Chapter 4

Connections between parabolic properties

This chapter is dedicated to correspondences between Kirchhoff Laplacians and discrete graph Laplacians on the level of Markovian extensions and parabolic properties (e.g., recurrence, stochastic completeness, on-diagonal heat kernel estimates).

4.1 Markovian extensions

As in Section 3.3, let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph (as a metric space). The discussion below is independent of the choice of a concrete model, however, one can, of course, choose a model $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ and look then at $(\mathcal{G}, \mu, \nu)$ as its metric realization. Let also $\mathbf{H}^0$ be the corresponding minimal Kirchhoff Laplacian in $L^2(\mathcal{G}; \mu)$. We start by collecting some basic properties of *Markovian extensions*, that is, of self-adjoint extensions whose quadratic form is a Dirichlet form (see Appendix B for definitions and further facts). First of all, recall that $H^1(\mathcal{G})$ is the weighted Sobolev space defined by (2.20). When equipped with the graph norm (2.21), it turns into a Hilbert space. It is clear that the energy form

$$\mathfrak{Q}[f] = \int_{\mathcal{G}} |\nabla f(x)|^2 \nu(\mathrm{d}x), \tag{4.1}$$

when restricted to $\mathrm{dom}(\mathfrak{Q}_N) = H^1(\mathcal{G})$, is a Dirichlet form on $L^2(\mathcal{G}; \mu)$ and hence the corresponding Neumann Laplacian $\mathbf{H}_N$ is a Markovian extension of $\mathbf{H}^0$. Moreover, the quadratic form $\mathfrak{Q}_D$ of the Friedrichs extension of $\mathbf{H}^0$, which coincides with the Dirichlet Laplacian $\mathbf{H}_D$, is the restriction of $\mathfrak{Q}$ to the subspace $H^1_0(\mathcal{G})$. Recall that $H^1_0(\mathcal{G})$ is defined as the closure of $\mathrm{dom}(\mathbf{H}) \cap C_c(\mathcal{G})$ with respect to $\|\cdot\|_{H^1(\mathcal{G})}$ and hence $\mathfrak{Q}_D$ is a regular Dirichlet form. It is well known that the Dirichlet and Neumann Laplacians play a rather distinctive role among the Markovian extensions of $\mathbf{H}^0$.

Lemma 4.1. *If* $\widetilde{\mathbf{H}}$ *is a Markovian extension of* $\mathbf{H}^0$, *then* $\mathrm{dom}(\widetilde{\mathbf{H}}) \subset H^1(\mathcal{G})$ *and*

$$\mathbf{H}_N \le \widetilde{\mathbf{H}} \le \mathbf{H}_D, \tag{4.2}$$

where the inequalities are understood in the sense of forms.[1] *Moreover, the following*

[1] We shall write $A \le B$ for two non-negative self-adjoint operators A and B if their quadratic forms $\mathfrak{t}_A$ and $\mathfrak{t}_B$ satisfy $\mathrm{dom}(\mathfrak{t}_B) \subseteq \mathrm{dom}(\mathfrak{t}_A)$ and $\mathfrak{t}_A[f] \le \mathfrak{t}_B[f]$ for every $f \in \mathrm{dom}(\mathfrak{t}_B)$. The latter is also equivalent to the fact that $(A+I)^{-1} - (B+I)^{-1}$ is a positive operator.

statements are equivalent:

(i) $\mathbf{H}^0$ *admits a unique Markovian extension,*

(ii) $\mathbf{H}_D = \mathbf{H}_N$,

(iii) $H^1_0(\mathcal{G}) = H^1(\mathcal{G})$,

(iv) *the Gaffney Laplacian* $\mathbf{H}_G$ *is self-adjoint.*

Proof. The proof of [99, Theorem 5.2] carries over to our setting (see also the proof of [78, Theorem 3.3.1]). ∎

An analogous result holds true for weighted graph Laplacians (see [99]). Namely, fix a model $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ and let $\mathbf{h}^0$ be the graph Laplacian defined in $\ell^2(\mathcal{V}; m)$ by (3.7) with the coefficients (3.5) and (3.6) (notice that $\alpha \equiv 0$). In most of this chapter we are going to consider exactly this graph Laplacian, which is related to the Kirchhoff Laplacian. We shall see in Chapter 6 that this is not at all a restriction. Following the considerations in Section 2.2, we can introduce the Dirichlet $\mathbf{h}_D$ and the Neumann $\mathbf{h}_N$ Laplacians. Namely, define the energy form by

$$\mathfrak{q}[\mathbf{f}] := \frac{1}{2} \sum_{u,v \in \mathcal{V}} b(u, v)|\mathbf{f}(u) - \mathbf{f}(v)|^2, \tag{4.3}$$

with the edge weight

$$b(u, v) = \begin{cases} \displaystyle\sum_{\vec{e} \in \vec{\mathcal{E}}_u : e \in \mathcal{E}_v} \frac{\nu(e)}{|e|}, & u \neq v, \\ 0, & u = v, \end{cases} \qquad (u, v) \in \mathcal{V} \times \mathcal{V}, \tag{4.4}$$

and denote by $\mathrm{dom}(\mathfrak{q}_N)$ the space of all $\ell^2(\mathcal{V}; m)$-functions $\mathbf{f}$ such that $\mathfrak{q}[\mathbf{f}]$ is finite. Clearly, the restriction $\mathfrak{q}_N$ of $\mathfrak{q}$ to $\mathrm{dom}(\mathfrak{q}_N)$ is a Dirichlet form. The corresponding self-adjoint operator $\mathbf{h}_N$ is a Markovian extension of $\mathbf{h}^0$ and we refer to it as the *Neumann extension*. Moreover, the Friedrichs extension $\mathbf{h}_D$ is also a Markovian extension of $\mathbf{h}^0$ and we call it the *Dirichlet extension*. Its quadratic form $\mathfrak{q}_D$ is obtained by restricting $\mathfrak{q}_N$ to the domain $\mathrm{dom}(\mathfrak{q}_D)$, which is the closure of $\mathrm{dom}(\mathbf{h}^0)$ with respect to the graph norm

$$\|\cdot\|^2_{H^1(\mathcal{V})} := \mathfrak{q}[\cdot] + \|\cdot\|^2_{\ell^2(\mathcal{V};m)}.$$

Let us also denote

$$H^1(\mathcal{V}) = H^1(\mathcal{V}, m; b) := \mathrm{dom}(\mathfrak{q}_N),$$
$$H^1_0(\mathcal{V}) = H^1_0(\mathcal{V}, m; b) := \mathrm{dom}(\mathfrak{q}_D).$$

The analog of Lemma 4.1 for the discrete operator $\mathbf{h}^0$ now reads (see [99, Theorem 5.2]): *If* $\widetilde{\mathbf{h}}$ *is a Markovian extension of* $\mathbf{h}^0$, *then* $\mathrm{dom}(\widetilde{\mathbf{h}}) \subseteq H^1(\mathcal{V})$ *and*

$$\mathbf{h}_N \leq \widetilde{\mathbf{h}} \leq \mathbf{h}_D. \tag{4.5}$$

4.2 Brownian motion and random walks

The framework of Dirichlet forms relates the energy forms (4.1) and (4.3) with stochastic processes (*Brownian motions* and, respectively, *random walks*) and we will review certain connections known on this level. We will not need these stochastic results in the sequel and hence restrict to a rather informal discussion. However, in our opinion this viewpoint is conceptually important and gives a good motivation for subsequent considerations.

We follow the setup in Section 4.1: $(\mathcal{G}, \mu, \nu)$ is a weighted metric graph and $\mathfrak{Q}_D$ is the corresponding (strongly local) Dirichlet form in $L^2(\mathcal{G})$. Moreover, we fix a model of $(\mathcal{G}, \mu, \nu)$ and consider the corresponding form $\mathfrak{q}_D$ in $\ell^2(\mathcal{V}; m)$ associated with (4.3) and (4.4), where $m\colon \mathcal{V} \to (0, \infty)$ is the vertex weight (3.5). By definition, both $\mathfrak{Q}_D$ and $\mathfrak{q}_D$ are regular Dirichlet forms and hence they correspond to two stochastic processes $(X_t^{\mathcal{G}})_{t\ge 0}$ and $(X_t^{\mathcal{V}})_{t\ge 0}$ (see Remark B.3).

The stochastic process $(X_t^{\mathcal{V}})_{t\ge 0}$ defined by $\mathfrak{q}_D$ is a *continuous-time random walk* (see [12, Remark 5.7], [136, Sections 0.10 and 2.5] and [174] for details and further information). Roughly speaking, a particle starting at some vertex $v \in \mathcal{V}$ first waits for a random waiting time, which is exponentially distributed with parameter

$$\frac{1}{m(v)} \sum_{u\in\mathcal{V}} b(u, v) = \operatorname{Deg}(v), \qquad v \in \mathcal{V} \tag{4.6}$$

(which is called the weighted degree in Section 2.2), and then jumps to a randomly chosen vertex $u \in \mathcal{V}$. Here, the probability of jumping from v to u is given by

$$p(u, v) = \frac{b(u, v)}{\sum_{u\in\mathcal{V}} b(u, v)}, \qquad u, v \in \mathcal{V}. \tag{4.7}$$

Repeating the same steps for the vertex u and continuing in this manner, we end up with a continuous-time random walk. Notice that the expected waiting time of the particle at the vertex v equals $1/\operatorname{Deg}(v)$. In particular, according to Lemma 2.9, the boundedness of $\mathbf{h}_D$ is equivalent to the existence of a uniform positive lower bound for expected waiting times.

On the other hand, the stochastic process $(X_t^{\mathcal{G}})_{t\ge 0}$ associated with $\mathfrak{Q}_D$ is a *Brownian motion* on a metric graph (see, e.g., [72, Section 2], [64, Section 2] and [154, Section 2]). It admits the following informal description: assume that the particle starts at the vertex $v \in \mathcal{V}$. Let $\mathcal{B} = (B_t)_{t\ge 0}$ denote the standard Brownian motion on $\mathbb{R}$ started at the origin. For each excursion of $\mathcal{B}$, we randomly pick an oriented edge $\vec{e} \in \vec{\mathcal{E}}_v$ with probability

$$P(v, \vec{e}) = \frac{\nu(e)}{\sum_{\vec{e}\in\vec{\mathcal{E}}_v} \nu(e)}, \qquad \vec{e} \in \mathcal{E}_v.$$

The excursions are then performed successively in the corresponding edges $e \in \mathcal{E}_v$, starting from v (for a loop edge, the orientation of $\vec{e}$ needs to be taken into account), however with different speeds. Namely, if $\vec{e}_1$ is the first chosen edge, then in the first excursion the particle is at position $|B_{\nu(e_1)t/\mu(e_1)}|$ instead of $|B_t|$ inside e_1 and so on. This is performed until we reach a new vertex $u \in \mathcal{V} \setminus \{v\}$. Then we repeat the construction with u as the starting vertex and continue in the same manner.

To make the connection between the two processes $(X_t^{\mathcal{G}})_{t\geq 0}$ and $(X_t^{\mathcal{V}})_{t\geq 0}$, we briefly recall the results of [72]. Denote by T the *first hitting time* of the Brownian motion, that is, the first time that the Brownian motion started at some vertex hits a different vertex. Then the expected value of T, if the Brownian motion starts at $v \in \mathcal{V}$, is given by (see [72, Theorem 2.2])

$$\mathbb{E}^v T = \frac{\sum_{\vec{e}\in\vec{\mathcal{E}}_v} |e|\mu(e)}{\sum_{w\neq v}\sum_{e\in\mathcal{E}_w\cap\mathcal{E}_v} \frac{\nu(e)}{|e|}}, \quad v \in \mathcal{V}. \tag{4.8}$$

Then the next natural question is which of the neighboring vertices gets hit at the time T. By [72, Theorem 2.1], if the Brownian motion starts at $v \in \mathcal{V}$, then for each $u \sim v$, $u \neq v$, the probability of being this next vertex is precisely

$$\mathbb{P}^v(X_T^{\mathcal{G}} = u) = \frac{\sum_{e\in\mathcal{E}_u\cap\mathcal{E}_v} \frac{\nu(e)}{|e|}}{\sum_{w\neq v}\sum_{e\in\mathcal{E}_w\cap\mathcal{E}_v} \frac{\nu(e)}{|e|}}. \tag{4.9}$$

Comparing (4.6) with (4.8) and (4.7) with (4.9), we see that if m is defined by (3.5) with the weight $r(e)$ given by (3.3) and b by (3.6), they coincide. In fact, the above discussion shows that to a certain extent the continuous-time random walk associated with $\mathfrak{q}_D$ is a discretization of the Brownian motion defined by $\mathfrak{Q}_D$. This can be taken as a first indication for connections between parabolic properties. However, we also stress that already the second moments of the hitting and waiting times differ (see [72, Theorem 2.3]).

