Chapter 7

From continuous to discrete and back

Our main goal in this chapter is to employ the established connections between graph Laplacians and metric graph Laplacians in order to prove new results for Laplacians on metric graphs as well as to provide another perspective on recent results for weighted graph Laplacians.

7.1 Self-adjointness

In this section we provide sufficient conditions for the self-adjoint uniqueness, that is, the self-adjointness of both the minimal and the maximal operator and hence the equality $\mathbf{H}_\alpha^0 = \mathbf{H}_\alpha$.

7.1.1 Kirchhoff Laplacians

We begin our study with the case $\alpha \equiv 0$. The next result is an immediate corollary of Sturm's extension of Yau's L^p-Liouville theorem for strongly local Dirichlet forms [198], see Theorem 6.56.

Theorem 7.1. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph and let ϱ_η be the corresponding intrinsic metric defined in Section 6.4.1. If $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space, then the minimal Kirchhoff Laplacian $\mathbf{H}^0$ is self-adjoint and $\mathbf{H}^0 = \mathbf{H}$.*

Proof. Assume that $\mathbf{H}^0$ is not self-adjoint. Since $\mathbf{H}^0$ is non-negative, this means that $\ker(\mathbf{H} + \mathrm{I}) \neq \{0\}$, that is, there is $0 \neq f \in \mathrm{dom}(\mathbf{H})$ such that $\Delta f = f$ (see [184, Theorem X.26]). However, by Lemma 6.53, $|f|$ is subharmonic. Moreover, we have $|f| \in L^2(\mathcal{G}; \mu)$ since $f \in \mathrm{dom}(\mathbf{H})$. On the other hand, if $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space, then Theorem 6.56 implies that $f \equiv 0$. This contradiction completes the proof of the theorem. ∎

Remark 7.2. A few remarks are in order.

(i) A different proof of Theorem 7.1 can be found in [97, Theorem 3.49]. Moreover, one more proof is provided by Theorem 7.9 below.

(ii) Simple examples show that the completeness with respect to the intrinsic path metric is only sufficient. Indeed, take a path graph and assume for simplicity that $\mu = \nu$. In this case, the intrinsic metric ϱ_η coincides with the natural path metric ϱ_0 and hence completeness is equivalent to the infinite length of the path. However, by Lemma 5.2, the self-adjointness of the

Kirchhoff Laplacian is equivalent to the divergence of the series (5.6). For another example see [68, Example 4.14].

(iii) Notice also that by the Hopf–Rinow theorem for metric graphs (see Theorem 6.38) completeness of $(\mathcal{G}, \varrho_\eta)$ is equivalent to bounded compactness (compactness of distance balls), as well as to geodesic completeness.

As an immediate corollary of Theorem 7.1 and the results in Section 6.4, we obtain the analog of the above result for graph Laplacians, which was first established in [115, Theorem 2]:

Corollary 7.3 ([115]). *Let b be a locally finite graph over $(\mathcal{V}, m)$ and let ϱ be an intrinsic metric which generates the discrete topology on $\mathcal{V}$. If $(\mathcal{V}, \varrho)$ is complete as a metric space, then $\mathbf{h}^0$ is self-adjoint and $\mathbf{h}^0 = \mathbf{h}$.*

Proof. We prove the claim in three steps.

(i) Assume first that ϱ is an intrinsic path metric of finite jump size such that $(\mathcal{V}, \varrho)$ is complete. Then, by Lemma 6.33 there is a cable system $(\mathcal{G}, \mu, \nu)$ for $(\mathcal{V}, m; b)$ such that $\rho = \rho_{\mathcal{V}}$ and $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space. Hence the corresponding minimal Kirchhoff Laplacian $\mathbf{H}^0$ is self-adjoint by Theorem 7.1 and it remains to apply Theorem 3.1 (i).

(ii) Suppose now that $\varrho = \varrho_p$ is a general intrinsic path metric with weight function $p \geq 0$ such that $(\mathcal{V}, \varrho)$ is complete. By the discrete Hopf–Rinow Theorem 6.39, the completeness is equivalent to the fact that

$$\sum_{n \geq 0} p(v_n, v_{n+1}) = \infty$$

for any infinite path $\mathcal{P} = (v_0, v_1, v_2, \dots)$ (i.e., $b(v_n, v_{n+1}) > 0$ for all $n \geq 0$, see (6.7)). However, introducing the new weight function $\widetilde{p} := \min\{1, p\}$, we arrive at another path metric $\widetilde{\varrho} := \varrho_{\widetilde{p}}$, which is strongly intrinsic with respect to $(\mathcal{V}, m; b)$ (by construction) and, moreover, has jump size at most 1. It is not hard to show (e.g., by employing the Hopf–Rinow Theorem 6.39 once again) that $(\mathcal{V}, \varrho)$ is complete exactly when so is $(\mathcal{V}, \widetilde{\varrho})$ and this finishes the proof in this case.

