# Chapter 6

# Graph Laplacians as boundary operators

The results in the preceding chapters lead to the following question:

> *Which graph Laplacians may arise as boundary operators (in the sense of Chapters 3 and 4) for a Kirchhoff Laplacian on a weighted metric graph?*

Let us be more specific in stating the above problem. Suppose a vertex set $\mathcal{V}$ is given. Each graph Laplacian (2.4) is determined by the vertex weight function $m\colon \mathcal{V} \to (0,\infty)$, edge weight function $b\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ having properties (i), (ii) and (iv) of Section 2.2, and the killing term $c\colon \mathcal{V} \to [0,\infty)$. We always assume that the underlying graphs are connected. With each such $b$ we can associate a locally finite simple graph $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ as described in Remark 2.7.

**Definition 6.1.** A *cable system* for a graph $b$ over $(\mathcal{V}, m)$ is a model of a weighted metric graph $(\mathcal{G}, \mu, \nu)$ having $\mathcal{V}$ as its vertex set and such that the functions defined by (3.1)–(3.5) and (3.6) coincide with $m$ and, respectively, $b$. If in addition the underlying graph $(\mathcal{V}, \mathcal{E})$ of the model coincides with $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$, then the cable system is called *minimal*.

**Remark 6.2.** Notice that the underlying combinatorial graph $(\mathcal{V}, \mathcal{E})$ of a cable system for $(\mathcal{V}, m; b)$ can always be obtained from the simple graph $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ by adding loops and multiple edges.

Since the killing term $c$ is nothing but the strength of $\delta$-couplings at the vertices in (3.7), we can restrict our considerations to the case $c \equiv 0$:

**Problem 6.1.** Which locally finite graphs $(\mathcal{V}, m; b)$ have a minimal cable system?

The case of a path graph shows that the answer to the above problem is not trivial (see Proposition 5.20). However, we stress that a general cable system may have loops and multiple edges and thus the simplicity assumption on the model of $(\mathcal{G}, \mu, \nu)$ (that is, the minimality of a cable system for $(\mathcal{V}, m; b)$) might be too restrictive. In fact, as discussed in Remark 2.11 and Remark 2.12, we can allow multi-graphs and this leads us to another question:

**Problem 6.2.** Which locally finite graphs $(\mathcal{V}, m; b)$ have a cable system?

Once the above problems will be resolved, the next natural question (also in context with possible applications) is:

**Problem 6.3.** How can one describe all cable systems of a locally finite graph $b$ over $(\mathcal{V}, m)$?

On the other hand, there is another closely connected class of second order difference operators on graphs, however, acting in $\ell^2(\mathcal{V})$. In particular, the operator defined in $\ell^2(\mathcal{V})$ by the difference expression (3.28) is a special case of

$$(\tau f)(v) = \beta(v) f(v) - \sum_{u \in \mathcal{V}} q(u,v) f(u), \quad v \in \mathcal{V},$$

where $\beta \colon \mathcal{V} \to \mathbb{R}$ and $q$ is a graph over $\mathcal{V}$ satisfying properties (i), (ii) and (iv) of Section 2.2. This leads to a similar problem:

**Problem 6.4.** Given a graph $q$ over $\mathcal{V}$, which of the above difference expressions arise as boundary operators for Laplacians with $\delta$-couplings on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ over $\mathcal{G}_q = (\mathcal{V}, \mathcal{E}_q)$?

Despite an obvious similarity and a clear connection between these problems, as we learned in Section 5.3, they have very different answers even in the case of a path graph (see Proposition 5.18 and Proposition 5.20).

**Remark 6.3.** Taking into account an obvious analogy between the above second order difference expression and Jacobi matrices, it is tempting to call them *Jacobi matrices on graphs* (cf., e.g., [8–10]).

## 6.1 Examples

Before studying Problems 6.1–6.4, let us first give several illustrative examples.

**Example 6.4** (Normalized Laplacians/simple random walks). Let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ be a locally finite simple graph. Let also $|\cdot| \colon \mathcal{E} \to (0, \infty)$ be given and define edge weights $\mu, \nu \colon \mathcal{E} \to (0, \infty)$ by setting

$$\mu \colon e \mapsto \frac{1}{|e|}, \quad \nu \colon e \mapsto |e|.$$

Notice that the intrinsic edge length is constant on $\mathcal{E}$, that is,

$$\eta(e) = |e| \sqrt{\frac{\mu(e)}{\nu(e)}} = 1$$

for all $e \in \mathcal{E}$ in this case, and hence (3.3), (3.5) and (3.6) give

$$m(v) = \sum_{u \sim v} |e| \mu(e) = \deg(v), \quad v \in \mathcal{V},$$

and

$$b(u,v) = \begin{cases} 1, & u \sim v, \\ 0, & u \not\sim v, \end{cases} \quad (u,v) \in \mathcal{V} \times \mathcal{V}.$$

The corresponding graph Laplacian (3.7) (with $\alpha \equiv 0$) has the form

$$(L_{\mathrm{norm}} f)(v) := \frac{1}{\deg(v)} \sum_{u \sim v} f(v) - f(u) = f(v) - \frac{1}{\deg(v)} \sum_{u \sim v} f(u)$$

for all $v \in \mathcal{V}$. It is known in the literature as a *normalized Laplacian* (or *physical Laplacian*). This operator has a venerable history. In particular, it appears as the generator of the simple random walk on $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$, where "simple" refers to the fact that the probabilities to move from $v$ to a neighboring vertex are all equal to $\frac{1}{\deg(v)}$ (see, e.g., [212]).

**Example 6.5** (Electrical networks/Random walks). Again, let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ be a locally finite simple graph. Suppose $|\cdot|\colon \mathcal{E} \to \{1\}$, that is, the corresponding metric graph $\mathcal{G}$ is equilateral (each $e \in \mathcal{E}$ can be identified with a copy of the interval $[0, 1]$). Next, suppose that the edge weights $\mu, \nu\colon \mathcal{E} \to (0, \infty)$ coincide, that is, $\mu(e) = \nu(e)$ for all $e \in \mathcal{E}$. Then

$$\eta(e) = \sqrt{\frac{\mu(e)}{\nu(e)}} = 1$$

for all $e \in \mathcal{E}$ and hence, by (3.3), (3.5) and (3.6),

$$b(u, v) = \begin{cases} \mu(e_{u,v}), & u \sim v, \\ 0, & u \not\sim v, \end{cases} \qquad m(v) = m_b(v) := \sum_{e \in \mathcal{E}_v} \mu(e).$$

The corresponding graph Laplacian (3.7) (with $\alpha \equiv 0$) is given explicitly by

$$(L_b f)(v) := \frac{1}{m_b(v)} \sum_{u \sim v} b(u, v)(f(v) - f(u)), \quad v \in \mathcal{V},$$

and arises in the study of random walks on $\mathcal{G}_d$ (a.k.a. reversible Markov chains), where the jump probabilities are defined by (see, e.g., [12, Chapter 1.2], [91])

$$p(u, v) = \frac{b(u, v)}{\sum_{x \in \mathcal{V}} b(u, x)}, \quad u, v \in \mathcal{V}.$$

On the other hand, considering informally an electrical network as a set of wires (edges) and nodes (vertices), we can interpret $b(u, v)$ as a *conductance* of a wire $e_{u,v}$ connecting $u$ with $v$, $r(u, v) = \frac{1}{b(u,v)}$ is the *resistance* of $e_{u,v}$ and $m(v)$ is the *total conductance* at $v$. Thus, the corresponding weighted Laplacian $L_b$ arises in the study of *pure resistor networks* (see [12, 195, 212]).

Therefore, *every electrical network operator/generator of a random walk (reversible Markov chains) on a locally finite graph arises as a boundary operator for a Kirchhoff Laplacian on a weighted metric graph*. Notice also that by Lemma 2.9 the corresponding graph Laplacian is bounded (in fact, its norm is at most 2).

**Remark 6.6.** The construction in Example 6.5 connecting a random walk on a graph with a Brownian motion on a weighted metric graph can be found in [205].

The above examples show that a very important class of graph Laplacians arises as boundary operators (in the sense of Proposition 3.11) for Laplacians on weighted metric graphs. However, as we shall see next, the answer to Problem 6.1 is far from trivial.

**Example 6.7** (Combinatorial Laplacians on antitrees). Again, let $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ be a locally finite simple graph. Set $m = \mathbb{1}$ on $\mathcal{V}$ and define a graph $b$ over $(\mathcal{V}, m)$ by

$$b(u,v) = \begin{cases} 1, & u \sim v, \\ 0, & u \not\sim v, \end{cases} \qquad (u,v) \in \mathcal{V} \times \mathcal{V}.$$

Notice in particular that the associated combinatorial graph $(\mathcal{V}, \mathcal{E}_b)$ coincides with $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ (see Remark 2.7). The corresponding graph Laplacian acts in $\ell^2(\mathcal{V})$ and is given by

$$(L_{\mathrm{comb}} f)(v) := \sum_{u \sim v} f(v) - f(u) = \deg(v) f(v) - \sum_{u \sim v} f(u). \tag{6.1}$$

This operator is known as the *combinatorial Laplacian*[1] and $A = (b(u,v))_{u,v \in \mathcal{V}}$ is nothing but the adjacency matrix of the graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$.

Suppose additionally that our graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a rooted *antitree* (see [48, 149, 213] and also Section 8.1), that is, fix a root vertex $o \in \mathcal{V}$ and then order the graph with respect to the combinatorial spheres $S_n$, $n \in \mathbb{Z}_{\geq 0}$ ($S_n$ consists of all vertices $v \in \mathcal{V}$ such that the combinatorial distance from $v$ to the root $o$, that is, the combinatorial length of the shortest path connecting $v$ with $o$, equals $n$; notice that $S_0 = \{o\}$). The graph $\mathcal{G}_d$ is called an *antitree* if it is simple and every vertex in $S_n$ is connected to every vertex in $S_{n+1}$ and there are no horizontal edges, i.e., there are no edges with all endpoints in the same sphere (see Figure 6.1). In this particular situation

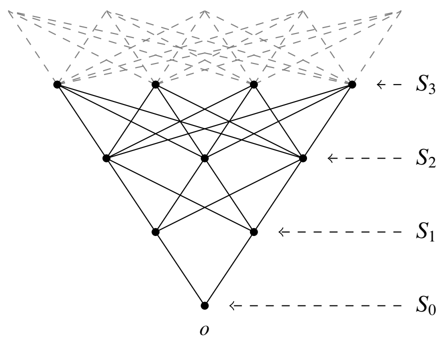


**Figure 6.1.** Example of an antitree with $s_n = \#S_n = n + 1$.

[1] It seems that there is no agreement how to call this difference operator and sometimes the name "physical Laplacian" is used instead. However, taking into account its obvious connection with the adjacency matrix, the name "combinatorial Laplacian" looks more appropriate to us.

(a combinatorial Laplacian on an infinite antitree) the next result provides a complete answer to Problem 6.1.

**Proposition 6.8.** *Let $\mathcal{A} = (\mathcal{V}, \mathcal{E})$ be an (infinite) antitree and let $s_n := \#S_n$, $n \in \mathbb{Z}_{\geq 0}$, be its sphere numbers. Then the corresponding combinatorial Laplacian* (6.1) *on $\mathcal{A}$ arises as a boundary operator for a minimal Kirchhoff Laplacian on a weighted metric antitree if and only if*

$$\sum_{k=0}^{n} (-1)^k s_{n-k} > 0 \tag{6.2}$$

*holds for all $n \in \mathbb{Z}_{\geq 0}$.*

We shall give the proof of this result in Section 6.2. Let us only mention the similarity between (6.2) and (5.33), which is, in fact, not at all surprising in view of connections between Laplacians on family preserving graphs and Jacobi matrices (see [30]).

## 6.2 Life without loops I: Graph Laplacians

We begin with Problem 6.1. Its importance stems from the fact that every regular Dirichlet form over $(\mathcal{V}, m)$ arises as the energy form $\mathfrak{q}_D$ for some graph $(b, c)$ over $(\mathcal{V}, m)$ (see [132, Theorem 7]).