4.3 Correspondence between quadratic forms

A more straightforward approach to establish connections between weighted Kirchhoff Laplacians and weighted graph Laplacians is to compare their quadratic forms. Fix a model $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ of $(\mathcal{G}, \mu, \nu)$ and consider the space of *continuous edgewise affine functions on* $\mathcal{G}$,

$$\mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) := \{f \in C(\mathcal{G}) : f|_e \text{ is affine for each edge } e \in \mathcal{E}\}.$$

The importance of $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ stems from the fact that it contains the kernel $\ker(\mathbf{H})$ of the maximal Kirchhoff Laplacian $\mathbf{H}$, as well as all harmonic functions on $\mathcal{G}$, as a subspace (see Section 6.5.2). Clearly, for each refinement of a given model the corresponding space of edgewise affine functions is larger.

Every function $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ can be identified with its values $f|_{\mathcal{V}} = (f(v))_{v\in\mathcal{V}}$ at the vertices. Conversely, we can identify each $\mathbf{f} \in C(\mathcal{V})$ with a continuous edgewise affine function $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ such that $\mathbf{f} = f|_{\mathcal{V}} = (f(v))_{v\in\mathcal{V}}$. This suggests to define the map

$$\begin{aligned} \iota_{\mathcal{V}} \colon C(\mathcal{G}) &\to C(\mathcal{V}), \\ f &\mapsto f|_{\mathcal{V}}. \end{aligned} \tag{4.10}$$

Notice that this map is linear. Moreover, it is bijective when restricted to $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$. In the following we shall denote by $\iota_{\mathcal{V}}^{-1}$ the inverse of its restriction to $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$. Clearly, when restricted to bounded edgewise affine functions, $\iota_{\mathcal{V}}$ is a bijection onto $\ell^\infty(\mathcal{V})$. The situation is not so trivial when $1 \le p < \infty$, as the next result shows. Recall that (see Definition 3.16) a model of a weighted metric graph has finite intrinsic size if

$$\eta^*(\mathcal{E}) = \sup_{e\in\mathcal{E}} \eta(e) = \sup_{e\in\mathcal{E}} |e| \sqrt{\frac{\mu(e)}{\nu(e)}} < \infty. \tag{4.11}$$

Moreover, we define the vertex weight m by (3.5) with r given by (3.3) for models having finite intrinsic size and by (3.4) otherwise.

Lemma 4.2. *If $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) \cap L^p(\mathcal{G};\mu)$, $1 \le p < \infty$, then $\mathbf{f} = \iota_{\mathcal{V}}(f) \in \ell^p(\mathcal{V};m)$, where m is the vertex weight (3.5), (3.3)–(3.4). If additionally the underlying model has finite intrinsic size, then the inclusion $\mathbf{f} \in \ell^p(\mathcal{V};m)$ implies that the corresponding continuous edgewise affine function $f = \iota_{\mathcal{V}}^{-1}(\mathbf{f})$ belongs to $L^p(\mathcal{G};\mu)$ and, moreover,*

$$\|f\|^p_{L^p(\mathcal{G};\mu)} \le \|\mathbf{f}\|^p_{\ell^p(\mathcal{V};m)} \le 4^p \|f\|^p_{L^p(\mathcal{G};\mu)}. \tag{4.12}$$

Proof. Consider the case $p = 1$ first. Then

$$\frac{\ell}{4}(|f(0)| + |f(\ell)|) \le \int_0^\ell |f(x)|\,\mathrm{d}x \le \frac{\ell}{2}(|f(0)| + |f(\ell)|), \tag{4.13}$$

for each affine function on $\mathcal{I}_\ell = [0,\ell]$ and hence

$$\begin{aligned} \|f\|_{L^1(\mathcal{G};\mu)} &= \int_{\mathcal{G}} |f(x)|\mu(\mathrm{d}x) = \sum_{e\in\mathcal{E}} \int_e |f(x)|\mu(\mathrm{d}x) \\ &\ge \frac{1}{4}\sum_{e\in\mathcal{E}} |e|\mu(e)(|f(e_\iota)| + |f(e_\tau)|), \end{aligned}$$

whenever $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$. However, by (3.3)–(3.4),

$$r(e) \le |e|\mu(e) \tag{4.14}$$

for all $e \in \mathcal{E}$, and hence (3.5) implies the estimate

$$\|f\|_{L^1(\mathcal{G};\mu)} \ge \frac{1}{4}\|\iota_{\mathcal{V}}(f)\|_{\ell^1(\mathcal{V};m)} = \frac{1}{4}\|\mathbf{f}\|_{\ell^1(\mathcal{V};m)}.$$

The case $p > 1$ easily follows from the above considerations. Indeed, applying Hölder's inequality to the left-hand side in (4.13) together with the simple inequality

$$(a+b)^p \ge a^p + b^p, \quad a, b, \ge 0,\ p \ge 1,$$

we get from (4.13) the following estimate for edgewise affine functions:

$$4^p \int_e |f(x)|^p \mu(\mathrm{d}x) \ge |e|\mu(e)(|f(e_\iota)|^p + |f(e_\tau)|^p), \quad e \in \mathcal{E}.$$

Summing up over all edges and taking into account (4.14), we finally arrive at the estimate

$$4^p \|f\|^p_{L^p(\mathcal{G};\mu)} \ge \|\imath_{\mathcal{V}}(f)\|^p_{\ell^p(\mathcal{V};m)} = \|\mathbf{f}\|^p_{\ell^p(\mathcal{V};m)}.$$

This proves the first claim as well as the second inequality in (4.12).

Assume now that the model has finite intrinsic size. Then r is defined by (3.3) and hence for $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ we get

$$\begin{aligned}\|f\|^p_{L^p(\mathcal{G})} &= \sum_{e\in\mathcal{E}} \int_e |f(x)|^p \mu(\mathrm{d}x) \\ &\le \sum_{e\in\mathcal{E}} |e|\mu(e) \max_{x\in e} |f(x)|^p \\ &\le \sum_{e\in\mathcal{E}} |e|\mu(e)(|f(e_\iota)|^p + |f(e_\tau)|^p) \\ &\le \sum_{v\in\mathcal{V}} |\mathbf{f}(v)|^p m(v) = \|\mathbf{f}\|^p_{\ell^p(\mathcal{V};m)}.\end{aligned}$$

This clearly implies the first estimate in (4.12) and finishes the proof. ■

Remark 4.3. A few remarks are in order.

(i) Considering $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) \cap L^p(\mathcal{G};\mu)$ as a Banach space with the corresponding L^p norm, the above result actually says that $\imath_{\mathcal{V}}$ is a bounded linear operator from $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) \cap L^p(\mathcal{G};\mu)$ to $\ell^p(\mathcal{V};m)$ for all $1 \le p < \infty$ (however, for $p = \infty$ this claim is trivial) and this is true for each model of a given weighted metric graph. However, this map has a bounded inverse exactly when the model has finite intrinsic size.

(ii) The estimate in (4.12) is not optimal. In particular, in the case $p = 2$ the arguments from [68, Remark 3.8] (see also [149, Section 2.5]) show that

$$2\|f\|^2_{L^2(\mathcal{G};\mu)} \le \|\mathbf{f}\|^2_{\ell^2(\mathcal{V};m)} \le 6\|f\|^2_{L^2(\mathcal{G};\mu)},$$

for any model of finite intrinsic size (for models of infinite intrinsic size, only the second inequality is valid).

(iii) Let us also mention that if $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ is non-negative, $f \geq 0$, then the second inequality in (4.13) turns into equality. Therefore, if the underlying model has finite intrinsic size, we end up with the equality

$$\|f\|_{L^1(\mathcal{G};\mu)} = \frac{1}{2}\|\imath_{\mathcal{V}}(f)\|_{\ell^1(\mathcal{V};m)} = \frac{1}{2}\|\mathbf{f}\|_{\ell^1(\mathcal{V};m)} \tag{4.15}$$

for all $0 \leq f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) \cap L^1(\mathcal{G};\mu)$.

The crucial fact for our further considerations is the observation that the above results can be extended to the H^1 setting:

Corollary 4.4. *If $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V}) \cap H^1(\mathcal{G})$, then $\mathbf{f} = \imath_{\mathcal{V}}(f)$ belongs to $H^1(\mathcal{V})$ and*

$$\mathfrak{Q}[f] = \mathfrak{q}[\mathbf{f}]. \tag{4.16}$$

Conversely, if $\mathbf{f} \in H^1(\mathcal{V})$ and the underlying model has finite intrinsic size, then $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) \in H^1(\mathcal{G})$.

Proof. Taking into account the relationship established in Lemma 4.2, we only need to mention that for $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ the energy forms (4.1) and (4.3) coincide upon identification (4.10):

$$\begin{aligned}
\mathfrak{Q}[f] &= \int_{\mathcal{G}} |\nabla f(x)|^2 \nu(\mathrm{d}x) \\
&= \sum_{e\in\mathcal{E}} \int_e |\nabla f(x)|^2 \nu(\mathrm{d}x) \\
&= \sum_{e\in\mathcal{E}} \frac{\nu(e)}{|e|} |f(e_\imath) - f(e_\tau)|^2 \\
&= \frac{1}{2} \sum_{u,v\in\mathcal{V}} b(v,u)|\mathbf{f}(v) - \mathbf{f}(u)|^2 = \mathfrak{q}[\mathbf{f}].
\end{aligned}$$

■

Every continuous function f on $\mathcal{G}$ can be uniquely decomposed as

$$f = f_{\mathrm{lin}} + f_0, \tag{4.17}$$

where both f_{lin} and f_0 are continuous functions on $\mathcal{G}$, however, f_{lin} is edgewise affine on $\mathcal{G}$, $f_{\mathrm{lin}} \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ and f_0 vanishes at all vertices, that is,

$$f_{\mathrm{lin}}|_{\mathcal{V}} = f|_{\mathcal{V}}, \quad f_0|_{\mathcal{V}} = 0.$$

Notice also the following identity $f_{\mathrm{lin}} = (\imath_{\mathcal{V}}^{-1} \circ \imath_{\mathcal{V}})(f)$ in terms of (4.10). Now we are in a position to state the key technical result connecting the energy forms (4.1) and (4.3). For convenience matters, let us introduce the following notation:

$$H_0^1(\mathcal{G} \setminus \mathcal{V}) = \{f \in H^1(\mathcal{G}) : f|_{\mathcal{V}} = 0\}.$$

Lemma 4.5. *Let $f \in H^1(\mathcal{G})$ and consider its decomposition* (4.17)*. If* (4.11) *is satisfied, then $f_0 \in H^1_0(\mathcal{G} \setminus \mathcal{V})$, $f_{\mathrm{lin}} \in H^1(\mathcal{G})$ and*

$$\mathfrak{Q}[f] = \mathfrak{Q}[f_{\mathrm{lin}}] + \mathfrak{Q}[f_0]. \tag{4.18}$$

Moreover, $\mathbf{f} = \imath_{\mathcal{V}}(f)$ *belongs to $H^1(\mathcal{V})$ and*

$$\mathfrak{Q}[f_{\mathrm{lin}}] = \mathfrak{q}[\mathbf{f}].$$

Proof. A straightforward edgewise integration by parts gives

$$\begin{aligned}\mathfrak{Q}[f] &= \sum_{e\in\mathcal{E}} \int_e |\nabla f(x)|^2 \nu(\mathrm{d}x) \\ &= \sum_{e\in\mathcal{E}} \int_e |\nabla f_{\mathrm{lin}}(x)|^2 + |\nabla f_0(x)|^2 \, \nu(\mathrm{d}x) \\ &= \int_{\mathcal{G}} |\nabla f_{\mathrm{lin}}(x)|^2 \nu(\mathrm{d}x) + \int_{\mathcal{G}} |\nabla f_0(x)|^2 \nu(\mathrm{d}x) = \mathfrak{Q}[f_{\mathrm{lin}}] + \mathfrak{Q}[f_0].\end{aligned}$$

The latter implies that if f is continuous and has finite energy (i.e., it is edgewise in H^1 and $\mathfrak{Q}[f] < \infty$), then both summands on the right-hand side in (4.17) have finite energy. In particular, (4.18) holds for all continuous edgewise H^1 functions on $\mathcal{G}$.

Taking into account the trivial estimate

$$\int_0^{|e|} |f(x)|^2 \, \mathrm{d}x \le \frac{|e|^2}{\pi^2} \int_0^{|e|} |f'(x)|^2 \, \mathrm{d}x,$$

which holds for any $f \in H^1_0([0, |e|])$, we get

$$\|f_0\|_{L^2(\mathcal{G};\mu)} \le \frac{\eta^*(\mathcal{E})}{\pi} \|\nabla f_0\|_{L^2(\mathcal{G};\nu)}. \tag{4.19}$$

Therefore, $f_0 \in L^2(\mathcal{G}; \mu)$ whenever (4.11) holds true and f_0 has finite energy. This immediately implies that $f_{\mathrm{lin}} \in H^1(\mathcal{G})$ if so is f and (4.11) holds. It remains to apply Corollary 4.4. ∎

Remark 4.6. The constant in (4.19) is optimal since so are the corresponding constants in one-dimensional inequalities for H^1_0 functions (see also (3.33)).