(iii) Finally, assume that ϱ is an intrinsic metric which generates the discrete topology on $\mathcal{V}$ and such that $(\mathcal{V}, \varrho)$ is complete. We show how to associate with ϱ an intrinsic path metric $\widetilde{\varrho}$ on $\mathcal{V}$ such that $(\mathcal{V}, \widetilde{\varrho})$ is complete as well. Consider the weight $p \colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ given by $p(x, y) := \varrho(x, y)$ whenever $x \sim y$ and $p(x, y) = 0$ if $x \not\sim y$. By construction, p is an intrinsic weight and the associated intrinsic path metric $\widetilde{\varrho} = \varrho_p$ satisfies $\rho \leq \widetilde{\varrho}$. Moreover, since both $\widetilde{\varrho}$ and ϱ generate the discrete topology on $\mathcal{V}$, the completeness of $(\mathcal{V}, \widetilde{\varrho})$ follows by comparison. This completes the proof in the general case. ∎

Remark 7.4. In the context of manifolds, Theorem 7.1 and Corollary 7.3 are known as Gaffney-type theorems.

The following results can be seen as a demonstration of the "from discrete to continuous" approach. First one can replace the completeness condition by a weaker one formulated in terms of the weighted degree function.

Lemma 7.5. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Suppose that for some model of finite intrinsic size the weighted degree function* (2.9) *with the vertex and edge weights defined by* (3.5) *and* (3.6) *is bounded on finite radius metric balls of $(\mathcal{V}, \varrho_{\mathcal{V}})$. Then the minimal Kirchhoff Laplacian $\mathbf{H}^0$ is self-adjoint. In particular, $\mathbf{H}^0$ is self-adjoint if* Deg *is bounded on $\mathcal{V}$.*

Here $\varrho_{\mathcal{V}}$ is the restriction of ϱ_η onto $\mathcal{V} \times \mathcal{V}$ defined by (6.10).

Proof. If Deg is bounded on $\mathcal{V}$, then, by Lemma 2.9, the corresponding graph Laplacian $\mathbf{h}^0$ is bounded and hence self-adjoint. Therefore, by Theorem 3.1 (i), the minimal Kirchhoff Laplacian $\mathbf{H}^0$ is also self-adjoint.

Assume now that Deg is bounded on distance balls of $(\mathcal{V}, \varrho_{\mathcal{V}})$. By Lemma 6.27 (see also Remark 6.28 (i)), $\varrho_{\mathcal{V}}$ is intrinsic and hence applying [115, Theorem 1] we conclude that $\mathbf{h}^0$ is self-adjoint. It remains to apply Theorem 3.1 (i). ∎

Remark 7.6. Notice that Lemma 7.5 improves Theorem 7.1. Indeed, the assumption of Lemma 7.5 is satisfied if $(\mathcal{G}, \varrho_\eta)$ is complete since in this case distance balls in $(\mathcal{V}, \varrho_{\mathcal{V}})$ are finite by the Hopf–Rinow Theorem 6.39.

Theorem 7.7. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Assume that for some model of $(\mathcal{G}, \mu, \nu)$, the vertex set $\mathcal{V}$ equipped with the star metric ϱ_m (defined by* (6.9) *and* (3.5)*) is a complete metric space. Then the minimal Kirchhoff Laplacian $\mathbf{H}^0$ is self-adjoint.*

Proof. By Theorem 3.1 (i) (see also Corollary 3.15), $\mathbf{H}^0$ is self-adjoint if and only if $\mathbf{h}^0$ is self-adjoint for some model of $(\mathcal{G}, \mu, \nu)$. However, by [132, Theorem 6], the minimal graph Laplacian defined by (3.35) in $\ell^2(\mathcal{V}; m)$ is self-adjoint if

$$\sum_{n \geq 0} m(v_n) = \infty$$

for any infinite path $\mathcal{P} = (v_0, v_1, v_2, \dots)$. However, our graph is locally finite and hence, by Theorem 6.39, the latter is equivalent to completeness of $(\mathcal{V}, \varrho_m)$ with respect to the star path metric (6.9). ∎

Remark 7.8. A few remarks are in order.