Suppose that a connected locally finite graph $(b, c)$ over $(\mathcal{V}, m)$ is given. Let $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ be the simple graph associated with $(b, c)$: $u \sim v$ exactly when $b(u, v) \neq 0$ (see Remark 2.7). Then for each weighted metric graph $(\mathcal{G}, \mu, \nu)$ over $(\mathcal{V}, \mathcal{E}_b)$ the functions defined by (3.1)–(3.5) and (3.6) take the following form:

$$m_{\mathcal{G}}(v) = \sum_{u: b(u,v) \neq 0} r(e_{u,v}), \tag{6.3}$$

where $r$ is defined by (3.1), (3.3)–(3.4), and

$$b_{\mathcal{G}}(u, v) = \begin{cases} \dfrac{\nu(e_{u,v})}{|e_{u,v}|}, & b(u, v) > 0, \\ 0, & b(u, v) = 0. \end{cases}$$

Comparing the form of the boundary operator (3.7) with (2.4), it is clear that the killing term $c$ is nothing but the strength of $\delta$-couplings at the vertices and hence we can restrict our considerations to the case $c \equiv 0$. In fact, the next result shows that Problem 6.1 can be reduced to a description of all possible vertex weights $m$:

**Proposition 6.9.** *A locally finite graph $(\mathcal{V}, m; b)$ admits a minimal cable system if and only if there is a function $r_b\colon \mathcal{E}_b \to (0, \infty)$ such that, for all $v \in \mathcal{V}$,*

$$m(v) = \sum_{e \in \mathcal{E}_v} r_b(e). \tag{6.4}$$

*Proof.* Necessity immediately follows from (6.3). Let us prove sufficiency. Suppose there is $r_b\colon \mathcal{E}_b \to (0,\infty)$ such that (6.4) holds true for all $v \in \mathcal{V}$. First of all, we set $|e_{u,v}| \equiv 1$ and $\nu(e_{u,v}) := b(u,v)$ for all edges $e_{u,v} \in \mathcal{E}_b$. If $\sup_{u,v} \frac{r_b(e_{u,v})}{b(u,v)} < \infty$, then we define $\mu(e_{u,v}) = r_b(e_{u,v})$ and otherwise set

$$\mu(e_{u,v}) = \begin{cases} r_b(e_{u,v}), & r_b(e_{u,v}) \le b(u,v), \\ \dfrac{r_b(e_{u,v})^2}{b(u,v)}, & r_b(e_{u,v}) > b(u,v), \end{cases}$$

for each $e_{u,v} \in \mathcal{E}_b$. It is then straightforward to check that the corresponding functions defined by (3.1)–(3.5) and (3.6) coincide with $m$ and $b$. ■

In fact, the above result shows that the answer to Problem 6.1 is analogous to the answer in the case of a path graph (see Proposition 5.20 and Remark 5.21). Indeed, let $\mathcal{G}_d = (\mathcal{V},\mathcal{E})$ be a simple locally finite graph and consider the map $D\colon C(\mathcal{V}) \to C(\mathcal{E})$ given by

$$(Df)(e_{u,v}) = f(u) + f(v).$$

If we define the Hilbert space $\ell^2(\mathcal{E})$ as

$$\ell^2(\mathcal{E}) = \Big\{\phi\colon \mathcal{E} \to \mathbb{C} : \sum_{e\in\mathcal{E}} |\phi(e)|^2 < \infty\Big\},$$

then $D$ defines a possibly unbounded operator from $\ell^2(\mathcal{V})$ to $\ell^2(\mathcal{E})$ (in fact, $D$ is bounded if and only if the graph $\mathcal{G}_d$ has bounded geometry, $\sup_{v\in\mathcal{V}} \deg(v) < \infty$). Its (formal) adjoint $D^*\colon C(\mathcal{E}) \to C(\mathcal{V})$ is given by

$$(D^*\phi)(v) = \sum_{e\in\mathcal{E}_v} \phi(e), \quad v \in \mathcal{V}.$$

Comparing this formula with (6.4), we immediately arrive at the following result:

**Corollary 6.10.** *A locally finite graph $(\mathcal{V}, m; b)$ admits a minimal cable system if and only if $m$ belongs to the image of the positive cone $C^+(\mathcal{E})$ under the map $D^*$.*

**Remark 6.11.** Taking into account Example 6.5, Corollary 6.10 admits the following reformulation: *A locally finite graph $(\mathcal{V}, m; b)$ admits a minimal cable system if and only if there are resistances $R\colon \mathcal{E}_b \to \mathbb{R}_{>0}$ such that total conductances on $\mathcal{V}$ coincide with $m$.*

Let us apply the above result to antitrees in order to prove Proposition 6.8.

*Proof of Proposition 6.8.* By Proposition 6.9, we need to show that for a given antitree $\mathcal{A} = (\mathcal{V},\mathcal{E})$ with sphere numbers $(s_n)_{n\ge 0}$ condition (6.2) holds for all $n \ge 0$ if and only if there is a strictly positive function $r\colon \mathcal{E} \to (0,\infty)$ such that $\sum_{e\in\mathcal{E}_v} r(e) = 1$ for all $v \in \mathcal{V}$.

Suppose first that (6.2) holds for all $n \geq 0$. Then setting

$$r(e) := \frac{1}{s_n s_{n+1}} \sum_{k=0}^{n} (-1)^k s_{n-k}$$

for all $e \in \mathcal{E}_n$, where $\mathcal{E}_n$ the set of edges connecting the spheres $S_n$ and $S_{n+1}$, we get for each $v \in S_n$, $n \geq 0$,

$$\begin{aligned}
\sum_{e \in \mathcal{E}_v} r(e) &= \sum_{e \in \mathcal{E}_n \cap \mathcal{E}_v} r(e) + \sum_{e \in \mathcal{E}_{n-1} \cap \mathcal{E}_v} r(e) \\
&= s_{n+1} \frac{1}{s_n s_{n+1}} \sum_{k=0}^{n} (-1)^k s_{n-k} + s_{n-1} \frac{1}{s_{n-1} s_n} \sum_{k=0}^{n-1} (-1)^k s_{n-1-k} \\
&= 1.
\end{aligned}$$

Conversely, suppose $r\colon \mathcal{E} \to (0,\infty)$ is such that $D^* r = \mathbb{1}_{\mathcal{V}}$. Then we have

$$\sum_{e \in \mathcal{E}_0} r(e) = \sum_{e \in \mathcal{E}_o} r(e) = 1 = \#S_0 = s_0,$$

and hence

$$\begin{aligned}
0 < \sum_{e \in \mathcal{E}_n} r(e) &= \sum_{v \in S_n} \sum_{e \in \mathcal{E}_v} r(e) - \sum_{e \in \mathcal{E}_{n-1}} r(e) \\
&= s_n - \sum_{e \in \mathcal{E}_{n-1}} r(e) \\
&= \sum_{k=0}^{n} (-1)^k s_{n-k}
\end{aligned}$$

for all $n \geq 0$, where the last equality follows immediately by induction. ■

**Remark 6.12.** A few remarks are in order.

(i) Proposition 6.8 can be generalized to family preserving graphs (see [30] for definitions).

(ii) We stress that, by the above results, the combinatorial Laplacian on an infinite path graph $\mathcal{G}_d = \mathbb{Z}_{\geq 0}$ has no minimal cable system. Indeed, every infinite path graph is an antitree with sphere numbers $s_n = 1$ for all $n \geq 0$ and (6.2) clearly fails to hold in this case (see also Proposition 5.20).

Despite its simple form, for a given vertex weight it is not so easy to verify the conditions in Proposition 6.9 and Corollary 6.10. In particular, returning to Example 6.7, the corresponding vertex weight $m$ is a constant function, $m = \mathbb{1}_{\mathcal{V}}$, and one may ask: *for which graphs* $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ *the constant function* $\mathbb{1}_{\mathcal{V}}$ *belongs to* $D^*(C^+(\mathcal{E}))$? The answer to this question is provided by the following elegant result:

**Lemma 6.13.** *Let $\mathscr{G}_d = (\mathcal{V}, \mathcal{E})$ be a simple graph satisfying Hypotheses 2.1. Then $\mathbb{1}_{\mathcal{V}} \in D^*(C^+(\mathcal{E}))$ if and only if for each $e \in \mathcal{E}$ there is a disjoint cycle cover of $\mathscr{G}_d$ containing $e$ in one of its cycles.*

Recall that a *disjoint cycle cover* of $\mathscr{G}_d$ is a collection of vertex-disjoint cycles in $\mathscr{G}_d$ such that every vertex in $\mathscr{G}_d$ lies on some edge in one of the cycles. Here, by a cycle of length $n \in \mathbb{Z}_{\geq 2}$ in a simple graph $\mathscr{G}_d$, we mean a path $\mathscr{P} = (v_k)_{k=0}^n$ such that $v_0 = v_n$ and all other vertices are distinct. Notice that this definition differs slightly from the one given in Section 2.1.1, that is, in the present section we allow for a moment cycles of length two (consisting of "going back and forth" along one fixed edge).

**Remark 6.14.** Lemma 6.13 is due to G. Zaimi and was published in *MathOverflow*[2] as the answer to a question posed by M. Folz. It is curious to mention that Folz came up in [72] with a problem similar to Problem 6.1 when studying stochastic completeness of weighted graphs and attempting to prove a volume growth test by employing connections between Dirichlet forms on graphs and metric graphs, which allow to transfer the results from strongly local Dirichlet forms to Dirichlet forms on graphs (see Sections 4.2 and 4.6 for further information).

**Remark 6.15.** Notice that in the case of finite graphs, for each $e \in \mathcal{E}$ there is a disjoint cycle cover containing $e$ in one of its cycles if and only if removing an edge decreases the permanent of the corresponding adjacency matrix. The appearance of permanents is not at all surprising since

$$(D^*Df)(v) = \sum_{u \sim v} f(v) + f(u) = \deg(v) f(v) + \sum_{u \sim v} f(u)$$

is the so-called *signless Laplacian*. Here the second summand is the usual adjacency matrix.

## 6.3 Life with loops

As we have seen in Section 6.2, a minimal cable system for $(\mathcal{V}, m; b)$ may not exist. Moreover, to verify its existence is a rather complicated task even in some simple cases. It turns out that the situation changes once we drop the minimality assumption. In particular, we obtain an affirmative answer to Problem 6.2:

**Theorem 6.16.** *Every locally finite graph $(\mathcal{V}, m; b)$ has a cable system.*

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[2] See https://mathoverflow.net/questions/59117/: *Assigning positive edge weights to a graph so that the weight incident to each vertex is 1*, (2011).

*Proof.* The proof is by construction. As before, denote by $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ the simple graph associated with $b$ (see Remark 2.7). Let $\mathcal{G}_{\text{loop}} = (\mathcal{V}, \mathcal{E}_{\text{loop}})$ be the (combinatorial) graph obtained from $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ by adding a loop $e_v = e_{v,v}$ at each vertex $v \in \mathcal{V}$. More precisely, its edge set is given by

$$\mathcal{E}_{\text{loop}} = \mathcal{E}_b \cup \{e_v : v \in \mathcal{V}\}.$$

Next, define the edge weight $p\colon \mathcal{E}_{\text{loop}} \to (0, \infty)$ by

$$p(e_{u,v})^2 = \begin{cases} \dfrac{1}{2\max\{1, \operatorname{Deg}(u), \operatorname{Deg}(v)\}}, & u \neq v, \\ 1, & u = v, \end{cases}$$

where Deg is the weighted degree function (2.9). The edge lengths are then defined by $|\cdot| = p(\cdot)$ on $\mathcal{E}_{\text{loop}}$ and the edge weights $\mu$ and $\nu$ are given by

$$\mu(e_{u,v}) = \nu(e_{u,v}) = \begin{cases} b(u,v)p(u,v), & u \neq v, \\ m(v) - \sum\limits_{u\sim v} b(u,v)p(e_{u,v})^2, & u = v. \end{cases}$$

By construction, $\mu(e_v) = \nu(e_v) > 0$ and hence we indeed obtain well-defined weights $\mu, \nu\colon \mathcal{E}_{\text{loop}} \to (0, \infty)$. Moreover, it is easy to check that $(\mathcal{G}_{\text{loop}}, |\cdot|, \mu, \nu)$ is a cable system for $(\mathcal{V}, m; b)$. ■

**Remark 6.17.** A few remarks are in order:

(i) The above construction is taken from [72, Remark 2, p. 2107], where it was suggested in context with synchronizing Brownian motions and random walks on graphs. However, we stress that, due to the presence of a loop at every vertex, this cable system is never minimal.

(ii) After establishing existence of cable systems, the next natural question is their uniqueness. In fact, every locally finite graph $b$ over $(\mathcal{V}, m)$ has a large number of cable systems. In particular, the above cable system is a special case of a general construction using different metrizations of discrete graphs. These connections will be discussed in the next section.

## 6.4 Intrinsic metrics

In this section we discuss connections between *intrinsic metrics* for the Kirchhoff Laplacian on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ and the associated discrete Laplacian on a fixed model. Notice that we cannot expect a close link between the properties of the length metric $\varrho_0$ (see Section 2.1) and Kirchhoff Laplacians on weighted metric graphs since $\varrho_0$ does not depend on $\mu$ and $\nu$. However, it is known that the spectral properties of an operator associated to a (regular) Dirichlet form relate closely to its associated *intrinsic metrics* (see, e.g., [74, 198] for precise definitions and further references).

Historically, intrinsic metrics appear first in context with strongly local forms (see [53, Chapter 3.2] and [27]). More precisely, to each strongly local, regular Dirichlet form there is an associated intrinsic metric and this notion allows to generalize many results known for the Laplace–Beltrami operator on a Riemannian manifold and the Riemannian metric (see [198–200] for details and further references).

A rather general notion of intrinsic metrics for arbitrary (regular) Dirichlet forms was introduced in [74]. With its help, a variety of results could be recovered also in the non-local setting (see, e.g., [18, 74, 113, 116, 129] and the references therein). One of the crucial differences is that it is no longer possible to associate a unique intrinsic metric to a general Dirichlet form. More precisely, if the Dirichlet form is strongly local, then the classical intrinsic metric is intrinsic in the sense of [74]. Moreover, it is in a certain sense the largest one among all such metrics (see [74, Theorem 6.1]) and hence provides a canonical choice. For a non-local Dirichlet form (including the setting of graph Laplacians), there is in general no largest intrinsic metric and hence it is not possible to make a canonical choice.

### 6.4.1 Intrinsic metrics on metric graphs

We define the intrinsic metric of a weighted metric graph $(\mathcal{G}, \mu, \nu)$ as the (largest) intrinsic metric of its Dirichlet Laplacian $\mathbf{H}_D$ (in particular, note that $\mathfrak{Q}_D$ is a strongly local, regular Dirichlet form). By [198, equation (1.3)] (see also [74, Theorem 6.1]), $\varrho_{\mathrm{intr}}$ is given by

$$\varrho_{\mathrm{intr}}(x, y) = \sup\{f(x) - f(y) : f \in \widehat{\mathfrak{D}}_{\mathrm{loc}}\}, \quad x, y \in \mathcal{G},$$

where the function space $\widehat{\mathfrak{D}}_{\mathrm{loc}}$ is defined as

$$\widehat{\mathfrak{D}}_{\mathrm{loc}} = \{f \in H^1_{\mathrm{loc}}(\mathcal{G}) : \nu(x)|\nabla f(x)|^2 \leq \mu(x) \text{ for a.e. } x \in \mathcal{G}\}.$$

It turns out that $\varrho_{\mathrm{intr}}$ admits a rather explicit description. First of all, the above suggest to define the *intrinsic weight* $\eta \colon \mathcal{G} \to (0, \infty)$,

$$\eta = \eta_{\mu,\nu} := \sqrt{\frac{\mu}{\nu}} \quad \text{on } \mathcal{G}.$$

This weight gives rise to a new measure on $\mathcal{G}$ whose density with respect to the Lebesgue measure is exactly $\eta$ (as in the case of the edge weights on a metric graph, we abuse the notation and denote with $\eta$ both the edge weight and the corresponding measure).