To emphasize the role of the map (4.10), let us provide another way to write down the correspondence between self-adjoint extensions of the minimal Kirchhoff Laplacian $\mathbf{H}^0$ and the corresponding minimal graph Laplacian $\mathbf{h}^0$established in Lemma 3.20. For a self-adjoint extension $\widetilde{\mathbf{H}} \in \mathrm{Ext}_S(\mathbf{H}^0)$ of $\mathbf{H}^0$ define the operator $\widetilde{\mathbf{h}}$ in $\ell^2(\mathcal{V}; m)$ by setting

$$\widetilde{\mathbf{h}} := \mathbf{h} \restriction \mathrm{dom}(\widetilde{\mathbf{h}}), \quad \mathrm{dom}(\widetilde{\mathbf{h}}) = \{\imath_{\mathcal{V}}(f) : f \in \mathrm{dom}(\widetilde{\mathbf{H}})\}, \tag{4.20}$$

where $\mathbf{h} = (\mathbf{h}^0)^*$ is the maximal graph Laplacian.

Lemma 4.7. *Let* $\mathbf{H}^0$ *be the minimal Kirchhoff Laplacian with possibly non-trivial deficiency indices,* $\mathrm{n}_{\pm}(\mathbf{H}^0) \geq 0$*. If* $\widetilde{\mathbf{H}} \in \mathrm{Ext}_S(\mathbf{H}^0)$*, then the operator* $\widetilde{\mathbf{h}}$ *defined by* (4.20) *is a self-adjoint extension of* $\mathbf{h}^0$*. Moreover, the induced map*

$$\begin{aligned} \mathrm{Ext}_S(\mathbf{H}^0) &\to \mathrm{Ext}_S(\mathbf{h}^0), \\ \widetilde{\mathbf{H}} &\mapsto \widetilde{\mathbf{h}} \end{aligned} \tag{4.21}$$

is a bijection. The inverse image of a self-adjoint extension $\widetilde{\mathbf{h}}$ *of* $\mathbf{h}^0$ *is the extension*

$$\widetilde{\mathbf{H}} := \mathbf{H} \restriction \mathrm{dom}(\widetilde{\mathbf{H}}), \quad \mathrm{dom}(\widetilde{\mathbf{H}}) = \{f \in \mathrm{dom}(\mathbf{H}) : \imath_{\mathcal{V}}(f) \in \mathrm{dom}(\widetilde{\mathbf{h}})\}. \tag{4.22}$$

Proof. First of all, let us show that the map is well defined, that is, $\widetilde{\mathbf{h}}$ is indeed a self-adjoint restriction of $\mathbf{h}$. Recall that $\widetilde{\mathbf{H}}$ admits the representation (3.38) and, moreover, by Lemma 3.20, there is a self-adjoint extension $\widehat{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}_0)$ such that

$$\widetilde{\Theta} = \Theta_{\mathrm{mul}} \oplus \{(\Phi f, \Phi \widehat{\mathbf{h}} f) : f \in \mathrm{dom}(\widehat{\mathbf{h}})\}.$$

The Kirchhoff conditions at vertices imply that (see (3.13) and (3.25), (3.30))

$$\Gamma_0^{\mathcal{V}} f = \sum_{v \in \mathcal{V}} f(v)\mathbf{f}^v = \Phi(\imath_{\mathcal{V}}(f)) \tag{4.23}$$

for all $f \in \mathrm{dom}(\mathbf{H})$. Therefore, by (3.38),

$$\mathrm{dom}(\widehat{\mathbf{h}}) = \Phi^{-1}(\mathrm{dom}(\widetilde{\Theta})) = \mathrm{dom}(\widetilde{\mathbf{h}}).$$

Thus, by (4.20), we have $\widetilde{\mathbf{h}} = \widehat{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$. Moreover, this also implies that the map (4.21) coincides with the inverse of the map (3.37) and hence (4.21) is a bijection by Lemma 3.20.

It remains to prove the last claim. However, by definition, we have

$$\begin{aligned} \mathrm{dom}(\widetilde{\mathbf{H}}) &\subseteq \{f \in \mathrm{dom}(\mathbf{H}) : \imath_{\mathcal{V}}(f) \in \mathrm{dom}(\widetilde{\mathbf{h}})\} \\ &= \{f \in \mathrm{dom}(\mathbf{H}_{\max}) : (\Gamma_0^{\mathcal{V}} f, \Gamma_1^{\mathcal{V}} f) \in \Theta,\ \imath_{\mathcal{V}}(f) \in \mathrm{dom}(\widetilde{\mathbf{h}})\}. \end{aligned}$$

Taking into account the decomposition

$$\Theta = \Theta_{\mathrm{mul}} \oplus \{(\Phi \mathbf{f}, \Phi\, \mathbf{hf}) : \mathbf{f} \in \mathrm{dom}(\mathbf{h})\},$$

as well as (4.23), it is clear that (4.22) coincides with (3.37), which proves the claim. ∎

Remark 4.8. Since the map (4.20)–(4.21) coincides with the inverse of the map (3.37), Theorem 3.22 (see also Remark 3.23) implies that (4.20) remains to be a bijection when it is further restricted to certain subclasses of self-adjoint extensions (e.g., semibounded, non-negative, etc.).

It turns out that the simple correspondence in Lemma 4.7 also prevails on the level of quadratic forms.

Corollary 4.9. *Suppose that* $\widetilde{\mathbf{H}} \in \mathrm{Ext}_S(\mathbf{H}^0)$ *is a self-adjoint extension of* $\mathbf{H}^0$ *and let* $\widetilde{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$ *be the self-adjoint extension of* $\mathbf{h}^0$ *defined by* (4.20)*. Then*

$$\langle \widetilde{\mathbf{H}} f, f \rangle_{L^2(\mathcal{G};\mu)} = \langle \widetilde{\mathbf{h}}\,\mathbf{f}, \mathbf{f} \rangle_{\ell^2(\mathcal{V};m)} + \int_{\mathcal{G}} |\nabla f_0(x)|^2 \, \nu(\mathrm{d}x) \tag{4.24}$$

for all $f \in \mathrm{dom}(\widetilde{\mathbf{H}})$*, where* $\mathbf{f} = \imath_{\mathcal{V}}(f)$ *and* f_0 *is the function defined by* (4.17)*. In particular,* f_0 *has finite energy,* $\mathfrak{Q}[f_0] = \|\nabla f_0\|^2_{L^2(\mathcal{G};\nu)} < \infty$ *for every* $f \in \mathrm{dom}(\widetilde{\mathbf{H}})$*.*

Proof. Take $f \in \mathrm{dom}(\widetilde{\mathbf{H}})$ and consider $\mathbf{f} = \imath_{\mathcal{V}}(f)$, which belongs to $\mathrm{dom}(\widetilde{\mathbf{h}})$ by definition. Using the same notation as in the proof of Lemma 3.20 and Lemma 4.7, we conclude from (4.23) that

$$\begin{aligned}
\langle \widetilde{\mathbf{h}}\,\mathbf{f}, \mathbf{f} \rangle_{\ell^2(\mathcal{V};m)} &= \langle \widetilde{\mathbf{h}} \Phi^{-1} \Gamma_0^{\mathcal{V}} f, \Phi^{-1} \Gamma_0^{\mathcal{V}} f \rangle_{\ell^2(\mathcal{V};m)} \\
&= \langle \Gamma_1^{\mathcal{V}} f, \Gamma_0^{\mathcal{V}} f \rangle_{\mathcal{H}_{\mathcal{V}}} \\
&= \langle \Gamma_1^{\mathcal{E}} f, \Gamma_0^{\mathcal{E}} f \rangle_{\mathcal{H}_{\mathcal{E}}}.
\end{aligned}$$

Here, $\Pi_{\mathcal{E}}$ and $\Pi_{\mathcal{V}}$ denote the edge-based and vertex-based boundary triplets introduced in Theorem 3.5 and Corollary 3.8 in Section 3.2.2. Decompose $f \in \mathrm{dom}(\widetilde{\mathbf{H}})$ as $f = f_0 + f_{\mathrm{lin}}$ (see (4.17)). A straightforward edgewise integration by parts gives (see (3.10))

$$\begin{aligned}
\langle \widetilde{\mathbf{H}} f, f \rangle_{L^2(\mathcal{G})} &= \sum_{e \in \mathcal{E}} -\langle \Delta f, f \rangle_{L^2(e;\mu)} \\
&= \sum_{e \in \mathcal{E}} \langle \Gamma_{1,e} f, \Gamma_{0,e} f \rangle_{\mathbb{C}^2} + \langle \nabla f_0, \nabla f \rangle_{L^2(e;\nu)} \\
&= \sum_{e \in \mathcal{E}} \langle \Gamma_{1,e} f, \Gamma_{0,e} f \rangle_{\mathbb{C}^2} + \sum_{e \in \mathcal{E}} \langle \nabla f_0, \nabla f \rangle_{L^2(e;\nu)} \\
&= \langle \Gamma_1^{\mathcal{E}} f, \Gamma_0^{\mathcal{E}} f \rangle_{\mathcal{H}_{\mathcal{E}}} + \sum_{e \in \mathcal{E}} \langle \nabla f_0, \nabla f \rangle_{L^2(e;\nu)}.
\end{aligned}$$

Notice that we can rearrange sums. Indeed, both $(\Gamma_{0,e} f)_{e \in \mathcal{E}}$ and $(\Gamma_{1,e} f)_{e \in \mathcal{E}}$ belong to $\mathcal{H}_{\mathcal{E}}$ by Theorem 3.5 and hence the first sum is absolutely convergent. Taking into account that f_0 vanishes on $\mathcal{V}$, we get

$$\langle \nabla f_0, \nabla f \rangle_{L^2(e;\nu)} = \langle \nabla f_0, \nabla f_0 \rangle_{L^2(e;\nu)} \geq 0$$

for all $e \in \mathcal{E}$, which implies that the second series is also absolutely convergent and equals the energy $\mathfrak{Q}[f_0]$ of f_0. This finishes the proof of equality (4.24). ∎

Remark 4.10. Notice that Theorem 3.1 (i) states that the sets of self-adjoint extensions of $\mathbf{H}^0$ and $\mathbf{h}^0$ are in one-to-one correspondence and the concept of boundary triplets provides the explicit correspondence which, however, requires a construction of a suitable boundary triplet. From this perspective, Lemma 4.7 and Corollary 4.9

connect self-adjoint extensions via quadratic forms and this approach has its roots in the pioneering works of M. G. Krein, M. I. Vishik and M. S. Birman in the 1950s on boundary value problems for elliptic PDEs (see, e.g., [56] for more details). However, let us emphasize that the decomposition (4.24) is usually established under the additional assumption that the corresponding symmetric operator is uniformly positive, see [158, f-la (25)] (in our setting this would mean that the Dirichlet Laplacian $\mathbf{H}_D$ has positive spectral gap).

4.4 Correspondence between Markovian extensions

According to (4.2) and (4.5), the sets $\mathrm{Ext}_M(\mathbf{H}^0)$ and $\mathrm{Ext}_M(\mathbf{h}^0)$ of Markovian extensions are nonempty. Lemma 3.20 as well as Lemma 4.7 show that first of all, the sets of self-adjoint extensions $\mathrm{Ext}_S(\mathbf{H}^0)$ and $\mathrm{Ext}_S(\mathbf{h}^0)$ are in bijection, and, what is more important, each self-adjoint extension of $\mathbf{h}^0$ can be seen as a boundary operator parameterizing the corresponding self-adjoint extension of $\mathbf{H}^0$. The further correspondence between their spectral properties indicates that one can hope that (4.20) and (4.21) induce a bijection between the sets $\mathrm{Ext}_M(\mathbf{H}^0)$ and $\mathrm{Ext}_M(\mathbf{h}^0)$ and we shall see that this is indeed the case.

It turns out that the correspondence between Markovian extensions can be conveniently explained using the notion of extended Dirichlet spaces (see Appendix B.3 for details) and we need to introduce the following function spaces. Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Recall that the energy of a continuous, edgewise H^1-function $f : \mathcal{G} \to \mathbb{C}$ is given by

$$\mathfrak{Q}[f] := \|\nabla f\|^2_{L^2(\mathcal{G};\nu)} = \int_{\mathcal{G}} |\nabla f(x)|^2 \, \nu(\mathrm{d}x). \tag{4.25}$$

The space of *functions of finite energy* is defined as

$$\dot{H}^1(\mathcal{G}) := \{f \in C(\mathcal{G}) : f|_e \in H^1(e) \text{ for all } e \in \mathcal{E},\ \mathfrak{Q}[f] < \infty\},$$

and its subspace of functions vanishing on the vertex set is denoted by $\dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$,

$$\dot{H}^1_0(\mathcal{G} \setminus \mathcal{V}) := \{f \in \dot{H}^1(\mathcal{G}) : \imath_{\mathcal{V}}(f) \equiv 0\}.$$

Let us stress at this point that in contrast to the Sobolev space $H^1(\mathcal{G})$ we do not require f to belong to $L^2(\mathcal{G};\mu)$ (for example, $\mathbb{1}$ always belongs to $\dot{H}^1(\mathcal{G})$, however, $\mathbb{1} \in H^1(\mathcal{G})$ exactly when $\mu(\mathcal{G}) < \infty$).

