corresponding map of complexes of abelian groups

$$C_*(Y)(X) \to C_*(F_Y)(X)$$

is a quasi-isomorphism

PROOF: Let $\Delta^n \cong \mathbf{A}^n$ be the standard algebraic simplex and $\partial\Delta^n$ the subpresheaf in Δ^n which is the union of the images of the face maps $\Delta^{n-1} \to \Delta^n$. Then the n-th homology group of the complex $C_*(F)(X)$ for any F is the group of homotopy classes of maps from $X \otimes (\Delta^n/\partial\Delta^n)$ to F. Our result now follows directly from 4.6.

COROLLARY 4.10 *Let k be a perfect field. Then for any K, L in $DM_-^{eff}(k)$ the map $Hom(K, L) \to Hom(K(1), L(1))$ is a bijection.*

PROOF: Since DM_-^{eff} is generated by objects of the form X it is enough to check that for smooth schemes X, Y over k and $n \in \mathbf{Z}$ one has

$$Hom(S^1_t X, S^1_t Y[n]) = Hom(X, Y[n])$$

By Corollary 4.9 we know that the map

$$Y \to F_Y = \underline{Hom}(S^1_t, S^1_t Y)$$

is an isomorphism in DM. Let us show now that for any sheaf with transfers F and any X one has

$$Hom_{DM}(S^1_t X, F[n]) = Hom_{DM}(X, \underline{Hom}(S^1_t, F)[n]) \tag{4.3}$$

The left hand side of (4.3) is the hypercohomology group $\mathbf{H}^n(\mathbf{G}_m X, C_*(F))$ modulo the subgroup $\mathbf{H}^n(X, C_*(F))$. The right hand side is the hypercohomology group $\mathbf{H}^n(X, C_*\underline{Hom}(\mathbf{G}_m, F))$ modulo similar subgroup. Let $p : \mathbf{G}_m X \to X$ be the projection. It is easy to see that (4.3) asserts that $\mathbf{R}p_*(C_*(F)) \cong C_*(p_*(F))$. There is a spectral sequence which converges to the cohomology sheaves of $\mathbf{R}p_*(C_*(F))$ and starts with the higher direct images $R^i p_*(\underline{H}^j(C_*(F)))$. We need to verify that $R^i p_*(\underline{H}^j(C_*(F))) = 0$ for $i > 0$ and that $p_*(\underline{H}^j(C_*(F))) = \underline{H}^j(C_*(p_*(F)))$. Both statements follow from [3, Prop. 4.34, p.124] and the comparison of Zariski and Nisnevich cohomology for homotopy invariant presheaves with transfers.

References

[1] Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[2] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.

[3] Vladimir Voevodsky. Cohomological theory of presheaves with transfers. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 87–137. Princeton Univ. Press, Princeton, NJ, 2000.

[4] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[5] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.

Vladimir Voevodsky
School of Mathematics
Institute for Advanced Study
Princeton NJ 08540
USA
vladimir@math.ias.edu