Recall from Remark 2.2 that a path $\mathcal{P}$ in $\mathcal{G}$ is a continuous and piecewise injective map $\mathcal{P}: I \to \mathcal{G}$ defined on an interval $I \subseteq \mathbb{R}$. In case that $\mathcal{I} = [a, b]$ is compact, we call $\mathcal{P}$ a path with starting point $x := \mathcal{P}(a)$ and endpoint $y := \mathcal{P}(b)$. The *(intrinsic) length* of such a path $\mathcal{P}$ in $\mathcal{G}$ is defined as

$$|\mathcal{P}|_\eta := \sum_j \int_{\mathcal{P}((t_j, t_{j+1}))} \eta(\mathrm{d}s), \tag{6.5}$$

where $a = t_0 < \cdots < t_n = b$ is any partition of $\mathcal{I} = [a, b]$ such that $\mathcal{P}$ is injective on each interval $(t_j, t_{j+1})$ (clearly, $|\mathcal{P}|_\eta$ is well defined).

**Lemma 6.18.** *The metric $\varrho_\eta$ defined by*

$$\varrho_\eta(x, y) := \inf_{\mathcal{P}} |\mathcal{P}|_\eta = \inf_{\mathcal{P}} \int_{\mathcal{P}} \eta(\mathrm{d}s), \quad x, y \in \mathcal{G}, \tag{6.6}$$

*where the infimum is taken over all paths $\mathcal{P}$ from $x$ to $y$, coincides with the intrinsic metric on $(\mathcal{G}, \mu, \nu)$ (with respect to $\mathfrak{Q}_D$), that is, $\varrho_{\mathrm{intr}} = \varrho_\eta$.*

Notice that in the case $\mu = \nu$, $\eta$ coincides with the Lebesgue measure and hence $\varrho_\eta$ is nothing but the length metric $\varrho_0$ on $\mathcal{G}$.

*Proof.* The proof is straightforward and can be found in, e.g., [97, Proposition 2.21], however, we decided to present it for the sake of completeness. First, observe that for any two points $x, y$ on $\mathcal{G}$ and every path $\mathcal{P}$ from $x$ to $y$, the estimate

$$|f(x) - f(y)| \le \int_{\mathcal{P}} |\nabla f| \mathrm{d}s \le \int_{\mathcal{P}} \sqrt{\frac{\mu}{\nu}}\, \mathrm{d}s = \int_{\mathcal{P}} \eta(\mathrm{d}s) = |\mathcal{P}|_\eta$$

holds true for every $f \in \widehat{\mathfrak{D}}_{\mathrm{loc}}$, and hence $\varrho_{\mathrm{intr}} \le \varrho_\eta$.

On the other hand, fixing some $y \in \mathcal{G}$, define $f \in H^1_{\mathrm{loc}}(\mathcal{G})$ by $f(x) = \varrho_\eta(x, y)$ for all $x \in \mathcal{G}$. It is immediate to see that $f$ is edgewise absolutely continuous and

$$|\nabla f| = \sqrt{\frac{\mu}{\nu}} \quad \text{a.e. on } \mathcal{G}.$$

Therefore, $f \in \widehat{\mathfrak{D}}_{\mathrm{loc}}$. Moreover, for each $x \in \mathcal{G}$ we clearly have

$$\varrho_\eta(x, y) = f(x) - f(y) = f(x),$$

which finishes the proof. ∎

**Remark 6.19.** According to the above definition of the intrinsic weight, we get for a path $\mathcal{P}_e$ consisting of a single edge $e \in \mathcal{E}$

$$|\mathcal{P}_e|_\eta = \int_e \eta(\mathrm{d}s) = |e| \sqrt{\frac{\mu(e)}{\nu(e)}} = \eta(e),$$

which connects the intrinsic path metric $\varrho_{\mathrm{intr}} = \varrho_\eta$ on $(\mathcal{G}, \mu, \nu)$ with the notion of the intrinsic edge length (3.1).

**Remark 6.20** (Eikonal/optical metric). Let us mention that the obtained intrinsic metric admits a mechanical interpretation. In terms of the wave equation, the weight $\sqrt{\frac{\mu}{\nu}}$ is precisely the reciprocal of the speed of wave propagation on a given edge. Moreover, the distance function $f := \varrho_\eta(x_0, \cdot)$ for a reference point $x_0$ satisfies the eikonal equation $|\nabla f| = \sqrt{\frac{\mu}{\nu}}$ on all edges. From this perspective, one may try to interpret the intrinsic distance between two points on a weighted metric graph as a time that the wave initiated at one point needs to reach the other one. In the physics literature, the latter is often called *eikonal or optical metric*.

### 6.4.2 Intrinsic metrics on discrete graphs

The idea to use different metrics on graphs can be traced back at least to [52] and versions of metrics adapted to weighted discrete graphs have appeared independently in several works, see, e.g., [71,72,92,165]. Let us now recall the definition of intrinsic metrics for graph Laplacians, where we follow [18, 74, 129, 136].

Given a connected graph $b$ over $(\mathcal{V}, m)$, a symmetric function $p\colon \mathcal{V}\times\mathcal{V}\to[0,\infty)$ such that $p(u,v)>0$ exactly when $b(u,v)>0$ is called a *weight function* for $(\mathcal{V},m;b)$. Every weight function $p$ generates a *path metric* $\varrho_p$ on $\mathcal{V}$ with respect to the graph $b$ via

$$\varrho_p(u,v) := \inf_{\mathcal{P}=(v_0,\dots,v_n):u=v_0,\, v=v_n} \sum_k p(v_{k-1}, v_k). \tag{6.7}$$

Here the infimum is taken over all paths in $b$ connecting $u$ and $v$, that is, all sequences $\mathcal{P} = (v_0, \dots, v_n)$ such that $v_0 = u$, $v_n = v$ and $b(v_{k-1}, v_k) > 0$ for all $k \in \{1, \dots, n\}$. We stress that we always assume that $b$ is locally finite (see Section 2.2) and hence $\varrho_p(u, v) > 0$ whenever $u \neq v$.

**Example 6.21.** Let us provide a few important examples.

(i) *Combinatorial distance:* Let $p\colon \mathcal{V} \times \mathcal{V} \to \{0, 1\}$ be given by

$$p(u,v) = \begin{cases} 1, & b(u,v) \neq 0, \\ 0, & b(u,v) = 0. \end{cases}$$

Then the corresponding path metric is nothing but the combinatorial distance $\varrho_{\text{comb}}$ (also known as the *word metric* in the context of Cayley graphs) on a graph $b$ over $\mathcal{V}$.

(ii) *Natural path metric:* Define $p_b\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ by

$$p_b(u,v) = \begin{cases} \dfrac{1}{b(u,v)}, & b(u,v) \neq 0, \\ 0, & b(u,v) = 0. \end{cases} \tag{6.8}$$

Then the corresponding path metric $\varrho_b$ depends only on the graph $b$ and not on the weight function $m$, and hence one may call it as a *natural path met-*

*ric*. Notice also that the edge weight (6.8) can be interpreted as resistances (see Example 6.5).

(iii) *Star path metric:* Let $m\colon \mathcal{V} \to (0,\infty)$ be a vertex weight. Set

$$p_m(u,v) = \begin{cases} m(u)+m(v), & b(u,v) \neq 0, \\ 0, & b(u,v) = 0. \end{cases} \tag{6.9}$$

Then the corresponding path metric $\varrho_m$ is called the *star metric* on the graph $b$ over $\mathcal{V}$. The following two choices of $m$ are of particular interest: the vertex weight

$$m_b(v) := \sum_{u\in\mathcal{V}} b(u,v), \qquad v \in \mathcal{V},$$

corresponds to a simple random walk on graph $b$ (see Remark 2.11). Another choice

$$m_{1/b}(v) := \sum_{u\sim v} \frac{1}{b(u,v)}, \qquad v \in \mathcal{V},$$

appears in [68]. In particular, if $b\colon \mathcal{V}\times\mathcal{V} \to \{0,1\}$, then both $m_b$ and $m_{1/b}$ coincide with the combinatorial degree function deg. In both cases the vertex weight can be considered as a weight (or length) of the corresponding star $\mathcal{E}_v$ at $v \in \mathcal{V}$, which explains the name.

Recall (see [74] and also [115, 129]) the following important notion:

**Definition 6.22.** A metric $\varrho$ on $\mathcal{V}$ is called *intrinsic* with respect to $(\mathcal{V}, m; b)$ if

$$\sum_{u\in\mathcal{V}} b(u,v)\varrho(u,v)^2 \le m(v)$$

holds for all $v \in \mathcal{V}$.

Similarly, a weight function $p\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ is called an *intrinsic weight* for $(\mathcal{V}, m; b)$ if

$$\sum_{u\in\mathcal{V}} b(u,v)p(u,v)^2 \le m(v), \qquad v \in \mathcal{V}.$$

If $p$ is an intrinsic weight, then the associated path metric $\varrho_p$ is called *strongly intrinsic* (it is obviously intrinsic in the sense of Definition 6.22).

**Remark 6.23.** For any given locally finite graph $(\mathcal{V}, m; b)$ an intrinsic metric always exists (see [115, Example 2.1], [129] and also [45]). Indeed, we obtain an intrinsic weight by setting

$$p(u,v) = \begin{cases} \dfrac{1}{\sqrt{\max\{1, \mathrm{Deg}(u), \mathrm{Deg}(v)\}}}, & b(u,v) \neq 0, \\ 0, & b(u,v) = 0, \end{cases}$$

where Deg is the weighted degree function (2.9), and hence the corresponding path metric $\varrho = \varrho_p$ is strongly intrinsic. We are going to provide further examples in the next sections.

**Example 6.24.** Let us continue with Example 6.21.

(i) If a graph $b\colon \mathcal{V} \times \mathcal{V} \to \{0, 1\}$ is locally finite and $m = \deg$ on $\mathcal{V}$, then the combinatorial distance $\varrho_{\mathrm{comb}}$ on $\mathcal{V}$ is intrinsic.

(ii) If $m = m_{1/b}$, then the path metric $\varrho_b$ is intrinsic. Moreover, the weight $p_b$ is intrinsic as well.

(iii) Let us stress that the star path metric $\varrho_m$ is not intrinsic in general since it does not contain any information on $b$ except the underlying combinatorial structure.

**Remark 6.25.** Let us emphasize that the combinatorial distance $\varrho_{\mathrm{comb}}$ is not intrinsic for the combinatorial Laplacian $L_{\mathrm{comb}}$ ($m \equiv \mathbb{1}$ on $\mathcal{V}$ in this case). However, $\varrho_{\mathrm{comb}}$ is equivalent to an intrinsic path metric if and only if deg is bounded on $\mathcal{V}$, that is, the corresponding graph has bounded geometry. If $\sup_{\mathcal{V}} \deg(v) = \infty$, then $L_{\mathrm{comb}}$ is unbounded in $\ell^2(\mathcal{V})$ and it turned out that $\varrho_{\mathrm{comb}}$ is not a suitable metric on $\mathcal{V}$ to study the properties of $L_{\mathrm{comb}}$ (in particular, this has led to certain controversies in the past, see [135, 213]).

**Remark 6.26.** In the discrete setting we are unaware of any mechanical interpretation of intrinsic metrics (cf. Remark 6.20). In particular, the relationship to wave propagation speed is unclear since waves on discrete graphs propagate with infinite speed, which is closely connected to the non-locality of the corresponding Dirichlet form. It seems to us that exactly these facts are the origin of many difficulties in analysis on weighted (discrete) graphs.

### 6.4.3 Connections between discrete and continuous

Consider a weighted metric graph $(\mathcal{G}, \mu, \nu)$ and its intrinsic metric $\varrho_\eta$ defined in Section 6.4.1. With each model of $(\mathcal{G}, \mu, \nu)$ we can associate the vertex set $\mathcal{V}$ together with the vertex weight $m\colon \mathcal{V} \to (0, \infty)$ and the graph $b\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$, see (3.1)–(3.6). The next result shows that the intrinsic metric $\varrho_\eta$ of $(\mathcal{G}, \mu, \nu)$ gives rise to a particular intrinsic metric for $(\mathcal{V}, m; b)$.

**Lemma 6.27.** *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and $\varrho_\eta$ its intrinsic metric. Fix further a model of $(\mathcal{G}, \mu, \nu)$ having finite intrinsic size and define the metric $\varrho_{\mathcal{V}}$ on $\mathcal{V}$ as a restriction of $\varrho_\eta$ onto $\mathcal{V} \times \mathcal{V}$,*

$$\varrho_{\mathcal{V}}(u, v) := \varrho_\eta(u, v), \quad (u, v) \in \mathcal{V} \times \mathcal{V}. \tag{6.10}$$

*Then:*

(i) $\varrho_{\mathcal{V}}$ *is an intrinsic metric for* $(\mathcal{V}, m; b)$.