Since $\dot{H}^1(\mathcal{G}) \subset C(\mathcal{G})$, each $f \in \dot{H}^1(\mathcal{G})$ can be decomposed into $f = f_{\mathrm{lin}} + f_0$ as in (4.17) and, moreover, we easily get (see the proof of Lemma 4.5)

$$\mathfrak{Q}[f] = \mathfrak{Q}[f_{\mathrm{lin}}] + \mathfrak{Q}[f_0], \tag{4.26}$$

which implies that $f_{\mathrm{lin}} \in \dot{H}^1(\mathcal{G})$ and $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$ whenever $f \in \dot{H}^1(\mathcal{G})$. Moreover, the calculations in the proof of Corollary 4.4 imply that

$$\mathfrak{Q}[f_{\mathrm{lin}}] = \mathfrak{q}[\mathbf{f}] = \frac{1}{2} \sum_{u,v \in \mathcal{V}} b(v,u)|\mathbf{f}(v) - \mathbf{f}(u)|^2,$$

where $\mathbf{f} = \imath_{\mathcal{V}}(f) = \imath_{\mathcal{V}}(f_{\mathrm{lin}})$. In particular, this means that a function $\mathbf{f} \in C(\mathcal{V})$ has finite energy, $\mathfrak{q}[\mathbf{f}] < \infty$ exactly when the corresponding edgewise affine function $f_{\mathrm{lin}} = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ has finite energy. In contrast to the usual Sobolev space $H^1(\mathcal{G})$, the above decomposition holds for all models of a given metric graph (see Lemma 4.5) and exactly this fact makes the use of extended Dirichlet spaces very convenient. In particular, a similar decomposition holds for all Markovian extensions and the corresponding extended Dirichlet spaces.

Lemma 4.11. *Let $\widetilde{\mathbf{H}}$ be a Markovian extension of the minimal Kirchhoff Laplacian $\mathbf{H}^0$ and $\widetilde{\mathfrak{Q}}_e\colon \mathrm{dom}(\widetilde{\mathfrak{Q}}_e) \to [0,+\infty)$ the corresponding extended Dirichlet form. Then:*

(i) $\mathrm{dom}(\widetilde{\mathfrak{Q}}_e) \subseteq \dot{H}^1(\mathcal{G})$.

(ii) $\dot{H}^1_0(\mathcal{G} \setminus \mathcal{V}) \subseteq \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$ *and for each* $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$

$$\widetilde{\mathfrak{Q}}_e[f_0] = \mathfrak{Q}[f_0].$$

(iii) *Each* $f \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$ *has an approximating sequence* $(f_n)_n \subset \mathrm{dom}(\widetilde{\mathbf{H}})$.

(iv) *If* $f = f_{\mathrm{lin}} + f_0 \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$*, then* $f_{\mathrm{lin}} \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$*,* $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$ *and*

$$\widetilde{\mathfrak{Q}}_e[f] = \widetilde{\mathfrak{Q}}_e[f_{\mathrm{lin}}] + \mathfrak{Q}[f_0].$$

Proof. (i) By Lemma 4.1, $\mathbf{H}_N \leq \widetilde{\mathbf{H}}$. Moreover, it is easy to observe that the extended Dirichlet space for $\mathfrak{Q}_N$ is contained in $\dot{H}^1(\mathcal{G})$, which implies the desired inclusion.

(ii) For each $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$ there exists a sequence $(f_n)_n \subset \mathrm{dom}(\mathbf{H}) \cap C_c(\mathcal{G})$ such that each f_n vanishes in a neighborhood of all vertices and

$$\lim_{n\to\infty} \mathfrak{Q}[f_0 - f_n] = 0.$$

The claim now follows easily from Corollary 4.9.

(iii) This is an immediate consequence of the fact that $\mathrm{dom}(\widetilde{\mathbf{H}})$ is a core of $\mathrm{dom}(\widetilde{\mathfrak{Q}})$ and convergence in the graph norm of $\widetilde{\mathfrak{Q}}$ implies uniform convergence on compact subsets of $\mathcal{G}$.

(iv) Take $f = f_{\mathrm{lin}} + f_0 \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$. By (i), $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$ and hence (ii) implies that $f_{\mathrm{lin}} \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$. By (iii), pick an approximating sequence $(f_n)_n \subset \mathrm{dom}(\widetilde{\mathbf{H}})$ for f with $f_n = f_{n,0} + f_{n,\mathrm{lin}}$ for each n. By the proof of (ii), there exists an approximating sequence $(g_n)_n \subset \mathrm{dom}(\mathbf{H}) \cap C_c(\mathcal{G})$ for f_0 such that $g_n|_{\mathcal{V}} \equiv 0$. Corollary 4.9 implies that $(f_{n,0})_n$ and $(g_n)_n$ are $\mathfrak{Q}$-Cauchy sequences. Moreover, it is straightforward to show that

$$\lim_{n\to\infty} \mathfrak{Q}[f_0 - f_{n,0}] = \lim_{n\to\infty} \mathfrak{Q}[f_0 - g_n] = 0.$$

Since $(f_n - g_n)_n$ is an approximating sequence for f_{lin}, by Corollary 4.9 we get

$$\begin{aligned}\widetilde{\mathfrak{Q}}[f_{\mathrm{lin}}] &= \lim_{n\to\infty} \langle \widetilde{\mathbf{h}}\,\mathbf{f}_n, \mathbf{f}_n\rangle + \mathfrak{Q}[f_{n,0} - g_n] \\ &= \lim_{n\to\infty} \langle \widetilde{\mathbf{h}}\,\mathbf{f}_n, \mathbf{f}_n\rangle + \mathfrak{Q}[f_{n,0}] - \mathfrak{Q}[f_{n,0}] \\ &= \widetilde{\mathfrak{Q}}[f] - \mathfrak{Q}[f_0].\end{aligned}$$

This completes the proof of Lemma 4.11. ■

Now we are in a position to state the main result of this section.

Theorem 4.12. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Then the map defined by* (4.20) *induces a bijection*

$$\begin{aligned}\mathrm{Ext}_M(\mathbf{H}^0) &\to \mathrm{Ext}_M(\mathbf{h}^0),\\ \widetilde{\mathbf{H}} &\mapsto \widetilde{\mathbf{h}}.\end{aligned}$$

Proof. By Lemma 4.7, the map (4.20) is a bijection between $\mathrm{Ext}_S(\mathbf{H}^0)$ and $\mathrm{Ext}_S(\mathbf{h}^0)$ and hence we only need to show that $\widetilde{\mathbf{H}} \in \mathrm{Ext}_S(\mathbf{H}^0)$ is Markovian exactly when so is the corresponding $\widetilde{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$. We divide the proof into several steps.

(i) First suppose that $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ and $\widetilde{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$ is defined by (4.20) with the corresponding quadratic form $\widetilde{\mathfrak{q}}$ in $\ell^2(\mathcal{V}; m)$. Let us show that $\widetilde{\mathbf{h}}$ is also Markovian. Define the quadratic form

$$\hat{\mathfrak{q}}_e[\mathbf{f}] := \widetilde{\mathfrak{Q}}_e[\imath_{\mathcal{V}}^{-1}(\mathbf{f})], \quad \mathbf{f} \in \mathrm{dom}(\hat{\mathfrak{q}}_e) := \{f \in C(\mathcal{V}) : \imath_{\mathcal{V}}^{-1}(f) \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)\}, \tag{4.27}$$

and also its $\ell^2(\mathcal{V}; m)$ restriction (compare with (B.3))

$$\hat{\mathfrak{q}} := \hat{\mathfrak{q}}_e \restriction \mathrm{dom}(\hat{\mathfrak{q}}), \quad \mathrm{dom}(\hat{\mathfrak{q}}) = \mathrm{dom}(\hat{\mathfrak{q}}_e) \cap \ell^2(\mathcal{V}; m). \tag{4.28}$$

Here $\widetilde{\mathfrak{Q}}_e$ is the extended Dirichlet form of $\widetilde{\mathfrak{Q}}$. It is straightforward to prove that $\hat{\mathfrak{q}}$ is closed, which basically follows from the fact that $\widetilde{\mathfrak{Q}}_e$ is closed under taking a.e. pointwise limits of $\widetilde{\mathfrak{Q}}_e$-Cauchy sequences. Moreover, $\hat{\mathfrak{q}}$ inherits the Markovian property from $\widetilde{\mathfrak{Q}}_e$. Indeed, take $\mathbf{f} \in \mathrm{dom}(\hat{\mathfrak{q}})$ and pick a normal contraction $\varphi: \mathbb{C} \to \mathbb{C}$. Then $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e)$ and hence $\varphi \circ \mathbf{f} = \imath_{\mathcal{V}}(\varphi \circ f)$ belongs to $\mathrm{dom}(\hat{\mathfrak{q}})$ since $\widetilde{\mathfrak{Q}}_e$ is Markovian (see Appendix B.3). Moreover, Lemma 4.11 implies

$$\begin{aligned}\hat{\mathfrak{q}}[\varphi \circ \mathbf{f}] = \hat{\mathfrak{q}}_e[\varphi \circ \mathbf{f}] &= \widetilde{\mathfrak{Q}}_e[\imath_{\mathcal{V}}^{-1}(\varphi \circ \mathbf{f})] \\ &\le \widetilde{\mathfrak{Q}}_e[\varphi \circ f] \le \widetilde{\mathfrak{Q}}_e[f] = \widetilde{\mathfrak{Q}}_e[\imath_{\mathcal{V}}^{-1}(\mathbf{f})] = \hat{\mathfrak{q}}[\mathbf{f}].\end{aligned}$$

Thus, $\hat{\mathfrak{q}}$ is a Dirichlet form in $\ell^2(\mathcal{V}; m)$ and the corresponding self-adjoint operator $\widehat{\mathbf{h}}$ is Markovian. Hence to prove the claim it suffices to show that $\widehat{\mathbf{h}} = \widetilde{\mathbf{h}}$ (or equivalently that $\hat{\mathfrak{q}} = \widetilde{\mathfrak{q}}$).

First of all, (4.20) implies that $\mathrm{dom}(\widetilde{\mathbf{h}}) \subseteq \mathrm{dom}(\hat{\mathfrak{q}})$ and $\widetilde{\mathfrak{q}} = \hat{\mathfrak{q}}$ on $\mathrm{dom}(\widetilde{\mathbf{h}})$ by Corollary 4.9. Therefore, $\widetilde{\mathbf{h}} \ge \widehat{\mathbf{h}}$.

To prove the converse, observe that $\widehat{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$. Indeed, take $\mathbf{f} \in \mathrm{dom}(\mathbf{h}')$ and $\mathbf{g} \in \mathrm{dom}(\widehat{\mathfrak{q}})$ and then pick an $f \in \mathrm{dom}(\mathbf{H}')$ with $\imath_{\mathcal{V}}(f) = \mathbf{f}$ and an approximating sequence $(g_n)_n \subset \mathrm{dom}(\widetilde{\mathbf{H}})$ for $g := \imath_{\mathcal{V}}^{-1}(\mathbf{g}) \in \mathrm{dom}(\widetilde{\mathfrak{Q}}_e) \cap \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$. Then by Lemma 4.11 (iv),

$$\begin{aligned}\widehat{\mathfrak{q}}[\mathbf{f}, \mathbf{g}] &= \widetilde{\mathfrak{Q}}_e[\imath_{\mathcal{V}}^{-1}(f), \imath_{\mathcal{V}}^{-1}(g)] = \widetilde{\mathfrak{Q}}_e[f, \imath_{\mathcal{V}}^{-1}(g)] \\ &= \lim_{n\to\infty} \widetilde{\mathfrak{Q}}[f, g_n] = \lim_{n\to\infty} \langle \mathbf{H}' f, g_n \rangle_{L^2}.\end{aligned}$$

Since $\widetilde{\mathbf{H}} \ge \mathbf{H}_N$ (see Lemma 4.1), it follows that g_n converges to g uniformly on compact subsets of $\mathcal{G}$. Using integration by parts and (4.26),

$$\widehat{\mathfrak{q}}[\mathbf{f}, \mathbf{g}] = \langle \mathbf{H}' f, g \rangle_{L^2} = \mathfrak{Q}[f, g] = \mathfrak{q}[\mathbf{f}, \mathbf{g}] = \langle \mathbf{h}'\mathbf{f}, \mathbf{g} \rangle_{\ell^2},$$

which shows that $\mathbf{h}' \subseteq \widehat{\mathbf{h}}$ and hence $\widehat{\mathbf{h}} \in \mathrm{Ext}_S(\mathbf{h}^0)$.

Let $\widehat{\mathbf{H}}$ be the non-negative self-adjoint extension of $\mathbf{H}^0$ corresponding to $\widehat{\mathbf{h}}$ via (4.20). Again, we infer from Lemma 4.7, Lemma 4.11 (iv) and Corollary 4.9 that (see also (B.3))

$$\mathrm{dom}(\widehat{\mathbf{H}}) \subseteq \mathrm{dom}(\widetilde{\mathfrak{Q}}_e) \cap L^2(\mathcal{G}; \mu) = \mathrm{dom}(\widetilde{\mathfrak{Q}})$$

and that $\widehat{\mathfrak{Q}} = \widetilde{\mathfrak{Q}}$ on $\mathrm{dom}(\widehat{\mathbf{H}})$. This implies that $\widehat{\mathbf{H}} \ge \widetilde{\mathbf{H}}$. However, the map between non-negative extensions of $\mathbf{H}^0$ and $\mathbf{h}^0$ is monotonic (this can easily be deduced from Krein's resolvent formula (A.5)), that is, $\widetilde{\mathbf{H}}_1 \ge \widetilde{\mathbf{H}}_2$ exactly when $\widetilde{\mathbf{h}}_1 \ge \widetilde{\mathbf{h}}_2$. Hence we conclude that $\widehat{\mathbf{h}} = \widetilde{\mathbf{h}}$.