(i) Theorem 7.7 can be seen as an extension of Corollary 5.3 to the graph setting (see also Remark 5.4). In turn, Corollary 5.3 shows that completeness with respect to the star path metric ϱ_m is only sufficient even in the simplest case of a path graph. It would be of great interest to find (at least some) conditions which would guarantee the necessity of completeness with respect to the star path metric for the self-adjointness of both $\mathbf{H}^0$ and $\mathbf{h}^0$.

(ii) It is not hard to see that the completeness conditions in Theorem 7.1 and Theorem 7.7 are different. For example, if $\mu = \nu$, then the intrinsic metric ϱ_η coincides with the natural path metric ϱ_0 and hence the completeness in Theorem 7.1 is independent of the weight μ. On the other hand, the completeness in Theorem 7.7 is independent of the weight ν. However, in certain cases, Theorem 7.1 is a corollary of Theorem 7.7 (e.g., if $\mu = \nu \equiv 1$, see [68, Section 4.2]).

7.1.2 Laplacians with δ-couplings

We begin with the following result proved recently in [145], which says that completeness combined with semiboundedness guarantees self-adjointness:

Theorem 7.9 (The Glazman–Povzner–Wienholtz theorem on metric graphs). *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph such that $(\mathcal{G}, \varrho_\eta)$ is complete. Assume that $\alpha\colon \mathcal{V} \to \mathbb{R}$ is such that the minimal Laplacian $\mathbf{H}_\alpha^0$ is bounded from below. Then $\mathbf{H}_\alpha^0$ is self-adjoint and $\mathbf{H}_\alpha^0 = \mathbf{H}_\alpha$.*

Remark 7.10. A few remarks are in order.

(i) The proof of Theorem 7.9, which also provides another proof of Theorem 7.1, can be found in [145] (see Theorem 5.1 there). The claim in Theorem 7.9 remains valid if we add an additive potential $V\colon \mathcal{G} \to \mathbb{R}$ to the operator $\mathbf{H}_\alpha^0$, which preserves the semiboundedness. Of course, some regularity assumptions on V must be imposed (e.g., $V \in L^2_{\mathrm{loc}}(\mathcal{G})$), however, it is proved in [145, Theorem 5.1] that one may even allow distributional potentials $V \in H^{-1}_{\mathrm{loc}}(\mathcal{G})$.

(ii) It is tempting to replace in Theorem 7.9 the completeness with respect to ϱ_η by the one with respect to the star path metric ϱ_m. However, simple counterexamples show that it is not possible in general (see Remark 7.18 (ii) and also the detailed discussion in [145, Section 6]).

(iii) In the simplest case of a path graph Theorem 7.9 was first proved in [4] (see Theorem I.1 and Remark III.2 there). However, notice also that in this case Theorem 7.9 is nothing but Lemma 5.16 (ii) (take into account also Remark 3.24).

(iv) The Glazman–Povzner–Wienholtz theorem has a venerable history. To the best of our knowledge (see [29, Appendix D.1] for further information), for Schrödinger operators in $\mathbb{R}^N$ the result was conjectured by I. M. Glazman and proved by A. Ya. Povzner in 1952 [183]. However, this paper was published in Russian and was not widely known in the West until its English translation in 1967. For instance, P. Hartman (1948) and F. Rellich (1951) proved a one-dimensional version of this result, and F. Rellich in his invited address at the ICM in Amsterdam (1954) posed a multi-dimensional result as an open problem, which was solved later by his student E. Wienholtz (see [210]).

As an immediate application of Theorem 7.9 and the results connecting metric graphs with weighted graphs, we arrive at the following version of the Glazman–Povzner–Wienholtz theorem for weighted graphs (see [145, Theorem 6.1]).