(ii) $(\mathcal{G}, \varrho_\eta)$ *is complete as a metric space exactly when* $(\mathcal{V}, \varrho_{\mathcal{V}})$ *is complete.*

*Proof.* (i) Fix a model of a weighted metric graph of $(\mathcal{G}, \mu, \nu)$ and consider the edge weight function $p_\eta\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ given by

$$p_\eta(u, v) = \begin{cases} \min_{e \in \mathcal{E}_{u,v}} \eta(e), & u \sim v \text{ and } u \neq v, \\ 0, & \text{else,} \end{cases} \qquad (u, v) \in \mathcal{V} \times \mathcal{V}. \tag{6.11}$$

Here $\mathcal{E}_{u,v}$ denotes the set of edges between $u$ and $v$ (recall that we allow multigraphs). Using (3.1)–(3.6), notice that for every $v \in \mathcal{V}$,

$$\begin{aligned} \sum_{u \in \mathcal{V}} b(u, v) p_\eta(u, v)^2 &= \sum_{u \in \mathcal{V} \setminus \{v\}} \sum_{e \in \mathcal{E}_{u,v}} \frac{\nu(e)}{|e|} p_\eta(u, v)^2 \\ &\leq \sum_{u \in \mathcal{V} \setminus \{v\}} \sum_{e \in \mathcal{E}_{u,v}} \frac{\nu(e)}{|e|} \eta(e)^2 \\ &= \sum_{u \in \mathcal{V} \setminus \{v\}} \sum_{e \in \mathcal{E}_{u,v}} |e| \mu(e) \\ &\leq m(v), \end{aligned}$$

where in the last inequality we used the fact that $(\mathcal{G}, \mu, \nu)$ has finite intrinsic size. Hence $p_\eta$ is an intrinsic weight for $(\mathcal{V}, m; b)$. It remains to notice that each path $\mathcal{P}$ without self-intersections from $u \in \mathcal{V}$ to $v \in \mathcal{V}$ in the metric graph $\mathcal{G}$ can be identified with a path $\mathcal{P}_d = (e_{u,v_1}, \dots, e_{v_{n-1},v})$ in the fixed model from $u = v_0$ to $v = v_n$ without self-intersections. With respect to this identification,

$$|\mathcal{P}|_\eta = \sum_{k=1}^{n} \eta(e_{v_{k-1},v_k})$$

which immediately implies that $\varrho_{p_\eta} = \varrho_\eta|_{\mathcal{V} \times \mathcal{V}}$ (notice that both the infima in (6.6) and (6.7) can be taken over paths without self-intersections).

(ii) The remaining equivalence of the metric space completeness is straightforward to verify directly (one can also immediately observe it by comparing geodesic completeness on both metric spaces and then using the corresponding versions of the Hopf–Rinow theorems, see Section 6.4.5). ∎

**Remark 6.28.** Notice that the proof also implies that (6.11) is an intrinsic weight for $(\mathcal{V}, m; b)$ and $\varrho_{\mathcal{V}} = \varrho_{p_\eta}$ is the corresponding strongly intrinsic path metric.

Let us mention that Lemma 6.27 also has an interpretation in terms of *quasi-isometries* (see, e.g., [12, Definition 1.12], [175, Section 1.3] and [187]).

**Definition 6.29.** A map $\phi\colon X_1 \to X_2$ between two metric spaces $(X_1, \varrho_1)$ and $(X_2, \varrho_2)$ is called *a quasi-isometry* if there are constants $a$, $R > 0$ and $d \geq 0$ such that

$$a^{-1}(\varrho_1(x, y) - d) \leq \varrho_2(\phi(x), \phi(y)) \leq a(\varrho_1(x, y) + d) \tag{6.12}$$

for all $x, y \in X_1$ and, moreover,

$$\bigcup_{x \in X_1} B_R(\phi(x); \varrho_2) = X_2. \tag{6.13}$$

One can check that quasi-isometries define an equivalence relation between metric spaces. It turns out that the map $\imath_{\mathcal{V}}$ defined in Section 4.3 is closely related with a quasi-isometry between weighted graphs and metric graphs:

**Lemma 6.30.** *Assume the conditions of Lemma 6.27. Then the map*

$$\varphi\colon \mathcal{V} \to \mathcal{G}, \quad \varphi(v) = v$$

*defines a quasi-isometry between the metric spaces* $(\mathcal{G}, \varrho_\eta)$ *and* $(\mathcal{V}, \varrho_{\mathcal{V}})$*. Moreover, the map* $\varphi$ *is bi-Lipschitz (i.e.,* $b$ *in* (6.12) *can be set equal to* 0*).*

*Proof.* The proof is a straightforward check of (6.12) and (6.13) for the map $\phi$ with $a = 1$, $b = 0$ and $R = \eta^*(\mathcal{E})$ and we leave it to the reader. ■

**Remark 6.31.** The notion of quasi-isometries was introduced in the works [94] of M. Gromov and [122, 123] of M. Kanai. It is well known in context with Riemannian manifolds and (combinatorial) graphs that quasi-isometric spaces share many important properties: e.g., geometric properties (such as volume growth and isoperimetric inequalities) [122], parabolicity/transience [47, 122, 160], Nash inequalities [47], Liouville-type theorems for harmonic functions of finite energy [47, 106, 107, 151, 160, 194] and parabolic/elliptic Harnack inequalities [14, 15, 47, 103]. However, we stress that most of these connections also require additional conditions on the local geometry of the spaces. Typically, one imposes a bounded geometry assumption for manifolds [122] and bounded geometry/controlled weights assumptions for graphs [12, 15], [195, Chapter VII].

Some of our conclusions are reminiscent of this notion (see, e.g., Theorem 4.17, Theorem 4.30 and Proposition 7.38), but in fact our results go beyond this framework. For instance, the strong/weak Liouville property (i.e., all positive/bounded harmonic functions are constant) is not preserved under bi-Lipschitz maps in general [155]. However, the equivalence holds true for our setting (this is a trivial consequence of Lemma 6.48 below). In addition, we stress that in contrast to the above works, we do not require any additional local conditions (e.g., bounded geometry). On the other hand, our results connect only two particular quasi-isometric spaces $(\mathcal{G}, \varrho_\eta)$ and $(\mathcal{V}, \varrho_{\mathcal{V}})$ and not the whole equivalence class of quasi-isometric weighted graphs or weighted metric graphs.

By Lemma 6.27, each cable system having finite intrinsic size[3] gives rise to an intrinsic metric $\varrho_{\mathcal{V}}$ for $(\mathcal{V}, m; b)$ using a simple restriction to vertices. In view of Problems 6.1–6.2, it is natural to ask which intrinsic metrics on graphs can be obtained as restrictions of intrinsic metrics on weighted metric graphs. It turns out that a rather large class can be covered in this way. Before stating the result, let us recall one more definition.

**Definition 6.32.** Let $b$ be a locally finite graph over $\mathcal{V}$. A metric $\varrho$ on $\mathcal{V}$ has *finite jump size* (with respect to $b$) if

$$s(\varrho) := \sup\{\varrho(u, v) : u, v \in \mathcal{V} \text{ with } b(u, v) > 0\}$$

is finite.

**Lemma 6.33.** *Let $(\mathcal{V}, m; b)$ be a locally finite graph and let $\rho\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ be an intrinsic path metric having finite jump size $s(\varrho) < \infty$. Then there is a cable system for $(\mathcal{V}, m; b)$ satisfying $\eta^*(\mathcal{E}) \leq \max\{s(\varrho), 1\}$ and such that $\rho_{\mathcal{V}} = \rho$. Moreover, $(\mathcal{V}, \varrho_{\mathcal{V}})$ is complete exactly when the corresponding weighted metric graph $(\mathcal{G}, \mu, \nu)$ of the cable system equipped with its intrinsic metric $\varrho_\eta$ is complete.*

*Proof.* Our proof follows closely the ideas of [114, p. 128] and [72]. The edge set $\mathcal{E}$ of the cable system $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu_{\mathcal{E}}, \nu_{\mathcal{E}})$ is defined as follows: first of all, we create an edge $e = e_{u,v}$ between each pair of vertices $u, v \in \mathcal{V}$ with $b(u, v) > 0$. Moreover, we add a loop edge at each vertex $v \in \mathcal{V}$ satisfying

$$\sum_{u \in \mathcal{V} \setminus \{v\}} b(u, v)\varrho(u, v)^2 < m(v).$$

Notice that the resulting combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ does not have any multiple edges. Specifying now the edge lengths and weight, assume first that $e_{u,v} \in \mathcal{E}$ is a non-loop edge, that is, $u \neq v$. Then we set

$$|e_{u,v}| = \varrho(u, v), \quad \mu(e_{u,v}) = \nu(e_{u,v}) = \varrho(u, v)b(u, v).$$

If $e \in \mathcal{E}$ is a loop at the vertex $v \in \mathcal{V}$, then we define

$$|e| = 1, \quad \mu(e) = \nu(e) = m(v) - \sum_{u \in \mathcal{V} \setminus \{v\}} b(u, v)\varrho(u, v)^2 > 0.$$

By definition, $\eta(e_{u,v}) = |e_{u,v}| = \varrho(u, v)$ for each non-loop edge $e_{u,v}$ and it is straightforward to check that $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu_{\mathcal{E}}, \nu_{\mathcal{E}})$ is a cable system for $(\mathcal{V}, m; b)$. Moreover, since $\varrho$ is a path metric, we easily infer that $\varrho = \varrho_{\mathcal{V}}$ (see Remark 6.28). ■

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[3] Since by definition a cable system is a model of a weighted metric graph, the notion of intrinsic size (see Definition 3.16) immediately extends to cable systems.

**Remark 6.34.** A few remarks are in order.

(i) Notice that an intrinsic path metric with jump size $s(\varrho) \le 1$ indeed exists for every graph $(\mathcal{V}, m; b)$ (e.g., take the path metric in Remark 6.23).

(ii) We stress that not every intrinsic metric is a path metric. However, in some sense intrinsic path metrics correspond to particularly large intrinsic metrics. Namely, for every intrinsic metric $\varrho$, the choice $p(u, v) := \varrho(u, v)$ whenever $b(u, v) > 0$ defines an intrinsic weight and the corresponding path metric clearly satisfies $\varrho \le \varrho_p$ on $\mathcal{V} \times \mathcal{V}$.

### 6.4.4 Description of cable systems

The results of the previous sections naturally lead us to Problem 6.3. It does not seem realistic to obtain a complete answer to this question since the class of all cable systems of $(\mathcal{V}, m; b)$ is rather large. Hence our strategy will be to restrict to a certain class of "well-behaved" cable systems and obtain a precise description of those.

**Definition 6.35.** A cable system $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ for a graph $b$ over $(\mathcal{V}, m)$ is called *canonical* if it satisfies the following additional assumptions:

(i) the underlying graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ has no multiple edges,

(ii) the edge weights $\mu$ and $\nu$ coincide,

$$\mu(e) = \nu(e), \quad e \in \mathcal{E},$$

(iii) $|e| = 1$ whenever $e$ is a loop and, moreover, $\sup_{e \in \mathcal{E}} |e| < \infty$.

The set of canonical cable systems of $(\mathcal{V}, m; b)$ is denoted by $\mathrm{Cab} = \mathrm{Cab}(\mathcal{V}, m; b)$.

Notice that conditions (ii) and (iii) imply that canonical cable systems have finite intrinsic size since in this case intrinsic edge length coincides with the edge length and hence

$$\eta^*(\mathcal{E}) = \sup_{e \in \mathcal{E}} |e|.$$

The importance of canonical cable systems stems from the fact that the intrinsic metric $\varrho_\eta$ of the corresponding weighted metric graph coincides with the length metric $\varrho_0$. Moreover, it turns out that canonical cable systems can be described in terms of intrinsic metrics. More precisely, denote by $\mathrm{W}(\mathcal{V}, m; b)$ the set of intrinsic weights for $(\mathcal{V}, m; b)$ having *finite jump size*, that is, all intrinsic weights $p\colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\ge 0}$ satisfying

$$\sup_{u,v : b(u,v) > 0} p(u, v) < \infty.$$

We already observed that for every canonical cable system, the choice

$$p(u, v) = \begin{cases} |e_{u,v}| & \text{if } u \ne v \text{ and } u \sim v, \\ 0 & \text{else}, \end{cases} \tag{6.14}$$

defines an intrinsic weight on $(\mathcal{V}, m; b)$ (see Remark 6.28). Hence (6.14) defines a map

$$\Psi\colon \mathrm{Cab}(\mathcal{V}, m; b) \to \mathrm{W}(\mathcal{V}, m; b). \tag{6.15}$$

In fact, this leads to a one-to-one correspondence between canonical cable systems and intrinsic weights.

**Theorem 6.36.** *Suppose $b$ is a locally finite connected graph over $(\mathcal{V}, m)$. Then the map $\Psi$ defined by (6.14) and (6.15) is a bijection between the set of canonical cable systems of $(\mathcal{V}, m; b)$ and intrinsic weights of $(\mathcal{V}, m; b)$ having finite jump size.*

*Proof.* As noticed above, the map $\Psi$ is well defined and, moreover, its surjectivity was established in Lemma 6.33. More precisely, if we replace $\varrho(u, v)$ by $p(u, v)$ in its proof, we obtain an explicit construction of a canonical cable system for every $p \in \mathrm{W}(\mathcal{V}, m; b)$.

To prove the injectivity of $\Psi$, we essentially invert the construction in Lemma 6.33. Let $C = (\mathcal{V}, \mathcal{E}, |\cdot|, \mu)$ be a canonical cable system for $(\mathcal{V}, m; b)$. First of all, notice that the non-loop edges of $\mathcal{E}$ are determined by (3.6): there is an edge $e_{u,v}$ between $u \neq v$ exactly when $b(u, v) > 0$. Moreover, if $\Psi(C) = p$, then equalities (6.14) and (3.6) imply that

$$|e_{u,v}| = p(u, v), \quad \mu(e) = b(u, v)p(u, v)$$

for each non-loop edge $e_{u,v}$ between $u \neq v$. However, this means that the location of the loop edges of $\mathcal{E}$ is determined by (3.5) and the finite intrinsic size assumption. Namely, it is easy to see that they are attached to exactly those vertices $v \in \mathcal{V}$ with

$$m(v) - \sum_{u : b(u,v)>0} b(u, v)p(u, v)^2 = m(v) - \sum_{u : b(u,v)>0} |e_{u,v}|\mu(e_{u,v}) > 0.$$

This proves that the edge set $\mathcal{E}$ of $C$ is uniquely determined by $p = \Psi(C)$. Moreover, since we required that $|e| = 1$ for loop edges, it follows that

$$2\mu(e_v) = m(v) - \sum_{u : b(u,v)>0} b(u, v)p(u, v)^2 > 0$$

if there is a loop $e_v$ at a vertex $v \in \mathcal{V}$. This shows that the weight $\mu\colon \mathcal{E} \to (0, \infty)$ is determined by $p = \Psi(C)$ as well and the injectivity of $\Psi$ is proven. ∎

**Remark 6.37.** Notice that from a cable system $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ of $(\mathcal{V}, m; b)$ we can construct further ones by scaling, that is, we set

$$|e|' = c(e)|e|, \quad \mu'(e) = c(e)^{-1}\mu(e), \quad \nu'(e) = c(e)\nu(e), \quad e \in \mathcal{E},$$

for some $(c(e))_{e\in\mathcal{E}} \subseteq (0, \infty)$. The corresponding Kirchhoff Laplacians and energy forms are (unitarily) equivalent as well. Among these equivalent cable systems there is a unique one satisfying $\mu \equiv \nu$ and this explains condition (ii) in Definition 6.35 (cf. also [97, Definition 2.18]). Conditions (ii) and (iii) exclude similar constructions (i.e., by replacing single edges with multiple ones and different normalizations of loop edges) and simplify the definition of $m$ (see (3.1)–(3.5)).