(ii) It remains to show that $\widetilde{\mathbf{H}}$ is a Markovian extension of $\mathbf{H}^0$ if $\widetilde{\mathbf{h}}$ is a Markovian extension of $\mathbf{h}^0$. The proof essentially consists in reversing the construction of the previous step. More precisely, we define the quadratic form

$$\begin{aligned}\widehat{\mathfrak{Q}}_e[f] &:= \widetilde{\mathfrak{q}}_e[\imath_{\mathcal{V}}(f)] + \mathfrak{Q}[f_0], \\ f \in \mathrm{dom}(\widehat{\mathfrak{Q}}_e) &:= \{g \in \dot{H}^1(\mathcal{G}) : \imath_{\mathcal{V}}(g) \in \mathrm{dom}(\widetilde{\mathfrak{q}}_e)\},\end{aligned} \tag{4.29}$$

and consider its restriction

$$\widehat{\mathfrak{Q}} := \widehat{\mathfrak{Q}}_e \restriction \mathrm{dom}(\widehat{\mathfrak{Q}}), \quad \mathrm{dom}(\widehat{\mathfrak{Q}}) = \mathrm{dom}(\widehat{\mathfrak{Q}}_e) \cap L^2(\mathcal{G}; \mu). \tag{4.30}$$

Similar to the previous step, it turns out that $\widehat{\mathfrak{Q}}$ is a Dirichlet form in $L^2(\mathcal{G}; \mu)$ and the associated operator coincides with $\widetilde{\mathbf{H}}$, that is, $\widehat{\mathbf{H}} = \widetilde{\mathbf{H}}$. Let us only prove that $\widehat{\mathfrak{Q}}$ verifies the Markovian property (B.1) since the other claimed properties can be verified without difficulty analogous to the previous step and we omit the details.

Take $f \in \widehat{\mathfrak{Q}}$ and pick a normal contraction $\varphi\colon \mathbb{C} \to \mathbb{C}$. By [133, Theorem 3.12] (see also (4.5)), the difference $\widetilde{\mathfrak{q}}_e - \mathfrak{q}$ satisfies the Markovian condition (B.1) on $\mathrm{dom}(\widetilde{\mathfrak{q}}_e)$. Setting $\mathbf{f} := \imath_{\mathcal{V}}(f)$, we see that

$$\imath_{\mathcal{V}}(\varphi \circ f) = \varphi \circ \mathbf{f}$$

and in particular $\varphi \circ f \in \operatorname{dom}(\widetilde{\mathfrak{q}})$. Moreover, it follows from (4.26) that

$$\begin{aligned}\widehat{\mathfrak{Q}}[f] &= \widetilde{\mathfrak{q}}_e[\mathbf{f}] + \mathfrak{Q}[f_0] = \widetilde{\mathfrak{q}}_e[\mathbf{f}] + \mathfrak{Q}[f] - \mathfrak{Q}[\imath_{\mathcal{V}}^{-1}(\mathbf{f})] = \widetilde{\mathfrak{q}}_e[\mathbf{f}] - \mathfrak{q}[\mathbf{f}] + \mathfrak{Q}[f] \\ &\geq (\widetilde{\mathfrak{q}}_e - \mathfrak{q})[\varphi \circ \mathbf{f}] + \mathfrak{Q}[\varphi \circ f] = \widehat{\mathfrak{Q}}[\varphi \circ f],\end{aligned}$$

which shows that $\widehat{\mathfrak{Q}}$ is Markovian. ∎

The proof of Theorem 4.12 in fact contains the following transparent correspondence between the extended Dirichlet forms (see (4.27)–(4.28) and (4.29)–(4.30)).

Corollary 4.13. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Let also $\widetilde{\mathbf{H}}$ be a Markovian extension of $\mathbf{H}^0$ and consider the associated Markovian extension $\widetilde{\mathbf{h}}$ of $\mathbf{h}^0$ defined by (4.20). The domains of the corresponding extended Dirichlet forms $\widetilde{\mathfrak{Q}}_e$ and $\widetilde{\mathfrak{q}}_e$ are related by*

$$\begin{aligned}\operatorname{dom}(\widetilde{\mathfrak{q}}_e) &= \{\imath_{\mathcal{V}}(f) : f \in \operatorname{dom}(\widetilde{\mathfrak{Q}}_e)\}, \\ \operatorname{dom}(\widetilde{\mathfrak{Q}}_e) &= \{f \in \dot{H}^1(\mathcal{G}) : \imath_{\mathcal{V}}(f) \in \operatorname{dom}(\widetilde{\mathfrak{q}}_e)\}.\end{aligned}$$

Moreover, for every function $f \in \operatorname{dom}(\widetilde{\mathfrak{Q}}_e)$,

$$\widetilde{\mathfrak{Q}}_e[f] = \widetilde{\mathfrak{q}}_e[\imath_{\mathcal{V}}(f)] + \mathfrak{Q}[f_0].$$

However, the above correspondence cannot be extended to the Dirichlet forms (and form domains) without further restrictions on the underlying model.

Corollary 4.14. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model having finite intrinsic size. Let $\widetilde{\mathbf{H}} \in \operatorname{Ext}_M(\mathbf{H}^0)$ and $\widetilde{\mathbf{h}} \in \operatorname{Ext}_M(\mathbf{h}^0)$ be given by (4.20). Then the corresponding Dirichlet forms $\widetilde{\mathfrak{Q}}$ and $\widetilde{\mathfrak{q}}$ are connected by*

$$\widetilde{\mathfrak{q}}[\mathbf{f}] = \widetilde{\mathfrak{Q}}[\imath_{\mathcal{V}}^{-1}(\mathbf{f})], \quad \mathbf{f} \in \operatorname{dom}(\widetilde{\mathfrak{q}}) = \{\imath_{\mathcal{V}}(f) : f \in \operatorname{dom}(\widetilde{\mathfrak{Q}})\},$$

and

$$\begin{aligned}\operatorname{dom}(\widetilde{\mathfrak{Q}}) &= \{\imath_{\mathcal{V}}^{-1}(\mathbf{f}) + f_0 : \mathbf{f} \in \operatorname{dom}(\widetilde{\mathfrak{q}}),\ f_0 \in H^1_0(\mathcal{G} \setminus \mathcal{V})\}, \\ \widetilde{\mathfrak{Q}}[f] &= \widetilde{\mathfrak{q}}[\mathbf{f}] + \mathfrak{Q}[f_0], \quad f = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) + f_0 \in \operatorname{dom}(\widetilde{\mathfrak{Q}}).\end{aligned}$$

Proof. Taking into account (B.3), the proof is a straightforward combination of Corollary 4.13, Lemma 4.11 and Lemma 4.5. ∎

Remark 4.15. It is easy to show that under the finite intrinsic size assumption (4.11), Corollary 4.14 holds true for non-negative extensions $\widetilde{\mathbf{H}} \in \operatorname{Ext}^+_S(\mathbf{H}^0)$ and $\widetilde{\mathbf{h}} \in \operatorname{Ext}^+_S(\mathbf{h}^0)$ as well. However, we restrict to the special case of Markovian extensions for the sake of a streamlined exposition.

Remark 4.16. The results of the present section remain valid for Laplacians with δ-couplings $\mathbf{H}^0_\alpha$ (see Section 2.4.3) and their associated discrete Laplacians $\mathbf{h}^0_\alpha$ (see (3.7) and Theorem 3.1), of course under the additional assumption that all strengths are non-negative, that is, $\alpha \colon \mathcal{V} \to [0, \infty)$.

4.5 Recurrence/transience

As it was explained in Section 4.2, the connection between a Brownian motion on a metric graph and a continuous time random walk on a graph indicates a connection between the corresponding heat semigroups. The main tool to confirm this intuition is the close relationship between the energy forms established in the previous sections. We begin with the study of *recurrence* and *transience* (see Appendix B.2 for definitions and further references).

Theorem 4.17. *Let* $(\mathcal{G}, \mu, \nu)$ *be a weighted metric graph together with a fixed model. Let also* $\widetilde{\mathbf{H}}$ *be a Markovian extension of* $\mathbf{H}^0$ *and* $\widetilde{\mathbf{h}}$ *the corresponding Markovian extension of* $\mathbf{h}^0$ *(see Theorem 4.12). Then the heat semigroup* $(e^{-\widetilde{\mathbf{H}}t})_{t>0}$ *is recurrent (respectively, transient) if and only if the semigroup* $(e^{-\widetilde{\mathbf{h}}t})_{t>0}$ *is recurrent (respectively, transient).*

Proof. The claim follows immediately from the recurrence characterization by means of extended Dirichlet spaces (see Lemma B.7) and the relationship between extended Dirichlet spaces established in Corollary 4.13. Notice also that $\mathcal{G}$ (and hence $\mathcal{G}_d$ for each model of $\mathcal{G}$) is connected and hence the corresponding Dirichlet form is irreducible, which implies the recurrence/transience dichotomy. ■

Remark 4.18. Let us stress that recurrence/transience is independent of the choice of a model of a weighted metric graph (one may even allow models having infinite intrinsic size). So, the situation is analogous to the self-adjoint uniqueness (cf. Corollary 3.15): *If* $(e^{-\widetilde{\mathbf{H}}t})_{t>0}$ *is recurrent, then* $(e^{-\widetilde{\mathbf{h}}t})_{t>0}$ *is recurrent for all models of* $(\mathcal{G}, \mu, \nu)$*. And conversely,* $(e^{-\widetilde{\mathbf{H}}t})_{t>0}$ *is recurrent if* $(e^{-\widetilde{\mathbf{h}}t})_{t>0}$ *is recurrent for one (and hence for all) models of* $(\mathcal{G}, \mu, \nu)$.

Remark 4.19. A similar approach connecting recurrence/transience on graphs and metric graphs was suggested in [97, Chapter 4].

For the two extremal Markovian extensions, the Dirichlet and Neumann Laplacians $\mathbf{H}_D$ and $\mathbf{H}_N$, we obtain the following characterizations.

Corollary 4.20. *Let* $(\mathcal{G}, \mu, \nu)$ *be a weighted metric graph together with a fixed model. The following statements are equivalent for the Neumann Laplacian* $\mathbf{H}_N$*:*

- (i) $(e^{-\mathbf{H}_N t})_{t>0}$ *is recurrent.*
- (ii) $(e^{-\mathbf{h}_N t})_{t>0}$ *is recurrent.*
- (iii) $\mathbb{1} \in \mathrm{dom}(\mathfrak{Q}_N^e)$*, where* $\mathrm{dom}(\mathfrak{Q}_N^e)$ *is the extended Dirichlet space of* $\mathfrak{Q}_N$*.*
- (iv) $\mathrm{dom}(\mathfrak{Q}_N^e) = \dot{H}^1(\mathcal{G})$.

Proof. Since $\mathbb{1} \in \dot{H}^1(\mathcal{G})$, in view of Theorem 4.12, Theorem 4.17 and Lemma B.7, we only need to prove the implication (iii) $\Rightarrow$ (iv). The arguments leading to their proofs are well known (see, e.g., [136, Proposition 6.11]), however, we repeat them for the sake of completeness.

Suppose (iii) holds true and let $(f_n)_n \subset H^1(\mathcal{G})$ be an approximating sequence for $\mathbb{1}$, that is, $\lim_{n\to\infty} f_n(x) = 1$ for a.e. $x \in \mathcal{G}$ and $\lim_{n\to\infty} \mathfrak{Q}[f_n] = 0$. Replacing f_n by $\widetilde{f_n} := 0 \vee \mathrm{Re}(f_n) \wedge 1$, if necessary, we can assume that $0 \le f_n \le 1$. Suppose also that $g \in \dot{H}^1(\mathcal{G})$ is bounded. Then $g_n := f_n g$ belongs to $H^1(\mathcal{G})$ as well for all $n \in \mathbb{Z}_{\ge 0}$. Moreover, the sequence $(g_n)_n$ converges to g pointwise a.e. on $\mathcal{G}$ and

$$\lim_{n\to\infty} \mathfrak{Q}[g - g_n] \le \lim_{n\to\infty} 2\|g\|_\infty^2 \mathfrak{Q}[f_n] + 2\int_{\mathcal{G}} (1 - f_n)^2 |\nabla g|^2 \nu(\mathrm{d}x) = 0.$$

Hence every bounded function $g \in \dot{H}^1(\mathcal{G})$ belongs to $\mathrm{dom}(\mathfrak{Q}_N^e)$ and satisfies

$$\mathfrak{Q}_N^e[g] = \mathfrak{Q}[g].$$

On the other hand, for every (real-valued) function $g \in \dot{H}^1(\mathcal{G})$, the sequence defined by

$$g_n := (-n) \vee g_n \wedge n, \quad n \in \mathbb{Z}_{\ge 0},$$

converges pointwise to g and, moreover, $\lim_{n\to\infty} \mathfrak{Q}[g - g_n] = 0$. In particular, it follows that (iv) holds true. ■

In the case of Dirichlet Laplacians, the characterization looks slightly differently. If $\mathbf{H}^0$ admits a unique Markovian extensions, then $\mathbf{H}_D$ coincides with $\mathbf{H}_N$ and in this case the above characterization applies. It turns out that Markovian uniqueness is necessary for $(e^{-\mathbf{H}_D t})_{t>0}$ to be recurrent.