### 6.4.5 Interlude: The Hopf–Rinow theorem on graphs

As it was already mentioned in Remark 2.1, a metric graph $\mathcal{G}$ equipped with its length metric $\varrho_0$ is a length metric space (or simply a length space, see [37] for definitions). Clearly, equipping a weighted metric graph $(\mathcal{G}, \mu, \nu)$ with the intrinsic metric $\varrho_\eta$, which is defined by (6.6), turns $\mathcal{G}$ into a length space as well. A path $\mathcal{P}$ in $\mathcal{G}$, a continuous and piecewise injective map $\mathcal{P}\colon I \to \mathcal{G}$ defined on an interval $I \subseteq \mathbb{R}$, is called *geodesic* if it is locally a distance minimizer, i.e., for each $x \in I$ there is a neighborhood $B(x) \subset I$ of $x$ such that $\mathcal{P}|_{B(x)}$ is a shortest path (with respect to the corresponding length metric). In the following it would be convenient to assume that each geodesic is parameterized by its arc length.

Complete length spaces enjoy a number of very important properties. For instance, if $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space (recall that we always assume $\mathcal{G}$ to be locally finite), then it is *a geodesic metric space* meaning that any two points $x, y \in \mathcal{G}$ can be connected by a minimal geodesic, that is, by a shortest path (see, e.g., [37, Theorem 2.5.23]). Moreover, the classical Hopf–Rinow theorem, which connects completeness with geodesic completeness, as well as with compactness of closed distance balls, extends from the smooth setting of Riemannian manifolds to locally compact length spaces [37, Theorem 2.5.28], and in the case of metric graphs it reads as follows.

**Theorem 6.38** (Hopf–Rinow's theorem on metric graphs). *Let $\mathcal{G}$ be a locally finite connected weighted metric graph and let $\varrho$ be a path metric on $\mathcal{G}$.*[4] *The following assertions are equivalent:*

(i) *$(\mathcal{G}, \varrho)$ is complete,*

(ii) *$(\mathcal{G}, \varrho)$ is boundedly compact (every closed metric ball in $(\mathcal{G}, \varrho)$ is compact),*

(iii) *every geodesic $\mathcal{P}\colon [0, a) \to \mathcal{G}$ extends to a continuous path $\overline{\mathcal{P}}\colon [0, a] \to \mathcal{G}$.*

It is natural to expect that the Hopf–Rinow theorem extends to the case of locally finite weighted graphs and this was done in [167] and [115, Theorem A.1] (see also [129]).

[4]In fact, we are going to use this result with only two particular metrics on $\mathcal{G}$: the length metric $\varrho_0$ and the intrinsic path metric $\varrho_\eta$.

**Theorem 6.39** (Hopf–Rinow's theorem on graphs). *Let $b$ be a locally finite graph over $\mathcal{V}$ and let $\varrho$ be a path metric for $(\mathcal{V}; b)$. The following assertions are equivalent:*

(i) *$(\mathcal{V}, \varrho)$ is complete as a metric space,*

(ii) *every closed metric ball in $(\mathcal{V}, \varrho)$ is finite,*

(iii) *every infinite geodesic has infinite length.*[5]

**Remark 6.40.** A few remarks are in order.

(i) Taking into account the connection between weighted graphs and cable systems, it is not difficult to derive Theorem 6.39 from Theorem 6.38. For instance, if additionally $\varrho$ is intrinsic for $(\mathcal{V}, m; b)$ and has finite jump size, then by Theorem 6.36 there is a canonical cable system $(\mathcal{G}, \mu, \mu)$ such that $\varrho$ coincides with the restriction of $\varrho_\eta = \varrho_0$ onto $\mathcal{V} \times \mathcal{V}$. By Lemma 6.27 (ii), $(\mathcal{V}, \varrho)$ is complete if and only if so is $(\mathcal{G}, \varrho_\eta)$ and hence it remains to apply Theorem 6.38. Notice that this approach was used in [167, p. 24].

(ii) For a version of the discrete Hopf–Rinow theorem for graphs which are *not locally finite* see the recent [137].

### 6.4.6 Volume growth

We finish this section with a simple but useful estimate between the volume of balls with respect to the intrinsic metrics $\varrho_\eta$ and $\varrho_{\mathcal{V}}$. For any $x \in \mathcal{G}$ and $r > 0$, we denote an intrinsic distance ball of radius $r$ by

$$B_r(x) := B_r(x; \varrho_\eta) = \{y \in \mathcal{G} : \varrho_\eta(x, y) < r\}.$$

Similarly, for any vertex $v \in \mathcal{V}$ and $r > 0$, the ball of radius $r$ in the induced metric $\varrho_{\mathcal{V}}$ on $\mathcal{V}$ is denoted by

$$B_r^{\mathcal{V}}(v) := B_r^{\mathcal{V}}(v; \varrho_{\mathcal{V}}) = \{u \in \mathcal{V} : \varrho_{\mathcal{V}}(u, v) < r\}.$$

In particular, we have the obvious relation $B_r^{\mathcal{V}}(v; \varrho_{\mathcal{V}}) = B_r(v; \varrho_\eta) \cap \mathcal{V}$ for every $r > 0$ and vertex $v \in \mathcal{V}$.

**Lemma 6.41.** *Assume the conditions of Lemma 6.27. Then*

$$\mu(B_r(v; \varrho_\eta)) \le m(B_r^{\mathcal{V}}(v; \varrho_{\mathcal{V}})) \le 2\mu(B_{r+\eta^*(\varepsilon)}(v; \varrho_\eta))$$

*for every $r > 0$ and vertex $v \in \mathcal{V}$.*

---

[5]In a discrete measure space, paths are parameterized by the combinatorial distance and "infinite geodesic" simply means that as a path it has infinite combinatorial length.

*Proof.* First of all, notice that

$$m(B_r^{\mathcal{V}}(v)) = \sum_{u \in B_r^{\mathcal{V}}(v)} \sum_{\vec{e} \in \vec{\mathcal{E}}_u} \mu(e)|e|$$
$$= \sum_{e \in \vec{\mathcal{E}}} \mu(e)|e|(\mathbb{1}_{B_r(v)}(e_\imath) + \mathbb{1}_{B_r(v)}(e_\tau)),$$

where as always $\mathbb{1}_{B_r(v)}$ denotes the characteristic function of the subset $B_r(v) \subseteq \mathcal{G}$. This implies the first inequality since clearly

$$m(B_r^{\mathcal{V}}(v)) \geq \sum_{e \in \mathcal{E}:\, e \cap B_r(v) \neq \varnothing} \mu(e)|e| \geq \sum_{e \in \mathcal{E}} \mu(e \cap B_r(v)) = \mu(B_r(v)).$$

Conversely, every edge $e \in \mathcal{E}$ with at least one endpoint in $B_r(v)$ is contained in the larger ball $B_{r+\eta^*(\mathcal{E})}(v)$. In particular,

$$m(B_r^{\mathcal{V}}(v)) \leq 2 \sum_{e \in \mathcal{E}} \mu(e \cap B_{r+\eta^*(\mathcal{E})}(v)) \leq 2\mu(B_{r+\eta^*(\mathcal{E})}(v)),$$

and the proof is complete. ■

**Remark 6.42.** On the one hand, Lemma 6.41 establishes connections between volume growth of large balls in $(\mathcal{G}, \varrho_\eta)$ and $(\mathcal{V}, \varrho_{\mathcal{V}})$ (e.g., their polynomial/subexponential/exponential growth rates are the same) and, in fact, this phenomenon is well known in context with quasi-isometries (indeed, a volume growth is one of the most important quasi-isometric invariants). On the other hand, Lemma 6.41 indicates a connection between small scales too and this is usually not a part of the quasi-isometric setting.

## 6.5 Harmonic functions on graphs

### 6.5.1 Harmonic functions on weighted graphs

Let us begin by briefly recalling basic definitions. Assume that $b$ is a connected graph over $(\mathcal{V}, m)$ satisfying assumptions (i)–(iii) of Section 2.2 (at this point there is no need to assume that $b$ is locally finite). Also, by $L$ we denote the corresponding formal Laplacian (2.4) (the killing term $c$ is assumed to be identically zero).

**Definition 6.43.** A function $f : \mathcal{V} \to \mathbb{C}$ is called *harmonic* (*subharmonic, superharmonic*) with respect to $(\mathcal{V}, m; b)$ (or, simply, $L$-harmonic, $L$-subharmonic, $L$-superharmonic) if $f$ belongs to $\mathcal{F}_b(\mathcal{V})$ and satisfies

$$(Lf)(v) = 0, \quad ((Lf)(v) \leq 0,\ (Lf)(v) \geq 0)) \tag{6.16}$$

for all $v \in \mathcal{V}$.

If $f \in \mathcal{F}_b(\mathcal{V})$ satisfies (6.16) on a subset $Y \subseteq \mathcal{V}$, then it is called *harmonic on* $Y$ (*subharmonic on* $Y$, etc.) with respect to $(\mathcal{V}, m; b)$.

**Remark 6.44.** Let us emphasize that the notion of harmonic/subharmonic/superharmonic functions is independent of the weight $m$ and hence one can simply set $m \equiv 1$ in Definition 6.43 and say *harmonic/subharmonic/superharmonic with respect to* $(\mathcal{V}; b)$. On the other hand, when considering the maximal Laplacian $\mathbf{h}$ (see (2.6)) in the Hilbert space $\ell^2(\mathcal{V}; m)$, its kernel consists of $L$-harmonic functions which belong to $\ell^2(\mathcal{V}; m)$, and this subspace of course depends on the weight $m$.

The following fact is trivial in the setting of weighted graphs.

**Lemma 6.45.** *Suppose $f \in \mathcal{F}_b(\mathcal{V})$ solves $Lf + \lambda f = 0$ for some $\lambda \in \mathbb{R}_{\geq 0}$.*[6] *Then $|f|$ is subharmonic with respect to $(\mathcal{V}, m; b)$. If in addition $f$ is real-valued, then both $f_+$ and $f_-$ are subharmonic with respect to $(\mathcal{V}, m; b)$. Here $f_\pm = \frac{1}{2}(|f| \pm f)$.*

*Proof.* First observe that $Lf + \lambda f = 0$ means that

$$f(v)\Big(\sum_{u\in\mathcal{V}} b(u,v) + \lambda m(v)\Big) = \sum_{u\in\mathcal{V}} b(u,v) f(u)$$

for all $v \in \mathcal{V}$. Since the second factor on the left-hand side is positive, we get

$$\begin{aligned} |f(v)|\Big(\sum_{u\in\mathcal{V}} b(u,v) + \lambda m(v)\Big) &= \Big|\sum_{u\in\mathcal{V}} b(u,v) f(u)\Big| \\ &\leq \sum_{u\in\mathcal{V}} b(u,v)|f(u)|, \end{aligned}$$

which immediately implies that

$$\begin{aligned} (L|f|)(v) &= \frac{1}{m(v)} \sum_{u\in\mathcal{V}} b(u,v)(|f(v)| - |f(u)|) \\ &= \frac{1}{m(v)}\Big(|f(v)| \sum_{u\in\mathcal{V}} b(u,v) - \sum_{u\in\mathcal{V}} b(u,v)|f(u)|\Big) \\ &\leq -\lambda |f(v)|. \end{aligned}$$

Therefore, $L|f| \leq -\lambda|f| \leq 0$ and hence $|f|$ is subharmonic with respect to $(\mathcal{V}, m; b)$. It remains to notice that for real-valued $f$ by linearity we have

$$Lf_\pm = \frac{1}{2}(L|f| \pm Lf) \leq \frac{1}{2}(-\lambda|f| \mp \lambda f) \leq 0. \qquad \blacksquare$$

[6]Usually, for $\lambda > 0$ such a function is called $\lambda$-harmonic.

### 6.5.2 Harmonic functions on metric graphs

In the case of metric graphs, one can start with the definition for strongly local Dirichlet forms (see, e.g., [198]).

**Definition 6.46.** A function $f: \mathcal{G} \to \mathbb{R}$ is called *harmonic* with respect to $(\mathcal{G}, \mu, \nu)$ if $f \in H^1_{\mathrm{loc}}(\mathcal{G})$ and

$$\int_{\mathcal{G}} \nabla f(x) \nabla g(x) \nu(\mathrm{d}x) = 0, \qquad (6.17)$$

for all $0 \le g \in H^1_c(\mathcal{G}) = H^1(\mathcal{G}) \cap C_c(\mathcal{G})$.

If for an open subset $Y \subseteq \mathcal{G}$, (6.17) holds for all $0 \le g \in H^1(\mathcal{G}) \cap C_c(Y)$ with compact support in $Y$, then $f$ is called *harmonic* on $Y$.

Replacing the equality in (6.17) by the inequality "$\le$" (resp., by "$\ge$"), one gets the definition of a *subharmonic* (resp., *superharmonic*) function on $Y \subseteq \mathcal{G}$ with respect to $(\mathcal{G}, \mu, \nu)$.

**Remark 6.47.** We stress that the notion of harmonic/subharmonic/superharmonic functions is independent of the weight $\mu: \mathcal{G} \to (0, \infty)$ (since this obviously holds for the space $H^1_{\mathrm{loc}}(\mathcal{G})$) and hence we could also call them harmonic/subharmonic/superharmonic functions with respect to $(\mathcal{G}, \nu)$. However, for our purposes we will mainly be interested in functions which additionally belong to $L^p(\mathcal{G}; \mu)$ and of course these spaces do depend on the edge weight $\mu$.