Corollary 4.21. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. The following statements are equivalent for the Dirichlet Laplacian $\mathbf{H}_D$:*

- (i) $(e^{-\mathbf{H}_D t})_{t>0}$ *is recurrent.*
- (ii) $(e^{-\mathbf{h}_D t})_{t>0}$ *is recurrent.*
- (iii) $\mathbb{1} \in \mathrm{dom}(\mathfrak{Q}_D^e)$, *where* $\mathrm{dom}(\mathfrak{Q}_D^e)$ *is the extended Dirichlet space of* $\mathfrak{Q}_D$.
- (iv) $\mathrm{dom}(\mathfrak{Q}_D^e) = \dot{H}^1(\mathcal{G})$.
- (v) $\mathbf{H}_D = \mathbf{H}_N$ *and* $\mathrm{dom}(\mathfrak{Q}_D^e) = \dot{H}^1(\mathcal{G})$.

Proof. Clearly, we only need to prove that $\mathbf{H}_D = \mathbf{H}_N$ if $(e^{-\mathbf{H}_D t})_{t>0}$ is recurrent. However, $\mathfrak{Q}_D$ is a regular Dirichlet form and the corresponding fact connecting recurrence and Markovian uniqueness is rather well known (see, e.g., [98, Theorem 5.20]). ■

Remark 4.22. A few remarks are in order.

- (i) Let us stress that Markovian uniqueness is not necessary for the Neumann Laplacian to be recurrent. Intuitively, this is explained by the fact that Neumann boundary conditions are considered as a reflecting boundary.

On the other hand, one can easily construct simple examples (see, e.g., Lemma 5.13).

(ii) For the Kirchhoff Laplacian $\mathbf{H}_\alpha$ with nonzero $\alpha \geq 0$ (which is equivalent to the presence of a nonzero killing term for $\mathbf{h}_\alpha$) the corresponding Dirichlet form is always transient.

(iii) As in the manifold case (see, e.g., [90]), transience/recurrence for both Kirchhoff Laplacians and graph Laplacians admits several equivalent reformulations in terms of harmonic and subharmonic functions. We shall return to this issue in Section 7.4.

4.6 Stochastic completeness

The preceding sections suggest a connection between stochastic completeness of the Kirchhoff Laplacian $\mathbf{H}$ on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ and its associated discrete Laplacian $\mathbf{h}$ on a fixed model. In fact, the results of [72, 114] imply that (assuming the model has finite intrinsic size and, for simplicity, that $\mathbf{H}$ and $\mathbf{h}$ are self-adjoint[2])

$$(e^{-t\mathbf{H}})_{t>0} \text{ stochastically complete } \Longrightarrow (e^{-t\mathbf{h}})_{t>0} \text{ stochastically complete.} \tag{4.31}$$

It can be shown by examples that the converse direction fails (even for models of finite intrinsic size). However, we are going to show that equivalence holds true in (4.31) if the corresponding model is in a certain sense fine enough.

Theorem 4.23. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph with a fixed model of finite intrinsic size. Let $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ be a Markovian extension of $\mathbf{H}^0$ together with the corresponding extension $\widetilde{\mathbf{h}} \in \mathrm{Ext}_M(\mathbf{h}^0)$ defined on $\ell^2(\mathcal{V}; m)$ by (4.20).*

(i) *If $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is stochastically complete, then $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is stochastically complete.*

(ii) *If $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is stochastically complete and the model additionally satisfies*

$$\sum_{e\in\mathcal{E}} \eta(e)\sqrt{|e|\mu(e)} < \infty, \tag{4.32}$$

then $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is stochastically complete.

Notice that one can always find a model satisfying (4.32) since by cutting a given edge e into N equal edges, the corresponding summand $\eta(e)\sqrt{|e|\mu(e)}$ in (4.32) is

[2] It is assumed in [72, 114] that $\mathcal{G}$ is complete as a metric space with respect to the corresponding intrinsic metric, which implies the self-adjointness of both $\mathbf{H}$ and $\mathbf{h}$, see Theorem 7.1.

replaced with $\frac{1}{\sqrt{N}}\eta(e)\sqrt{|e|\mu(e)}$. Taking this into account we end up with the following immediate corollary.

Corollary 4.24. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and let $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ be a Markovian extension of $\mathbf{H}^0$. Then:*

(i) *The heat semigroup $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is stochastically complete exactly when for each model of $(\mathcal{G}, \mu, \nu)$ having finite intrinsic size the heat semigroup $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ with the generator $\widetilde{\mathbf{h}}$ defined by (4.20) is stochastically complete.*

(ii) *The heat semigroup $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is not stochastically complete exactly when for each model of $(\mathcal{G}, \mu, \nu)$ having finite intrinsic size and satisfying (4.32) the corresponding heat semigroup $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is not stochastically complete.*

Remark 4.25. From Corollary 4.24 (i), we know that stochastic incompleteness of $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is equivalent to the existence of a model of finite intrinsic size such that $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is not stochastically complete. The point of Corollary 4.24 (ii) is to provide an explicit class of models for which $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{h}}$ are simultaneously stochastically complete.

Proof of Theorem 4.23. (i) This was essentially obtained in [72, 114] and we only slightly adapt the proof of [114, pp. 137–140] to our setting. Suppose $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is stochastically complete and consider the operator $\widetilde{\mathbf{h}}$ (see (4.20)) for some fixed model of $(\mathcal{G}, \mu, \nu)$ satisfying (4.11). By Lemma B.6, there exists a sequence $(f_n) \subset \mathrm{dom}(\widetilde{\mathfrak{Q}})$ such that $0 \le f_n \le 1$ for all $n \ge 0$, $\lim_{n\to\infty} f_n = 1$ a.e. on $\mathcal{G}$, and

$$\lim_{n\to\infty} \widetilde{\mathfrak{Q}}[f_n, g] = 0$$

for all $g \in \mathrm{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G}; m)$. By Corollary 4.14, $\mathbf{f}_n = \imath_{\mathcal{V}}(f_n) \in \mathrm{dom}(\widetilde{\mathfrak{q}})$ and, clearly, $0 \le \mathbf{f}_n \le 1$ for all $n \ge 0$. Moreover, using additionally Lemma 4.2, we see that

$$\lim_{n\to\infty} \widetilde{\mathfrak{q}}[\mathbf{f}_n, \mathbf{g}] = \lim_{n\to\infty} \widetilde{\mathfrak{Q}}[\imath_{\mathcal{V}}^{-1}(\mathbf{f}_n), \imath_{\mathcal{V}}^{-1}(\mathbf{g})] = \lim_{n\to\infty} \widetilde{\mathfrak{Q}}[f_n, \imath_{\mathcal{V}}^{-1}(\mathbf{g})] = 0$$

for all $\mathbf{g} \in \mathrm{dom}(\widetilde{\mathfrak{q}}) \cap \ell^1(\mathcal{V}; m)$. Taking into account again Lemma B.6, it remains to show that $\lim_{n\to\infty} \mathbf{f}_n(v) = 1$ for all vertices $v \in \mathcal{V}$. We decompose $f_n = f_{n,\mathrm{lin}} + f_{n,0}$ as in (4.17), where $f_{n,\mathrm{lin}} \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ and $f_{n,0} \in H_0^1(\mathcal{G} \setminus \mathcal{V})$. Denote by g_n^e the restriction of $f_{n,0}$ to the edge $e \in \mathcal{E}$ and extended by zero to the rest of $\mathcal{G}$. Clearly, g_n^e belongs to $\mathrm{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G})$ and taking into account Corollary 4.14, we see that

$$\lim_{n\to\infty} \int_e |\nabla g_n^e|^2\, \nu(\mathrm{d}x_e) = \lim_{n\to\infty} \widetilde{\mathfrak{Q}}[g_n^e, g_n^e] = \lim_{n\to\infty} \widetilde{\mathfrak{Q}}[f_n, g_n^e] = 0.$$

Since g_n^e has support contained in the edge e, this implies that $\lim_{n\to\infty} g_n^e(x) = 0$ for all $x \in e$ and hence $\lim_{n\to\infty} f_{n,0}(x) = 0$ for all $x \in \mathcal{G}$. Thus $\lim_{n\to\infty} f_{n,\mathrm{lin}}(x) = 1$ on $\mathcal{G}$, which implies the desired property of $(\mathbf{f}_n)$.

(ii) Suppose now that $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is stochastically complete for some model of $(\mathcal{G},\mu,\nu)$ satisfying (4.11). By Lemma B.6, there exists a sequence $(\mathbf{f}_n) \subset \operatorname{dom}(\widetilde{\mathfrak{q}})$ such that $0 \le \mathbf{f}_n \le 1$, $\lim_{n\to\infty} \mathbf{f}_n(v) = 1$ for all $v \in \mathcal{V}$ and $\lim_{n\to\infty} \widetilde{\mathfrak{q}}[\mathbf{f}_n, g] = 0$ for all $g \in \operatorname{dom}(\widetilde{\mathfrak{q}}) \cap \ell^1(\mathcal{V}; m)$. Define $f_n := \imath_{\mathcal{V}}^{-1}(\mathbf{f}_n) \in \mathrm{CA}(\mathcal{G}\setminus\mathcal{V})$ and notice that (f_n) is a sequence in $\operatorname{dom}(\widetilde{\mathfrak{Q}})$ with $0 \le f_n \le 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all $x \in \mathcal{G}$. Moreover, by Corollary 4.14 we have

$$\widetilde{\mathfrak{Q}}[f_n, g] = \widetilde{\mathfrak{Q}}[f_n, g_{\mathrm{lin}}] = \widetilde{\mathfrak{q}}[\mathbf{f}_n, \imath_{\mathcal{V}}(g_{\mathrm{lin}})]$$

for all $g \in \operatorname{dom}(\widetilde{\mathfrak{Q}})$. Hence, by Lemma B.6, the stochastic completeness of $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ would follow if we could prove that $\mathbf{g} := \imath_{\mathcal{V}}(g_{\mathrm{lin}})$ belongs to $\operatorname{dom}(\widetilde{\mathfrak{q}}) \cap \ell^1(\mathcal{V}; m)$ for all $g \in \operatorname{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G})$. Taking into account Corollary 4.14 and Lemma 4.2 with $p = 1$, it suffices to show that $g_{\mathrm{lin}} \in L^1(\mathcal{G};\mu)$ and the additional assumption (4.32) is needed exactly for this purpose. Indeed, for an edge $e \in \mathcal{E}_v$, the estimate

$$\begin{aligned}|g_{\mathrm{lin}}(x) - g(x)| &\le |g_{\mathrm{lin}}(x) - g_{\mathrm{lin}}(v)| + |g(x) - g(v)| \\ &\le |e|^{1/2}\Big(\int_e |\nabla g_{\mathrm{lin}}(x_e)|^2\,\mathrm{d}x_e\Big)^{1/2} + |e|^{1/2}\Big(\int_e |\nabla g(x_e)|^2\,\mathrm{d}x_e\Big)^{1/2}\end{aligned}$$

holds for all $x \in e$. Taking into account Corollary 4.14 this implies

$$\int_e |g_{\mathrm{lin}}(x) - g(x)|\mu(\mathrm{d}x) \le 2\eta(e)\sqrt{|e|\mu(e)}\sqrt{\widetilde{\mathfrak{Q}}[g]}, \quad e \in \mathcal{E},$$

and hence

$$\int_{\mathcal{G}} |g_{\mathrm{lin}}(x)|\mu(\mathrm{d}x) \le \|g\|_{L^1(\mathcal{G};\mu)} + 2\sqrt{\widetilde{\mathfrak{Q}}[g]}\sum_{e\in\mathcal{E}} \eta(e)\sqrt{|e|\mu(e)},$$

which proves the claim. ∎

Remark 4.26. A few remarks are in order.

(i) As in the manifold case (see, e.g., [90, Theorem 6.2]), stochastic completeness for both Kirchhoff Laplacians and graph Laplacians admits several equivalent reformulations in terms of λ-harmonic or λ-subharmonic functions and the uniqueness for the heat equation in L^∞ or ℓ^∞ (Khas'minskii-type theorems). Therefore, both Theorem 4.23 and Corollary 4.24 can be reformulated in these terms. For further details we refer to Section 7.5.

(ii) Condition (4.32) in Theorem 4.23 is far from being optimal. Actually, what one needs in proving the converse implication to (i) in Theorem 4.23 is the boundedness of $\imath_{\mathcal{V}}$ as a map from $\operatorname{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G};\mu)$ to the set $\operatorname{dom}(\widetilde{\mathfrak{q}}) \cap \ell^1(\mathcal{V}; m)$ equipped with the corresponding norms.

(iii) Theorem 4.23 can be extended in an obvious way to the case of non-trivial δ-couplings, of course under the positivity assumption that $\alpha \ge 0$ on $\mathcal{V}$.