If it is clear from the context which graph (weighted graph or weighted metric graph) is meant, we shall simply say harmonic, subharmonic, etc. Notice also that on each edge the structure of the corresponding Sobolev space is very well understood and hence we can rewrite the above definition in a more convenient form. Recall (see Section 4.3) that for each fixed model of $(\mathcal{G}, \mu, \nu)$, $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ denotes the space of continuous edgewise affine functions on $\mathcal{G}$.

**Lemma 6.48.** *A function $f: \mathcal{G} \to \mathbb{R}$ is harmonic with respect to $(\mathcal{G}, \mu, \nu)$ exactly when $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ for some model of $(\mathcal{G}, \mu, \nu)$ and, moreover, $f$ satisfies Kirchhoff conditions at each vertex $v \in \mathcal{V}$.*

*Proof.* Clearly, we only need to prove the "only if" claim. Fix an arbitrary model of $(\mathcal{G}, \mu, \nu)$. Upon choosing test functions $g \in H^1_c(\mathcal{G})$ whose support is contained in single edges, it is straightforward to see that $f$ is affine on each edge $e \in \mathcal{E}$ (indeed, one simply needs to use the fact that a distributional solution to $f'' = 0$ is a classical solution). Next, for each vertex $v \in \mathcal{V}$, choosing test functions supported in a sufficiently small vicinity of $v$, a straightforward integration by parts shows that $f$ must satisfy Kirchhoff conditions at $v \in \mathcal{V}$. ∎

**Remark 6.49.** Let us stress that by Lemma 6.48 the set of harmonic functions is independent of the choice of a model of $\mathcal{G}$.

Using the same arguments one can easily show the following result:

**Lemma 6.50.** *A function $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ is subharmonic (superharmonic) with respect to $(\mathcal{G}, \mu, \nu)$ exactly when, for all $v \in \mathcal{V}$,*

$$\sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} f(v) \geq 0, \quad \Big( \sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} f(v) \leq 0 \Big). \tag{6.18}$$

**Remark 6.51.** A few remarks are in order.

(i) Similar to the discrete situation, Definition 6.46 can be reformulated in terms of the Laplacian $\Delta$ (see (2.11)). More specifically, the left-hand side in (6.17) allows us to define $\Delta$ on locally $H^1$ functions in a standard way (as a distribution on the test function space $H^1_c(\mathcal{G})$). Then a locally $H^1$ function $f$ is called harmonic (resp., subharmonic, superharmonic) if $\Delta f = 0$ on $\mathcal{G}$ (resp., $\Delta f$ is a nonpositive/non-negative distribution on $\mathcal{G}$). This definition becomes transparent for edgewise affine functions. Indeed, if $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ for some model of $(\mathcal{G}, \mu, \nu)$, then a straightforward integration by parts shows that, as a distribution,

$$\Delta f = \sum_{v \in \mathcal{V}} \Big( \sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} f(v) \Big) \delta_v. \tag{6.19}$$

Comparing (6.19) with Lemma 6.48 and Lemma 6.50, one concludes that $f$ is harmonic (subharmonic or superharmonic) if and only if $\Delta f = 0$ (respectively, $\Delta f \geq 0$ or $\Delta f \leq 0$).

(ii) We stress that there are sub-/superharmonic functions which are not edgewise affine. For instance, it is easy to check that a continuous, edgewise $H^2$-function $f$ is subharmonic exactly when $f$ satisfies (6.18) and is subharmonic on every edge. However, for our purposes it will suffice to consider only edgewise affine sub-/superharmonic functions.

It is not difficult to notice that the above results immediately connect harmonic, subharmonic, and superharmonic functions on graphs and on metric graphs.

**Lemma 6.52.** *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Let also $(\mathcal{V}, m; b)$ be the corresponding weighted graph defined by (3.3)–(3.6). Then $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ is harmonic (resp., subharmonic, superharmonic) if and only if $\mathbf{f} = \imath_{\mathcal{V}}(f) = f|_{\mathcal{V}}$ is harmonic (resp., subharmonic, superharmonic) with respect to $(\mathcal{V}, m; b)$. Here the map $\imath_{\mathcal{V}}$ is defined by (4.10).*

*Proof.* Notice that for an edgewise affine function $f$, its slope at $v$ on an oriented edge $\vec{e} \in \vec{\mathcal{E}}_v$ having vertices $v$ and $u$ is simply given by

$$\partial_{\vec{e}} f(v) = \frac{f(u) - f(v)}{|e|}.$$

Thus, comparing (6.19) with (3.6) and then using Lemma 6.48 (resp., Lemma 6.50), one finishes the proof. ■

We also need the following analog of Lemma 6.45.

**Lemma 6.53.** *Suppose $f \in H^1_{\mathrm{loc}}(\mathcal{G})$ solves $\Delta f = \lambda f$ edgewise for some $\lambda \in \mathbb{R}_{\geq 0}$ and, moreover, satisfies Kirchhoff conditions at all the vertices. Then $|f|$ is subharmonic. If in addition $f$ is real-valued, then both $f_+$ and $f_-$ are subharmonic.*

*Proof.* Due to linearity, we can assume without loss of generality that $f$ is real-valued. Fix some model of $(\mathcal{G}, \mu, \nu)$. Then the equality $\Delta f = \lambda f$ implies that $f$ is a classical solution to $\nu(e) f'' = \lambda \mu(e) f$ on each edge $e \in \mathcal{E}$ (upon an identification of $e$ with the interval $\mathcal{I}_e = [0, |e|]$). Hence it is easy to show that

$$|f|'' \geq \lambda \frac{\mu(e)}{\nu(e)} |f|,$$

where the inequality is understood in the distributional sense (e.g., use the Kato inequality [184, Theorem X.27]). It remains to notice that

$$\sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} |f|(v) \geq 0$$

for all vertices $v \in \mathcal{V}$. Since $f$ is continuous at $v \in \mathcal{V}$, in the case $f(v) \neq 0$, $|f|$ coincides with $\mathrm{sign}(f(v)) f$ in a small vicinity of $v$ and hence Kirchhoff conditions would imply that

$$\sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} |f|(v) = \sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} f(v) = 0$$

at every such vertex. If $f(v) = 0$, then it is straightforward to see that in this case

$$0 = \sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} f(v) \leq \sum_{\vec{e} \in \vec{\mathcal{E}}_v} \nu(e) \partial_{\vec{e}} |f|(v),$$

which finishes the proof. ■

The following result is a standard characterization via the mean value property.

**Lemma 6.54** (Mean value property). *Let $f \in \mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ be real-valued. Then $f$ is harmonic (subharmonic, superharmonic) if and only if for each $v \in \mathcal{V}$*

$$\frac{1}{\mu(B_r(v; \varrho_\eta))} \int_{B_r(v;\varrho_\eta)} f(x) \mu(\mathrm{d}x) = f(v) \quad (\geq f(v), \leq f(v)) \tag{6.20}$$

*for all sufficiently small $r > 0$. Here $\varrho_\eta$ is the intrinsic metric on $(\mathcal{G}, \mu, \nu)$ and $B_r(v; \varrho_\eta)$ is the distance ball in $(\mathcal{G}, \varrho_\eta)$ of radius $r > 0$ with the center at $v$.*

*Proof.* In fact, the mean value property is a straightforward consequence of Lemma 6.48 (resp., and Lemma 6.50). Indeed, suppose $r > 0$ is such that the corresponding distance ball $B_r(v; \varrho_\eta)$ is isomorphic to a star-shaped set (2.3). Then taking into account that $f$ is edgewise affine, we easily get

$$\begin{aligned}\int_{B_r(v;\varrho_\eta)} f(x)\mu(\mathrm{d}x) &= \sum_{e\in\mathcal{E}_v} \int_{e\cap B_r(v;\varrho_\eta)} f(x_e)\mu(\mathrm{d}x_e)\\ &= \sum_{\vec{e}\in\vec{\mathcal{E}}_v} \frac{1}{2}\Big(2f(v) + \partial_{\vec{e}} f(v)\frac{r|e|}{\eta(e)}\Big)\frac{r|e|}{\eta(e)}\mu(e)\\ &= f(v)r\sum_{\vec{e}\in\vec{\mathcal{E}}_v}\sqrt{\mu(e)\nu(e)} + \frac{r^2}{2}\sum_{\vec{e}\in\vec{\mathcal{E}}_v}\nu(e)\partial_{\vec{e}} f(v).\end{aligned}$$

It remains to notice that

$$\mu(B_r(v;\varrho_\eta)) = \sum_{\vec{e}\in\vec{\mathcal{E}}_v}\mu(e)\frac{r|e|}{\eta(e)} = r\sum_{\vec{e}\in\vec{\mathcal{E}}_v}\sqrt{\mu(e)\nu(e)}. \qquad \blacksquare$$

**Remark 6.55.** We stress that the mean-value property on weighted metric graphs holds only locally. That is, even for a harmonic function $f$ on $(\mathcal{G}, \mu, \nu)$, the equality (6.20) can fail when the integral is taken over a ball $B_r(v; \varrho_\eta)$ with large radius $r$. Indeed, problems arise already if $B_r(v; \varrho_\eta)$ contains more than one vertex of degree $\geq 3$ and the latter is not at all surprising since these vertices can be considered as singularities of one-dimensional manifolds (see Remark 2.4).

### 6.5.3 Liouville-type properties on graphs

An important question is which subspaces of harmonic functions are trivial, that is, which conditions ensure the uniqueness of solutions to the Helmholtz equation

$$\Delta u = \lambda u.$$

Such results are referred to as *Liouville-type theorems*. In Riemannian geometry $L^p$-Liouville theorems for harmonic functions were studied, for example, by S.T. Yau [217], L. Karp [124], P. Li and R. Schoen [153] and many others. Karp's and Yau's theorems were later generalized by K.-T. Sturm [198] to the setting of strongly local, regular Dirichlet forms. In particular, in the case of metric graphs Sturm's result reads as follows (cf. [198, Corollary 1 (a)]).

**Theorem 6.56** (Yau's $L^p$-Liouville theorem on metric graphs [198]). *If $(\mathcal{G}, \mu, \nu)$ is a locally finite weighted metric graph such that $(\mathcal{G}, \varrho_\eta)$ is complete, then every non-negative subharmonic function which belongs to $L^p(\mathcal{G}; \mu)$ for some $p \in (1, \infty)$ is identically zero. In particular, if $f \in L^p(\mathcal{G}; \mu)$ is harmonic, then $f \equiv 0$.*

In the case of weighted graphs, Liouville-type theorems have been investigated in, e.g., [108, 110, 164, 186] and the analogs of Yau's and Karp's theorems were established quite recently by B. Hua and M. Keller [113].

**Theorem 6.57** (Yau's $L^p$-Liouville theorem on graphs [113]). *Let $b$ be a locally finite connected graph over $(\mathcal{V}, m)$ and let $\varrho$ be an intrinsic path metric of finite jump size. If $(\mathcal{V}, \varrho)$ is complete as a metric space, then every non-negative $L$-subharmonic function which belongs to $\ell^p(\mathcal{V}; m)$ for some $p \in (1, \infty)$ is identically zero. In particular, if $f \in \ell^p(\mathcal{V}; m)$ is $L$-harmonic, then $f \equiv 0$.*

**Remark 6.58.** We stated Corollary 1.2 from [113] in a weaker form in order to simplify considerations. In fact, the assumption that $\varrho$ is a path metric can be weakened. More precisely, the conclusion remains valid for a general intrinsic metric $\varrho$ of finite jump size such that $\varrho$ generates the discrete topology on $\mathcal{V}$ and $(\mathcal{V}, \varrho)$ is complete (the latter follows by a simple comparison argument with the path metric $\varrho_p$ constructed in Remark 6.34 (ii)).

In fact, the connection between intrinsic metrics on weighted graphs and cable systems shows that Theorem 6.57 easily follows from Theorem 6.56:

*Proof of Theorem 6.57.* Let $\varrho$ be an intrinsic path metric for $(\mathcal{V}, m; b)$ having finite jump size. Then by Lemma 6.33 there is a canonical cable system $(\mathcal{G}, \mu, \mu)$ such that $\varrho$ coincides with the restriction of $\varrho_\eta = \varrho_0$ onto $\mathcal{V} \times \mathcal{V}$. Clearly, $(\mathcal{V}, \varrho)$ is complete if and only if so is $(\mathcal{G}, \varrho_\eta)$.

Take now a non-negative function $\mathbf{f}\colon \mathcal{V} \to \mathbb{R}_{\geq 0}$ which is $L$-subharmonic. By Lemma 6.52, the corresponding function $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f})$ is non-negative and subharmonic with respect to $(\mathcal{G}, \mu, \nu)$. If $\mathbf{f} \in \ell^p(\mathcal{V}; m)$ for some $p \in (1, \infty)$, then $f \in L^p(\mathcal{G}; \mu)$ according to Lemma 4.2. Applying Theorem 6.56, we conclude that $f$ is trivial and hence so is $\mathbf{f} = \imath_{\mathcal{V}}(f)$. ■

**Remark 6.59.** Using the same line of reasoning and also connections between volume growth of metric graphs and weighted graphs (see Lemma 6.41), one can easily connect, for example, Karp's $L^p$ Liouville theorems for metric graphs and weighted graphs (see Section 7.4), Grigor'yan's $L^1$ theorem, etc.

## 6.6 Life without loops II: Jacobi matrices on graphs

This section deals with Problem 6.4. For a given $\beta\colon \mathcal{V} \to \mathbb{R}$ and a connected graph $q$ over $\mathcal{V}$ satisfying properties (i), (ii) and (iv) of Section 2.2, consider a second order symmetric difference expression

$$(\tau f)(v) = \beta(v) f(v) - \sum_{u \in \mathcal{V}} q(u, v) f(u), \quad v \in \mathcal{V}. \tag{6.21}$$

Alternatively, its action can be described by the infinite symmetric matrix

$$\mathrm{H} = (h_{uv})_{u,v\in\mathcal{V}}$$

given by

$$h_{uv} = \begin{cases} \beta(v), & u = v, \\ -q(u,v), & u \neq v. \end{cases}$$

As described in Section 2.2, we can associate in $\ell^2(\mathcal{V})$ the minimal and maximal operator with the difference expression (6.21).