(iv) In [117] and [116], a "refinement" of a graph $(\mathcal{V}, m; b)$ was suggested (see [116, Definition 1.4] and [117, Definition 1.10]). It is very much similar to the construction induced by (3.5)–(3.6) when refining a weighted metric graph, however, the corresponding difference can be seen as adding loops at the end vertices of a refined edge in order to keep the same vertex weights. Moreover, the construction from [116, 117] enjoys the same important stability property with respect to stochastic completeness: *If a refined graph is stochastically complete, then so is the original graph* $(\mathcal{V}, m; b)$ (see [116, Theorem 1.5]).

4.7 Spectral estimates

Recall that in Theorem 3.22 (v) we observed the following equivalence between strict positivity of spectra:

$$\lambda_0(\widetilde{\mathbf{H}}) = \inf \sigma(\widetilde{\mathbf{H}}) > 0 \iff \lambda_0(\widetilde{\mathbf{h}}) = \inf \sigma(\widetilde{\mathbf{h}}) > 0$$

for a non-negative extension $\widetilde{\mathbf{H}}$ of $\mathbf{H}^0$ on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ and the associated non-negative extension $\widetilde{\mathbf{h}}$ of $\mathbf{h}^0$ on a fixed model having finite intrinsic size. In this section we present a simple two-sided estimate between $\lambda_0(\widetilde{\mathbf{H}})$ and $\lambda_0(\widetilde{\mathbf{h}})$ based on the results of Section 4.3.

Theorem 4.27. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Suppose $\widetilde{\mathbf{H}} \in \operatorname{Ext}_S(\mathbf{H}^0)$ is a non-negative extension of $\mathbf{H}^0$ and consider in $\ell^2(\mathcal{V}; m)$ the non-negative extension $\widetilde{\mathbf{h}} \in \operatorname{Ext}_S(\mathbf{h}^0)$ of $\mathbf{h}^0$ defined by (4.20). Then*

$$\min\left\{\lambda_0(\widetilde{\mathbf{h}}), \frac{1}{2}\left(\frac{\pi}{\eta^*(\mathcal{E})}\right)^2\right\} \le \lambda_0(\widetilde{\mathbf{H}}) \le \min\left\{6\lambda_0(\widetilde{\mathbf{h}}), \left(\frac{\pi}{\eta^*(\mathcal{E})}\right)^2\right\}. \tag{4.33}$$

Proof. First of all, recall from Theorem 3.22 (ii) that $\widetilde{\mathbf{H}} \ge 0$ exactly when $\widetilde{\mathbf{h}} \ge 0$. Moreover, since $\widetilde{\mathbf{H}}$ is a non-negative extension of $\mathbf{H}_{\min} = \mathbf{H}_{\max}^*$, whose Friedrichs extension $\mathbf{H}^F$ is given by (3.32), we conclude from (3.34) that

$$\lambda_0(\widetilde{\mathbf{H}}) \le \lambda_0(\mathbf{H}^F) = \frac{\pi^2}{\eta^*(\mathcal{E})^2}.$$

In particular, (4.33) trivially holds if the model has infinite intrinsic size since all three terms vanish in this case (see also Corollary 3.18 (iii)). Hence in the following, we assume $\eta^*(\mathcal{E}) < \infty$.

Recall the following variational characterization via the Rayleigh quotient:

$$\lambda_0(\widetilde{\mathbf{H}}) = \inf_{f \in \operatorname{dom}(\widetilde{\mathbf{H}})} \frac{\langle \widetilde{\mathbf{H}} f, f\rangle_{L^2(\mathcal{G};\mu)}}{\|f\|^2_{L^2(\mathcal{G};\mu)}}, \quad \lambda_0(\widetilde{\mathbf{h}}) = \inf_{\mathbf{f} \in \operatorname{dom}(\widetilde{\mathbf{h}})} \frac{\langle \widetilde{\mathbf{h}}\mathbf{f}, \mathbf{f}\rangle_{\ell^2(\mathcal{V};m)}}{\|\mathbf{f}\|^2_{\ell^2(\mathcal{V};m)}}.$$

Turning to the upper estimate in terms of $\lambda_0(\widetilde{\mathbf{h}}_0)$, let $\mathbf{f} \in \operatorname{dom}(\widetilde{\mathbf{h}})$ be fixed. By Corollary 4.9, there is $f = f_{\mathrm{lin}} + f_0 \in \operatorname{dom}(\widetilde{\mathbf{H}})$ such that $\imath_{\mathcal{V}}(f) = \mathbf{f}$ and $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$. Moreover, by (4.19) and (4.11), $\dot{H}^1_0(\mathcal{G} \setminus \mathcal{V}) = H^1_0(\mathcal{G} \setminus \mathcal{V})$ algebraically and topologically. Modifying f by edgewise H^2-functions vanishing in a neighborhood of $\mathcal{V}$, we readily construct a sequence $(f_n) \subseteq \operatorname{dom}(\widetilde{\mathbf{H}})$, $f_n = f_{n,\mathrm{lin}} + f_{n,0}$ such that $\imath_{\mathcal{V}}(f_n) = \imath_{\mathcal{V}}(f_{n,\mathrm{lin}}) = \mathbf{f}$ and

$$\lim_{n\to\infty} \mathfrak{Q}[f_{n,0}] + \|f_{n,0}\|_{L^2(\mathcal{G};\mu)} = 0.$$

Hence we conclude from Corollary 4.9 that

$$\lambda_0(\widetilde{\mathbf{H}}) \le \lim_{n\to\infty} \frac{\langle \widetilde{\mathbf{H}} f_n, f_n\rangle_{L^2(\mathcal{G};\mu)}}{\|f_n\|^2_{L^2(\mathcal{G};\mu)}} = \frac{\langle \widetilde{\mathbf{h}}\,\mathbf{f}, \mathbf{f}\rangle_{\ell^2(\mathcal{V};m)}}{\|\imath_{\mathcal{V}}^{-1}(\mathbf{f})\|^2_{L^2(\mathcal{G};\mu)}},$$

and Remark 4.3 (ii) finishes the proof of the upper estimate in (4.33).

It remains to prove the lower inequality in (4.33). By Corollary 4.9, every function $f \in \operatorname{dom}(\widetilde{\mathbf{H}})$ admits a decomposition into $f = f_{\mathrm{lin}} + f_0$ with $f_{\mathrm{lin}} \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ and $f_0 \in \dot{H}^1_0(\mathcal{G} \setminus \mathcal{V})$ (see also (4.17)). Setting $\mathbf{f} := \imath_{\mathcal{V}}(f)$, (4.24) together with (4.19) imply

$$\begin{aligned}\langle \widetilde{\mathbf{H}} f, f\rangle_{L^2(\mathcal{G};\mu)} &\ge \langle \widetilde{\mathbf{h}}\,\mathbf{f}, \mathbf{f}\rangle_{\ell^2(\mathcal{V};m)} + \frac{\pi^2}{\eta^*(\mathcal{E})^2}\|f_0\|^2_{L^2(\mathcal{G};\mu)} \\ &\ge \lambda_0(\widetilde{\mathbf{h}})\|\mathbf{f}\|^2_{\ell^2(\mathcal{V};m)} + \frac{\pi^2}{\eta^*(\mathcal{E})^2}\|f_0\|^2_{L^2(\mathcal{G};\mu)}.\end{aligned}$$

The lower estimate in (4.33) now follows from Remark 4.3 (ii) and the trivial inequality $\|f\|^2_{L^2(\mathcal{G};\mu)} \le 2\|f_{\mathrm{lin}}\|^2_{L^2(\mathcal{G};\mu)} + 2\|f_0\|^2_{L^2(\mathcal{G};\mu)}$. ∎

We shall continue the study of the positivity of spectral gaps in Section 7.3 and now we complete this section with a few remarks.

Remark 4.28. The constant in the second estimate in (4.33) can be improved. For instance, a modified version of [180, Corollary 2.2 and Remark 2.3] yields the bound

$$\lambda_0(\widetilde{\mathbf{H}}) \le \frac{\pi^2}{2}\lambda_0(\widetilde{\mathbf{h}}).$$

Remark 4.29. Theorem 4.27 remains valid for Laplacians with δ-couplings $\mathbf{H}^0_\alpha$ (see Section 2.4.3) and their associated discrete Laplacians $\mathbf{h}^0_\alpha$ (see (3.7) and Theorem 3.1 and Remark 3.24), of course under the additional assumption that all strengths are non-negative, that is, $\alpha\colon \mathcal{V} \to [0,\infty)$.

4.8 Ultracontractivity estimates

Theorem 4.27 shows that under the additional assumption (4.11), there is a connection between the decay of heat semigroups $\mathrm{e}^{-t\widetilde{\mathbf{H}}}$ and $\mathrm{e}^{-t\widetilde{\mathbf{h}}}$ since $\|\mathrm{e}^{-t\widetilde{\mathbf{H}}}\|_{L^2} = \mathrm{e}^{-t\lambda_0(\widetilde{\mathbf{H}})}$

and $\|e^{-t\widetilde{\mathbf{h}}}\|_{\ell^2} = e^{-t\lambda_0(\widetilde{\mathbf{h}})}$ for all $t > 0$. Our next result indicates that the connection between the decay of heat semigroups can be specified further if $\lambda_0(\mathbf{H}) = \lambda_0(\mathbf{h}) = 0$. More specifically, we are going to relate small and large time behavior of the heat kernels by studying the corresponding ultracontractivity estimates.

Theorem 4.30. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model having finite intrinsic size. Let also $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ be a Markovian extension of $\mathbf{H}^0$ and consider the associated Markovian extension $\widetilde{\mathbf{h}}$ of $\mathbf{h}^0$ on $\ell^2(\mathcal{V}; m)$ defined by* (4.20)*.*

(i) *If $(e^{-t\widetilde{\mathbf{H}}})_{t>0}$ is ultracontractive and there are $D \geq 1$ and $C_1 > 0$ such that*

$$\|e^{-t\widetilde{\mathbf{H}}}\|_{L^1 \to L^\infty} \leq C_1 t^{-D/2} \tag{4.34}$$

holds for all $t > 0$, then $(e^{-t\widetilde{\mathbf{h}}})_{t>0}$ is ultracontractive and

$$\|e^{-t\widetilde{\mathbf{h}}}\|_{\ell^1 \to \ell^\infty} \leq C_2 t^{-D/2} \tag{4.35}$$

holds for all $t > 0$ with some positive constant $C_2 > 0$.

(ii) *If there is $D > 2$ such that the heat kernel of $\widetilde{\mathbf{h}}$ satisfies* (4.35) *for all $t > 0$ and, in addition, the underlying model satisfies*

$$\sup_{e \in \mathcal{E}} (|e|\mu(e))^{1-2/D} \frac{|e|}{\nu(e)} < \infty, \tag{4.36}$$

then the heat kernel of $\widetilde{\mathbf{H}}$ satisfies (4.34) *for all $t > 0$ with some positive constant $C_1 > 0$.*

Proof. (i) Suppose that (4.34) holds true for all $t > 0$ with some fixed $D \geq 1$. Then, by Theorem C.4, the Nash-type inequality

$$\|f\|_{L^2(\mathcal{G};\mu)}^{2+4/D} \leq C\, \widetilde{\mathfrak{Q}}[f]\, \|f\|_{L^1(\mathcal{G};\mu)}^{4/D} \tag{4.37}$$

holds true for all $0 \leq f \in \mathrm{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G}; \mu)$, where $\widetilde{\mathfrak{Q}}$ is the Dirichlet form associated with $\widetilde{\mathbf{H}}$. However, restricting in (4.37) to edgewise affine functions and then using Corollary 4.14 and the second inequality in (4.12) with $p = 2$ together with the first one with $p = 1$ (see also Remark 4.3 (iii)), one easily concludes that (4.37) implies

$$\|\mathbf{f}\|_{\ell^2(\mathcal{V};m)}^{2+4/D} \leq \widetilde{C}\, \widetilde{\mathfrak{q}}[\mathbf{f}]\, \|\mathbf{f}\|_{\ell^1(\mathcal{V};m)}^{4/D}, \quad \widetilde{C} = 4^{2+4/D} C,$$

for all $0 \leq \mathbf{f} \in \mathrm{dom}(\widetilde{\mathfrak{q}}) \cap \ell^1(\mathcal{V}; m)$, where $\widetilde{\mathfrak{q}}$ is the Dirichlet form associated with $\widetilde{\mathbf{h}}$. By Theorem C.4, this implies (4.35) for all $t > 0$.