**Remark 6.60.** Every difference operator (6.21) is a Schrödinger-type operator on $\ell^2(\mathcal{V})$ in the sense of Remark 2.10: the weight function $m = \mathbb{1}_{\mathcal{V}}$ on $\mathcal{V}$ and its coefficients are explicitly given by

$$b(u,v) = q(u,v), \quad c(v) = \beta(v) - \sum_{u\in\mathcal{V}} q(u,v).$$

Symmetric difference expressions (6.21) are also known as *Jacobi matrices on graphs* (see, e.g., [8–10]).

On the other hand, every Schrödinger-type operator in $\ell^2(\mathcal{V};m)$ is unitarily equivalent (by means of the map $\mathcal{U}\colon \ell^2(\mathcal{V};m) \to \ell^2(\mathcal{V})$ defined by (3.29)) to a Schrödinger operator in $\ell^2(\mathcal{V})$ and hence from this perspective the class of Schrödinger-type operators on $\ell^2(\mathcal{V})$ is sufficiently large.

The next result answers Problem 6.4 in the affirmative.

**Theorem 6.61.** *Let $q\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ be a locally finite connected graph over $\mathcal{V}$ and let $\mathcal{G}_q = (\mathcal{V}, \mathcal{E}_q)$ be the underlying simple graph (see Remark 2.7). Then there exist edge weights $\nu\colon \mathcal{E}_q \to (0,\infty)$ and edge lengths $|\cdot|\colon \mathcal{E}_q \to (0,\infty)$ such that*

$$|e|^2 \leq \nu(e) \tag{6.22}$$

*for all $e \in \mathcal{E}_q$, and*

$$q(u,v) = \frac{\nu(e_{u,v})}{|e_{u,v}|(\sum_{e\in\mathcal{E}_u} |e|)^{1/2}(\sum_{e\in\mathcal{E}_v} |e|)^{1/2}} \tag{6.23}$$

*for all $e_{u,v} \in \mathcal{E}_q$.*

Notice that the difference expression (3.28) is a special case of (6.21):

$$\beta(v) = \frac{1}{m(v)}\Big(\alpha(v) + \sum_{u\in\mathcal{V}} b(u,v)\Big), \quad q(u,v) = \frac{b(u,v)}{\sqrt{m(u)}\sqrt{m(v)}}. \tag{6.24}$$

Moreover, the minimal operator $\widetilde{\mathbf{h}}_\alpha$ associated with (6.21), (6.24) shares many of its basic spectral properties with the Laplacian $\mathbf{H}_\alpha$ (see Theorem 3.1 and its proof), however, there is in general no connection between their parabolic properties. Theorem 6.61 implies the following result.

**Corollary 6.62.** *Every second-order difference operator* (6.21) *arises as a boundary operator of a Laplacian with* $\delta$*-couplings. More precisely, there is a weighted metric graph* $(\mathcal{G}, \mu, \nu)$ *such that for its simple model* $(\mathcal{V}, \mathcal{E}_q, |\cdot|, \mu, \nu)$ *and* $\alpha\colon \mathcal{V} \to \mathbb{R}$ *the relations* (6.24) *holds true, where the graph* $(\mathcal{V}, m; b)$ *is given by* (3.1)–(3.5) *and* (3.6)*.*

The proof of Theorem 6.61 is based on the following two lemmas, however, first we need to recall a few basic notions. A connected simple graph $(\mathcal{V}, \mathcal{E})$ without cycles is called *a tree*. We shall denote trees by $\mathcal{T}$. Notice that for any two vertices $u$, $v$ on a tree $\mathcal{T}$ there is exactly one path $\mathcal{P}$ connecting $u$ and $v$, and hence the combinatorial distance on $\mathcal{T}$ is exactly the number of edges in the path connecting $u$ and $v$. A tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with a distinguished vertex $o \in \mathcal{V}$ is called *a rooted tree* and $o$ is called *the root*. Each vertex $v \in \mathcal{V}$ having degree 1 is called a *leaf*.

**Lemma 6.63.** *Let* $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ *be a locally finite infinite tree. Then there is an infinite subtree* $\mathcal{T}_\infty = (\mathcal{V}_\infty, \mathcal{E}_\infty) \subseteq \mathcal{T}$ *such that* $\mathcal{T}_\infty$ *has at most one leaf and* $\mathcal{T}$ *is obtained by attaching to each vertex* $v \in \mathcal{V}_\infty$ *a (possibly empty) finite tree* $\mathcal{T}_v$*.*

*Proof.* The proof is by construction, which can informally be considered as "cutting away" finite subtrees from a given tree. Fix a root $o \in \mathcal{V}$ for $\mathcal{T}$ and order the vertices of $\mathcal{T}$ according to combinatorial spheres. The latter also introduces a natural orientation on $\mathcal{T}$: for every edge $e$ its initial vertex $e_\iota$ belongs to the smaller combinatorial sphere.

Next, let us define the standard partial ordering on $\mathcal{T}$. For two edges $e, \widetilde{e} \in \mathcal{E}$, we write $\widetilde{e} \prec e$, if the path from the root $o$ to the terminal vertex $e_\tau$ of $e$ passes through $\widetilde{e}$. For any $e \in \mathcal{E}$, denote by $\mathcal{T}_e \subseteq \mathcal{T}$ the subtree with the edge set

$$\mathcal{E}(\mathcal{T}_e) = \{\widetilde{e} \in \mathcal{E} : e \prec \widetilde{e}\}.$$

Since "$\prec$" is transitive on $\mathcal{E}$, $e \in \mathcal{T}_{\widetilde{e}}$ implies that $\mathcal{T}_e \subseteq \mathcal{T}_{\widetilde{e}}$. Moreover, define

$$\mathcal{E}_v^\infty = \{e \in \mathcal{E}_v^+ : \mathcal{T}_e \text{ is infinite}\},$$

where $\mathcal{E}_v^+$ is the set of outgoing edges at $v$, see (2.1), and then for each $v \in \mathcal{V}$ denote by $\mathcal{T}_v$ the (possibly empty) finite subtree of $\mathcal{T}$ with the edge set

$$\mathcal{E}(\mathcal{T}_v) = \bigcup_{e \in \mathcal{E}_v^+ \setminus \mathcal{E}_v^\infty} \mathcal{E}(\mathcal{T}_e). \tag{6.25}$$

After all these lengthy preparations, we finally begin our construction. For every edge $e \in \mathcal{E}_o^+ = \mathcal{E}_o$ consider the subtree $\mathcal{T}_e$. Since $\mathcal{T}$ is infinite, there is at least one edge $e \in \mathcal{E}_o^+$ such that the corresponding subtree $\mathcal{T}_e$ is infinite and hence the set

$\mathcal{E}_o^\infty$ is non-empty. Denote the set of terminal vertices of all edges $e \in \mathcal{E}_o^\infty$ by $\mathcal{V}_1^\infty$. Notice that $\mathcal{V}_1^\infty$ is a subset of the first combinatorial sphere $S_1$. Next for each $v \in \mathcal{V}_1^\infty$ consider the corresponding edge sets $\mathcal{E}_v^\infty$. Again all of them are non-empty since, by construction, each $\mathcal{T}_e$ is infinite. The union of all terminal vertices of $e \in \mathcal{E}_v^\infty$ with $v \in \mathcal{V}_1^\infty$ is denoted by $\mathcal{V}_2^\infty$. Clearly, $\mathcal{V}_2^\infty$ is a non-empty subset of the second combinatorial sphere $S_2$. Continuing this process, we end up with an infinite sequence of vertex sets $\mathcal{V}_n^\infty \subseteq S_n$, $n \geq 1$. Since our initial tree $\mathcal{T}$ is infinite but locally finite, every vertex set $\mathcal{V}_n^\infty$, $n \geq 1$ is non-empty.

Now we define $\mathcal{T}_\infty$ as the subtree of $\mathcal{T}$ with the vertex set $\mathcal{V}_\infty := \{o\} \cup \{\mathcal{V}_n^\infty\}_{n\geq 1}$. It follows from our construction that $\mathcal{T}_\infty$ is an infinite tree with the only possible leaf $o$ (this happens exactly when $\#\mathcal{E}_o^\infty = 1$). Moreover, it is immediate to see that attaching to each $v \in \mathcal{V}_\infty$ the finite subtree $\mathcal{T}_v$ defined by (6.25) we recover the given tree $\mathcal{T}$. ■

The next result proves Theorem 6.61 for trees:

**Lemma 6.64.** *Let $q$ be a locally finite graph over $\mathcal{V}$ such that the associated simple graph $\mathcal{G}_q$ (see Remark 2.7) is an infinite tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$. Then there exist edge weights $\nu\colon \mathcal{E} \to (0,\infty)$ and edge lengths $|\cdot|\colon \mathcal{E} \to (0,\infty)$ such that (6.22) and (6.23) hold true for all $e \in \mathcal{E}$.*

*Proof.* We divide the proof into several steps.

(i) First of all, notice that the existence of $\nu$ and $|\cdot|$ satisfying (6.22) and (6.23) for all $e \in \mathcal{E}$ is equivalent to the existence of edge lengths $|\cdot|$ satisfying

$$T(e_{u,v}) := \frac{|e_{u,v}|}{(\sum_{e\in\mathcal{E}_u} |e|)^{1/2}(\sum_{e\in\mathcal{E}_v} |e|)^{1/2}} \leq q(u,v) \tag{6.26}$$

for each $u \sim v$, since in this case a suitable choice of the edge weight $\nu$ is simply given by

$$\nu(e) := |e|^2 \frac{q(e)}{T(e)}, \quad e \in \mathcal{E}. \tag{6.27}$$

Here and below we use the obvious notation $q(e_{u,v}) = q(u,v)$ for each $e = e_{u,v} \in \mathcal{E}$.

(ii) Next, by Lemma 6.63, we can find an infinite rooted subtree $\mathcal{T}_\infty = (\mathcal{V}_\infty, \mathcal{E}_\infty)$ of $\mathcal{T}$ such that $\mathcal{T}_\infty$ has at most one leaf at its root $o$ and such that $\mathcal{T}$ is obtained by attaching to each $v \in \mathcal{V}_\infty$ a (possibly empty) finite tree $\mathcal{T}_v$. Clearly,

$$\mathcal{E} \setminus \mathcal{E}_\infty = \bigcup_{v\in\mathcal{V}_\infty} \mathcal{E}(\mathcal{T}_v).$$

(iii) We start by assigning edge lengths to each finite non-empty subtree $\mathcal{T}_v$, $v \in \mathcal{V}_\infty$. Consider $\mathcal{T}_v$ as a rooted tree with the root at $v$, $o(\mathcal{T}_v) = v$. Let $h(v)$ be the height of $\mathcal{T}_v$, i.e., the maximal combinatorial distance of a vertex in $\mathcal{T}_v$ to $v$. For $n \in \{1,\dots,h(v)\}$, denote by $\mathcal{E}^n(\mathcal{T}_v)$ the set of edges $e \in \mathcal{E}(\mathcal{T}_v)$ between the combinatorial spheres $S_{n-1}(\mathcal{T}_v)$ and $S_n(\mathcal{T}_v)$ of $\mathcal{T}_v$. We will assign lengths for the sets

$\mathcal{E}^n(\mathcal{T}_v)$ inductively in $n$ starting from the top of $\mathcal{T}_v$ and going downwards to $o(\mathcal{T}_v)$. More precisely, we define positive reals $\ell_1, \dots, \ell_{h(v)}$ by first setting $\ell_{h(v)} = 1$ and, if $h(v) > 1$, inductively

$$\ell_{k-1} := \max_{e \in \mathcal{E}^k(\mathcal{T}_v)} \frac{\ell_k}{q(e)^2} = \frac{\ell_k}{(\min_{e \in \mathcal{E}^k(\mathcal{T}_v)} q(e))^2}$$

for all $k \in \{2, \dots, h(v)\}$. Next, we put $|e| := \ell_k$ for all $e \in \mathcal{E}^k(\mathcal{T}_v)$, $k \in \{1, \dots, h(v)\}$. Clearly, with this choice of lengths we have

$$\begin{aligned} T(e) &= \frac{\ell_k}{(\sum_{e \in \mathcal{E}_{e_\iota}} |e|)^{1/2} (\sum_{e \in \mathcal{E}_{e_\tau}} |e|)^{1/2}} \\ &\le \frac{\ell_k}{(\sum_{e \in \mathcal{E}_{e_\iota}^-} |e|)^{1/2} (\sum_{e \in \mathcal{E}_{e_\tau}^-} |e|)^{1/2}} \le \sqrt{\frac{\ell_k}{\ell_{k-1}}} \le q(e) \end{aligned}$$

for all $e \in \mathcal{E}^k(\mathcal{T}_v)$ and $k \in \{2, \dots, h(v)\}$.

(iv) It remains to define edge lengths for edges in $\mathcal{T}_\infty$ such that (6.26) then holds true on $\mathcal{E}_\infty$ and also on each non-empty edge set $\mathcal{E}^1(\mathcal{T}_v)$, $v \in \mathcal{V}$. Again, we will assign edge lengths inductively for the sets $\mathcal{E}^n(\mathcal{T}_\infty)$, but now moving “upwards” the tree $\mathcal{T}_\infty$. Here $\mathcal{E}^n(\mathcal{T}_\infty)$, $n \ge 1$, is the set of edges $e \in \mathcal{E}_\infty$ between the combinatorial spheres $S_{n-1}(\mathcal{T}_\infty)$ and $S_n(\mathcal{T}_\infty)$ in $\mathcal{T}_\infty$.