(ii) Suppose now that (4.35) holds true for all $t > 0$ with some fixed $D > 2$. Then, by Varopoulos' theorem (Theorem C.2), the Sobolev-type inequality

$$\|\mathbf{f}\|_{\ell^q(\mathcal{V};m)}^2 \leq C\widetilde{\mathfrak{q}}[\mathbf{f}], \quad \mathbf{f} \in \mathrm{dom}(\widetilde{\mathfrak{q}}), \tag{4.38}$$

is valid, where $q = q(D) := \frac{2D}{D-2}$. Since the model satisfies (4.11), by Corollary 4.14, every $f \in \mathrm{dom}(\widetilde{\mathfrak{Q}})$ admits a unique decomposition $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f}) + f_0$ with $\mathbf{f} \in \mathrm{dom}(\widetilde{\mathfrak{q}})$, $f_0 \in H_0^1(\mathcal{G} \setminus \mathcal{V})$ and, moreover,

$$\widetilde{\mathfrak{Q}}[f] = \widetilde{\mathfrak{q}}[\mathbf{f}] + \mathfrak{Q}[f_0] = \widetilde{\mathfrak{q}}[\mathbf{f}] + \|\nabla f_0\|^2_{L^2(\mathcal{G};\nu)}.$$

Using Lemma 4.2, the first inequality in (4.12) with $p = q$ together with (4.38) imply that

$$\|\imath_{\mathcal{V}}^{-1}(\mathbf{f})\|^2_{L^q(\mathcal{G};\mu)} \le C\, \widetilde{\mathfrak{q}}[\mathbf{f}]. \tag{4.39}$$

Next, using the simple estimate

$$\left(\int_0^\ell |f(s)|^q \,\mathrm{d}s\right)^{\frac{2}{q}} \le \ell^{\frac{2}{q}} \sup_{0\le x\le \ell} |f(x)|^2 \le \ell^{1+\frac{2}{q}} \int_0^\ell |f'(s)|^2 \,\mathrm{d}s,$$

which holds true for all $f \in H_0^1(0,\ell)$ and $\ell > 0$, we obtain

$$\left(\int_e |f(x)|^q \mu(\mathrm{d}x)\right)^{\frac{2}{q}} \le |e|^{1+\frac{2}{q}} \frac{\mu(e)^{\frac{2}{q}}}{\nu(e)} \int_e |\nabla f(x)|^2 \nu(\mathrm{d}x), \quad f \in H_0^1(\mathcal{G} \setminus \mathcal{V}),$$

for each edge $e \in \mathcal{E}$. Since $q > 2$, this immediately implies the inequality

$$\|f_0\|^2_{L^q(\mathcal{G};\mu)} \le C \|\nabla f_0\|^2_{L^2(\mathcal{G};\nu)} \tag{4.40}$$

for all $f_0 \in H_0^1(\mathcal{G} \setminus \mathcal{V})$, where the constant $C = C(\mathcal{E}, \mu, \nu)$ depends only on the model and edge weights μ, ν and is given by

$$C(\mathcal{E}, \mu, \nu) = \sup_{e\in\mathcal{E}} |e|^{1+\frac{2}{q}} \frac{\mu(e)^{\frac{2}{q}}}{\nu(e)} = \sup_{e\in\mathcal{E}} (|e|\mu(e))^{1-\frac{2}{D}} \frac{|e|}{\nu(e)}.$$

Thus, combining (4.40) with (4.39), we arrive at the Sobolev-type inequality

$$\|f\|^2_{L^q(\mathcal{G};\mu)} \le \widetilde{C}\, \widetilde{\mathfrak{Q}}[f], \quad f \in \mathrm{dom}(\widetilde{\mathfrak{Q}}).$$

Applying Theorem C.2 once again, we conclude that $(\mathrm{e}^{-t\widetilde{\mathbf{H}}})_{t>0}$ is ultracontractive and (4.34) holds true for all $t > 0$. ■

Remark 4.31. In the special case $\mu = \nu \equiv \mathbb{1}$ on $\mathcal{G}$, Theorem 4.30 was proved in [68, Section 5]. However, the proof of Theorem 4.30 (i) in [68] was based on the use of Varopoulos' theorem and hence was restricted to the case $D > 2$. Notice that Theorem 4.30 (i) with $\mu = \nu \equiv \mathbb{1}$ was observed by G. Rozenblum and M. Solomyak (see [189, Theorem 4.1]), however, for a different discrete Laplacian (the vertex weight m is defined in [189] as the vertex degree function deg: $v \mapsto \#(\vec{\mathcal{E}}_v)$).

The proof of Theorem 4.30 (ii) indicates that (4.36) is necessary for the validity of (4.34) for $t > 0$. As the next result shows, it is indeed necessary for all $D > 0$.

Lemma 4.32. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and let $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ be a Markovian extension of $\mathbf{H}^0$. Assume also that $(\mathrm{e}^{-t\widetilde{\mathbf{H}}})_{t>0}$ is ultracontractive. If there is a model of $(\mathcal{G}, \mu, \nu)$ such that (4.36) fails to hold for a given $D > 0$, then*

$$\sup_{t\in(0,1)} t^{D/2} \|\mathrm{e}^{-t\widetilde{\mathbf{H}}}\|_{L^1\to L^\infty} = \infty. \tag{4.41}$$

In particular, (4.41) always holds for $D \in (0, 1)$.

Proof. Assume the converse, that is, (4.34) holds for all $t \in (0, 1)$ with some fixed $D > 0$. Then, by Theorem C.4, this implies that the Nash-type inequality

$$\|f\|^{2+4/D}_{L^2(\mathcal{G};\mu)} \le C(\widetilde{\mathfrak{Q}}[f] + \|f\|^2_{L^2(\mathcal{G};\mu)})\,\|f\|^{4/D}_{L^1(\mathcal{G};\mu)} \tag{4.42}$$

holds true for all $0 \le f \in \mathrm{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G}; \mu)$. In particular, this inequality holds for all $0 \le f \in H^1_0(\mathcal{G} \setminus \mathcal{V}) \cap L^1(\mathcal{G}; \mu)$. It remains to apply a scaling argument. Indeed, take a positive function $0 \ne f_0 \in H^1_0([0, 1])$ with $\|f_0\|_{L^1} = 1$ and choose a model of $(\mathcal{G}, \mu, \nu)$ satisfying (4.11). Next define $f_e \in H^1_0(\mathcal{G} \setminus \mathcal{V})$ as $f_0(\cdot/|e|)$ on e (upon identification of $e \in \mathcal{E}$ with $\mathcal{I}_e = [0, |e|]$) and then extend it by 0 to the rest of $\mathcal{G} \setminus e$. Clearly, $0 \le f_e \in \mathrm{dom}(\widetilde{\mathfrak{Q}}) \cap L^1(\mathcal{G}; \mu)$ for all $e \in \mathcal{E}$ and

$$\begin{aligned} \|f_e\|_{L^1(\mathcal{G};\mu)} &= |e|\mu(e), \\ \|f_e\|^2_{L^2(\mathcal{G};\mu)} &= |e|\mu(e)\|f_0\|^2_2, \\ \mathfrak{Q}[f_e] &= \frac{\nu(e)}{|e|}\|f_0'\|^2_2. \end{aligned} \tag{4.43}$$

Plugging f_e into (4.42), we get

$$\begin{aligned} C &\ge \frac{(|e|\mu(e))^{1+2/D}\|f_0\|_2^{2+4/D}}{\big(\frac{\nu(e)}{|e|}\|f_0'\|_2^2 + |e|\mu(e)\|f_0\|_2^2\big)(|e|\mu(e))^{4/D}} \\ &\ge \frac{(|e|\mu(e))^{1-2/D}\|f_0\|_2^{2+4/D}}{\frac{\nu(e)}{|e|}\big(\|f_0'\|_2^2 + \eta^*(\mathcal{E})^2\|f_0\|_2^2\big)} \\ &= \left(\frac{\eta(e)^{2D-2}}{\mu(e)\nu(e)}\right)^{\frac{1}{D}} \frac{\|f_0\|_2^{2+4/D}}{\|f_0'\|_2^2 + \eta^*(\mathcal{E})^2\|f_0\|_2^2} \end{aligned}$$

for all $e \in \mathcal{E}$. Since $\eta^*(\mathcal{E}) < \infty$, the latter is unbounded from above if (4.36) fails to hold, and hence we arrive at a contradiction, which proves the first claim.

To prove the last claim it suffices to mention that $2D - 2 < 0$ if $D \in (0, 1)$ and hence we can always find a model such that (4.36) is not true with $D \in (0, 1)$. ∎

By using Theorem C.6, it is possible to extend the above connections to subexponential scales. In the next result we shall always assume that $s\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is a decreasing differentiable bijection such that its logarithmic derivative has polyno-*

mial growth (see (C.5)). A typical example are functions that behave like $t^{-d/2}$ with $d > 0$ for small t, and $\mathrm{e}^{-ct^{\alpha}}$ with $\alpha \in (0, 1]$ for large t (notice that $\alpha > 1$ is also allowed, however, heat semigroups cannot have such a fast decay at infinity).

Theorem 4.33. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model having finite intrinsic size. Let also $\widetilde{\mathbf{H}} \in \mathrm{Ext}_M(\mathbf{H}^0)$ be a Markovian extension of $\mathbf{H}^0$ and consider the associated Markovian extension $\widetilde{\mathbf{h}}$ of $\mathbf{h}^0$ on $\ell^2(\mathcal{V}; m)$ defined by* (4.20)*.*

(i) *If $(\mathrm{e}^{-t\widetilde{\mathbf{H}}})_{t>0}$ is ultracontractive and*

$$\|\mathrm{e}^{-t\widetilde{\mathbf{H}}}\|_{L^1\to L^\infty} \le s(t), \quad t > 0, \tag{4.44}$$

then $(\mathrm{e}^{-t\widetilde{\mathbf{h}}})_{t>0}$ is ultracontractive and

$$\|\mathrm{e}^{-t\widetilde{\mathbf{h}}}\|_{\ell^1\to \ell^\infty} \le s(ct) \tag{4.45}$$

holds for all $t > 0$ with some positive constant $c > 0$.

(ii) *If* (4.44) *holds true, then there is a positive constant $C > 0$ such that*

$$\left(\frac{8|e|\mu(e)}{\pi^2}\right)^2 \theta_s\left(\frac{\pi^2}{8\,|e|\mu(e)}\right) \le 8\frac{\nu(e)}{|e|}, \qquad \theta_s := -s' \circ s^{-1}, \tag{4.46}$$

for all $e \in \mathcal{E}$.

Proof. (i) For simplicity we assume that $\mathbf{H}$ is self-adjoint. Our proof is based on the use of Theorem C.6 and its proof in [46]. First of all, by [46, Proposition II.2], (4.44) implies that

$$\gamma_s(\|f\|^2_{L^2(\mathcal{G};\mu)}) \le \mathfrak{Q}[f]$$

for all $0 \le f \in \mathrm{dom}(\mathfrak{Q})$ with $\|f\|_{L^1(\mathcal{G};\mu)} \le 1$. Here the function $\gamma_s\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is given by

$$\gamma_s(x) := \sup_{r>0} \frac{x}{2r} \log\left(\frac{x}{s(r)}\right).$$

In particular, the latter holds for edgewise affine functions and hence restricting to $0 \le f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ we get by taking into account (4.15) and (4.16) that

$$\gamma_s(4^{-1}\|\mathbf{f}\|^2_{\ell^2(\mathcal{V};m)}) \le 4\mathfrak{q}[\mathbf{f}]$$

for all $0 \le \mathbf{f} \in \mathrm{dom}(\mathfrak{q})$ with $\|\mathbf{f}\|_{L^1(\mathcal{V};m)} \le 1$. Here we also used the estimate (4.12) with $p = 2$ together with the monotonicity of the logarithm. Now, taking into account that $\theta_{4s}(x) = 4\theta_s(x/4)$, by [46, Lemma II.3], there is $\widetilde{C} > 0$ such that

$$\theta(4^{-1}\|\mathbf{f}\|^2_{\ell^2(\mathcal{V};m)}) \le \widetilde{C}\mathfrak{q}[\mathbf{f}]$$

for all $0 \le \mathbf{f} \in \mathrm{dom}(\mathfrak{q})$ with $\|\mathbf{f}\|_{\ell^1(\mathcal{V};m)} \le 1$. It remains to use Theorem C.6 once again.

(ii) By Theorem C.6, (4.44) implies the Nash-type inequality

$$\theta_s(\|f\|^2_{L^2(\mathcal{G};\mu)}) \leq C\,\mathfrak{Q}[f] \tag{4.47}$$

for all $f \in \operatorname{dom}(\mathfrak{Q})$ with $\|f\|_{L^1(\mathcal{G};\mu)} = 1$. Pick $0 \leq f_0 \in H^1_0([0,1])$ with $\|f_0\|_1 = 1$. For each $e \in \mathcal{E}$, define $f_e \in H^1_0(\mathcal{G})$ as in the proof of Lemma 4.32. After plugging $f = \frac{1}{|e|\mu(e)} f_e$ into (4.47) and taking into account (4.43), we get

$$\theta_s\left(\frac{\|f_0\|_2^2}{|e|\mu(e)}\right) \leq \frac{\nu(e)}{|e|}\left(\frac{\|f_0'\|_2}{|e|\mu(e)}\right)^2 \tag{4.48}$$

for all $e \in \mathcal{E}$ and each $0 \leq f_0 \in H^1_0([0,1])$ with $\|f_0\|_1 = 1$. Finally, upon choosing $f_0(x) = \frac{\pi}{2}\sin(\pi x)$ in (4.48), we end up with (4.46). ■

Remark 4.34. We are convinced that (4.45) together with (4.46) should imply estimate (4.44), however, we have not succeeded in proving it by applying T. Coulhon's extension of Theorem C.4. Let us also stress that in the case of a polynomial decay our proof of Theorem 4.30 (ii) is based on Varopoulos' theorem (Theorem C.2) and hence the range of the corresponding exponent is restricted to $D > 2$.