For $n = 1$, we set $|e| = 1$ for all $e \in \mathcal{E}^1(\mathcal{T}_\infty)$ if $\mathcal{E}^1(\mathcal{T}_\infty) = \mathcal{E}_o^\infty = \mathcal{E}_o$ (that is, if $\mathcal{T}_o$ is empty). Otherwise, we define

$$\widetilde{\ell}_1 := \max_{e \in \mathcal{E}^1(\mathcal{T}_o)} \frac{|e|}{q(e)^2} = \frac{\ell_1(o)}{(\min_{e \in \mathcal{E}^1(\mathcal{T}_o)} q(e))^2},$$

and then set $|e| = \widetilde{\ell}_1$ for all $e \in \mathcal{E}^1(\mathcal{T}_\infty)$. Hence for each $e \in \mathcal{E}^1(\mathcal{T}_o)$ we get

$$\begin{aligned} T(e) &= \frac{\ell_1(o)}{(\sum_{e \in \mathcal{E}_o} |e|)^{1/2} (\sum_{e \in \mathcal{E}_{e_\tau}} |e|)^{1/2}} \\ &\le \frac{\ell_1(o)}{(\sum_{e \in \mathcal{E}^1(\mathcal{T}_o)} |e|)^{1/2} (\sum_{e \in \mathcal{E}^1(\mathcal{T}_\infty)} |e|)^{1/2}} \le \sqrt{\frac{\ell_1(o)}{\widetilde{\ell}_1}} \le q(e). \end{aligned}$$

Now assume we have already defined edge lengths for edges in $\mathcal{E}^k(\mathcal{T}_\infty)$ for all $k \le n$, such that (6.26) holds true on each

$$\widetilde{\mathcal{E}}^k := \mathcal{E}^{k-1}(\mathcal{T}_\infty) \cup \bigcup_{v \in S_{k-1}} \mathcal{E}^1(\mathcal{T}_v)$$

for $k \le n$. Now we define again

$$\widetilde{\ell}_{n+1} := \max_{e \in \widetilde{\mathcal{E}}^{n+1}} \frac{|e|}{q(e)^2},$$

and then we set $|e| = \widetilde{\ell}_{n+1}$ for all $e \in \mathcal{E}^{n+1}(\mathcal{T}_\infty)$. By our choice of the root, every vertex $v \in S_n(\mathcal{T}_\infty)$ is adjacent to at least one $e \in \mathcal{E}^{n+1}(\mathcal{T}_\infty)$. Hence $T(\widetilde{e}) \le q(\widetilde{e})$ for all $\widetilde{e}$ in $\widetilde{\mathcal{E}}^{n+1}$. Since $\bigcup_{n\ge 1} \mathcal{E}^n(\mathcal{T}_\infty) = \mathcal{E}_\infty$, by induction we obtain edge lengths on $\mathcal{E}$ such that (6.26) holds true for all $e \in \mathcal{E}$. ■

Now we are ready to prove Theorem 6.61 and Corollary 6.62.

*Proof of Theorem 6.61.* As in the proof of Lemma 6.64, it suffices to show the existence of lengths $|\cdot|$ satisfying (6.26) since in this case a suitable choice of edge weights is provided by (6.27). The main idea behind our construction is the observation that we assign weights and lengths to edges, and hence we can "transform" in a suitable way our graph to a tree and then apply Lemma 6.64 .

Suppose that $\mathcal{T}$ is a spanning tree for the underlying combinatorial graph $\mathcal{G}_q$. Denote the edge set of $\mathcal{T}$ by $\mathcal{E}(\mathcal{T}) \subseteq \mathcal{E}_q$. Now we decouple each remaining edge $e_{u,v} \in \mathcal{E}_q \setminus \mathcal{E}(\mathcal{T})$ at exactly one vertex (say, $v$) and thereby transform it to a leaf attached to the remaining vertex $u$.

Applying this to all edges $e \in \mathcal{E}_q \setminus \mathcal{E}(\mathcal{T})$ yields a new graph $\widetilde{\mathcal{G}}_q$. Clearly, $\widetilde{\mathcal{G}}_q$ is a tree and its edge set $\widetilde{\mathcal{E}}_q$ can be identified in the above way with $\mathcal{E}_q$. Hence every choice of edge lengths $|\cdot|$ on $\mathcal{G}_q$ corresponds to a respective choice on $\widetilde{\mathcal{G}}_q$. Moreover, by construction we have

$$T_{\mathcal{G}_q}(e) \le T_{\widetilde{\mathcal{G}}_q}(e)$$

for all $e \in \mathcal{E}_q$, where $T_{\widetilde{\mathcal{G}}_q}(e)$ and $T_{\mathcal{G}_q}(e)$ are given by (6.26). More precisely, within the identification we have $\widetilde{\mathcal{E}}_v \subseteq \mathcal{E}_v$ for every $v \in \mathcal{V}$ and $\widetilde{\mathcal{E}}_{v_e} = \{e\}$ for each of the new vertices $v_e$, $e \in \mathcal{E}_q \setminus \mathcal{E}(\mathcal{T})$. Hence

$$T_{\mathcal{G}_d}(e_{u,v}) = \frac{|e_{u,v}|}{(\sum_{e\in\mathcal{E}_u} |e|)^{1/2}(\sum_{e\in\mathcal{E}_v} |e|)^{1/2}} \le \frac{\sqrt{|e_{u,v}|}}{(\sum_{e\in\mathcal{E}_u} |e|)^{1/2}} = T_{\widetilde{\mathcal{G}}_d}(e_{u,v})$$

for every $e_{u,v} \in \mathcal{E}_q \setminus \mathcal{E}(\mathcal{T})$ and similar for each $e \in \mathcal{E}(\mathcal{T})$. Thus every choice of edge lengths satisfying (6.26) for $\widetilde{\mathcal{G}}_q$ defines a suitable choice of edge lengths for $\mathcal{G}_q$. It remains to apply Lemma 6.64. ■

*Proof of Corollary 6.62.* We simply need to set $\mu(e) = 1$ for each $e \in \mathcal{E}_q$ and then choose $\nu$ and $|\cdot|$ as in Theorem 6.61. By construction, this implies $\eta(e) \le 1$ for all edges $e \in \mathcal{E}_q$. Taking into account (6.23), it follows that $q$ coincides with (6.24). Moreover, choosing the function $\alpha\colon \mathcal{V} \to \mathbb{R}$ in a suitable way, we can achieve that $\beta$ coincides with (6.24) as well. ■

**Remark 6.65.** A few remarks are in order.

(i) Theorem 6.61 can be seen as an extension of Proposition 5.18 to an arbitrary locally finite graph.

(ii) According to the proof of Theorem 3.1, the graph Laplacian $\mathbf{h}^0_\alpha$ associated in $\ell^2(\mathcal{V}; m)$ with (3.7) is unitarily equivalent (by means of the map

$\mathcal{U}\colon \ell^2(\mathcal{V};m) \to \ell^2(\mathcal{V})$ defined by (3.29)) to the minimal symmetric operator $\widetilde{\mathbf{h}}_\alpha^0$ defined in $\ell^2(\mathcal{V})$ by (6.21) with the coefficients (6.24) and therefore, by Theorem 3.1, $\widetilde{\mathbf{h}}_\alpha^0$ shares its basis spectral properties with the Laplacian $\mathbf{H}_\alpha^0$. However, the map $\mathcal{U}$ does not preserve the Dirichlet form structure (e.g., the quadratic form of $\widetilde{\mathbf{h}}_\alpha^0$ may fail to be a Dirichlet form even if $\alpha \equiv 0$) and hence there is in general no connection between their parabolic properties.

## 6.7 Further comments and open problems

We would like to conclude this part with a few comments.

**1.** The results of this chapter suggest viewing connections between weighted graphs and metric graphs from geometric perspective. Namely, it is proved that with every weighted locally finite graph $(\mathcal{V}, m; b)$ one can always associate at least one cable system, that is, a weighted metric graph $(\mathcal{G}, \mu, \nu)$ such that for one of its models the weight $m$ and the graph $b$ are expressed via (3.1)–(3.5) and (3.6). Next, $(\mathcal{G}, \mu, \nu)$ is always equipped with the intrinsic path metric $\varrho_\eta$ and it turns out that the induced metric $\varrho_{\mathcal{V}} = \varrho_\eta|_{\mathcal{V}\times\mathcal{V}}$ is intrinsic with respect to the corresponding graph $(\mathcal{V}, m; b)$. Moreover, the spaces $(\mathcal{V}, \varrho_{\mathcal{V}})$ and $(\mathcal{G}, \varrho_\eta)$ are quasi-isometric and this fact connects their large scale geometric properties. However, their local combinatorial structures are also connected in an obvious way and these facts together provide a partial explanation for the close connections between graph Laplacians and metric graph Laplacians established in Chapters 3 and 4. Notice also that $(\mathcal{G}, \varrho_\eta)$ is a length space, a widely studied class of metric spaces, and this provides a lot of tools and techniques. This is reminiscent of the following common practice in geometric group theory: a finitely generated group can be turned into a length space by viewing its Cayley graph as an equilateral metric graph equipped with the length metric $\varrho_0$; moreover, the word metric $\varrho_{\mathrm{comb}}$ in this case is nothing but the induced metric $\varrho_0|_{\mathcal{V}\times\mathcal{V}}$.

**2.** It is hard to overestimate the role of intrinsic metrics in the progress achieved for weighted graph Laplacians during the last decade. Surprisingly, the above described procedure to construct an intrinsic metric for $(\mathcal{V}, m; b)$ in fact provides a way to obtain all finite jump size intrinsic path metrics on $(\mathcal{V}, m; b)$. Moreover, upon some normalization assumptions on cable systems (e.g., canonical cable systems) the correspondence between intrinsic weights on $(\mathcal{V}, m; b)$ and cable systems becomes bijective (Theorem 6.36).

**3.** Let us also briefly mention the following perspective on the results of Chapter 6 and on Problems 6.1–6.4. Suppose a vertex set $\mathcal{V}$ is given and consider a weighted metric graph $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu_{\mathcal{E}}, \nu_{\mathcal{E}})$ over $\mathcal{V}$, i.e., a model of a weighted metric graph having $\mathcal{V}$ as its vertex set. To this weighted metric graph, equations (3.5) and (3.6) associate a vertex weight $m: \mathcal{V} \to (0, \infty)$ and an edge weight $b: \mathcal{V} \times \mathcal{V} \to [0, \infty)$ with the properties (i)–(iv) of Section 2.2. In other words, we obtain a map

$$\Phi_{\mathcal{V}}: \mathrm{Graph}_{\mathrm{metr}}(\mathcal{V}) \to \mathrm{Graph}_{\mathrm{discr}}(\mathcal{V}),$$

where $\mathrm{Graph}_{\mathrm{metr}}$ and $\mathrm{Graph}_{\mathrm{discr}}$ denote the sets of all connected, locally finite weighted metric graphs and connected, locally finite weighted graphs over $\mathcal{V}$, respectively.

From this point of view, the results in Chapters 3 and 4 say that the map $\Phi_{\mathcal{V}}$ connects the basic spectral and parabolic properties of the respective Laplacian-type operators, as well as spectral properties of Laplacians with $\delta$-couplings on weighted metric graphs and Schrödinger operators on weighted discrete graphs. Moreover, the results of Section 6.4 connect certain basic geometric features (see also Proposition 7.38). In terms of this map, the results of Sections 6.2–6.3 and Section 6.4.4 can be formulated as follows:

- The map $\Phi_{\mathcal{V}}$ is surjective (see Theorem 6.16).
- When restricted to simple metric graphs, the map $\Phi_{\mathcal{V}}$ is no longer surjective (Section 6.2).
- Unfortunately, the map $\Phi_{\mathcal{V}}$ is not injective, that is, the correspondence between weighted metric and weighted discrete graphs is not one-to-one. However, after restricting $\Phi_{\mathcal{V}}$ further to the class of canonical weighted metric graphs over $\mathcal{V}$, we can at least describe the preimage $\Phi_{\mathcal{V}}^{-1}(m, b)$ of a locally finite graph $(\mathcal{V}, m; b)$ using intrinsic weights (see Theorem 6.36 and the map $\Psi$ given by (6.15)).

**4.** The results of Section 6.6 show that similar connections work for Jacobi matrices on graphs. We decided not to proceed in this direction and demonstrate it by only one application in the next chapter. More specifically, in Section 7.1.3 we briefly discuss the self-adjointness problem for the minimal operator associated with (6.21) in $\ell^2(\mathcal{V})$ and prove the analogs of some classical self-adjointness tests for the usual Jacobi matrices, which also improve several recent results (Theorem 7.17).

**5.** Taking into account the said above, the following problems remain open.

**Problem 6.5.** Given a locally finite $b$ graph over $(\mathcal{V}, m)$, is there an efficient way to decide whether it admits a minimal cable system?

This problem can be reformulated in other terms (e.g., given a simple graph, how can one describe the image of the positive cone $C^+(\mathcal{E})$ under the map $D^*$?).

Of course, stated this way, Problem 6.5 is too complicated to obtain a complete answer and hence it makes sense either to restrict to some classes of weights (for constant weights the answer is given by means of a disjoint cycle cover) or to particular classes of graphs (seems, for antitrees the answer depends on sphere numbers in a rather non-trivial way).

Taking into account the fact that each graph admits an infinite family of cable systems, one can specify the above problem:

**Problem 6.6.** Given a locally finite $b$ graph over $(\mathcal{V}, m)$, is there an efficient procedure/algorithm to construct a cable system with certain desirable properties?

The same kind of questions can be asked about Jacobi matrices on graphs:

**Problem 6.7.** Given a Jacobi matrix (6.21) on a graph, is there an efficient procedure/algorithm to construct a weighted metric graph such that Jacobi parameters admit the representation (6.24)?

The direction "from $(\mathcal{V}, m; b)$ to a cable system" seems to be rather non-trivial despite the fact that we have provided some constructions. Namely, Problems 6.6 and 6.7 are of practical importance since it is desirable to get as accurate information as possible regarding the properties of the obtained cable system. For instance, in Theorem 7.19 it is desirable to know the qualitative behavior of the corresponding length function $|\cdot|$, however, even for the usual Jacobi matrix it is not trivial to get this information out of its Jacobi parameters using the construction in Proposition 5.18 (see (5.28)).