Theorem 7.11 (The Glazman–Povzner–Wienholtz theorem on graphs). *Let b be a locally finite graph over $(\mathcal{V}, m)$ and assume that there exists an intrinsic metric ϱ which generates the discrete topology on $\mathcal{V}$ and such that $(\mathcal{V}, \varrho)$ is complete. Assume also that $\alpha\colon \mathcal{V} \to \mathbb{R}$ is such that the minimal Schrödinger operator $\mathbf{h}_\alpha^0$ is bounded from below in $\ell^2(\mathcal{V}; m)$. Then $\mathbf{h}_\alpha^0$ is self-adjoint and $\mathbf{h}_\alpha^0 = \mathbf{h}_\alpha$.*

Proof. Arguing as in the proof of Corollary 7.3, it suffices to consider the case when ϱ is an intrinsic path metric of finite jump size. Then applying Lemma 6.33, we obtain a cable system $(\mathcal{G}, \mu, \nu)$ for $(\mathcal{V}, m; b)$ such that $\rho = \rho_{\mathcal{V}}$ and $(\mathcal{G}, \varrho_\eta)$ is complete. Moreover, by Theorem 3.22 (i) and Remark 3.24, the corresponding operator $\mathbf{H}_\alpha^0$ is bounded from below. Applying Theorem 7.9, we conclude that $\mathbf{H}_\alpha^0$ is self-adjoint. It remains to apply Theorem 3.1 (i). ■

Remark 7.12. To the best of our knowledge the Glazman–Povzner–Wienholtz theorem for graphs was established first in [167, Theorem 1.3] and [202, Theorem 6.1] (however, under the additional bounded geometry assumption on $(\mathcal{V}, b)$) and then independently in [7, Theorem 1] and [96, Theorem 2.16] (the latter allows non-locally finite graphs, see also [190]).

Usually, it is not an easy task to find necessary and sufficient conditions which guarantee semiboundedness. We begin with the simplest situation.

Lemma 7.13. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Assume that the weighted degree function (2.9) with the vertex and edge weights defined by (3.5) and (3.6) is bounded on $\mathcal{V}$. Then the Laplacian $\mathbf{H}_\alpha$ with δ-couplings on $\mathcal{V}$ is self-adjoint for any $\alpha\colon \mathcal{V} \to \mathbb{R}$. Moreover, $\mathbf{H}_\alpha$ is bounded from below exactly when*

$$\inf_{v\in\mathcal{V}} \frac{\alpha(v)}{m(v)} > -\infty. \tag{7.1}$$

Proof. It suffices to notice that $\mathbf{h}_\alpha = \mathbf{h} + \frac{\alpha}{m}$. Indeed, $\frac{\alpha}{m}$ is a multiplication operator in $\ell^2(\mathcal{V}; m)$ and hence it is self-adjoint since α is real-valued. Moreover, it is bounded from below exactly when (7.1) holds true. Since $\mathbf{h}$ is a bounded operator by Lemma 2.9, and both self-adjointness and semiboundedness are stable under bounded perturbations, we complete the proof by applying Theorem 3.1. ■

As an immediate corollary we arrive at the following result.

Corollary 7.14. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. If*

$$\eta_*(\mathcal{G}) := \inf_{e \in \mathcal{E}} \eta(e) > 0, \tag{7.2}$$

then $\mathbf{H}_\alpha$ is self-adjoint for any $\alpha \colon \mathcal{V} \to \mathbb{R}$. Moreover, it is bounded from below exactly when (7.1) is satisfied.

Proof. Without loss of generality we can assume that the model is simple and has finite intrinsic size (we can "cut" each loop and multiple edge in the middle, and also each long edge by adding inessential vertices; clearly, this would not change $\mathbf{H}_\alpha$ and also the corresponding conditions (7.1) and (7.2) would hold true as well). Since (7.2) means that

$$|e|\mu(e) \geq \eta_*(\mathcal{G})^2 \frac{\nu(e)}{|e|}$$

for all $e \in \mathcal{E}$ by (7.2), it follows that

$$\begin{aligned} \operatorname{Deg}(v) &= \frac{\sum_{e \in \mathcal{E}_v} \frac{\nu(e)}{|e|}}{\sum_{e \in \mathcal{E}_v} |e|\mu(e)} \leq \frac{\sum_{e \in \mathcal{E}_v} \frac{\nu(e)}{|e|}}{\sum_{e \in \mathcal{E}_v} \eta_*(\mathcal{G})^2 \frac{\nu(e)}{|e|}} \\ &= \frac{1}{\eta_*(\mathcal{G})^2} < \infty, \end{aligned}$$

and hence Lemma 7.13 applies. ■

Remark 7.15. The most common restriction imposed in the quantum graphs literature is that $\mu = \nu \equiv 1$ and $\inf_{\mathcal{E}} |e| > 0$ on $\mathcal{G}$ (see, e.g., [25]). For non-trivial weights, a similar assumption is sometimes imposed: $\mu = \nu$ on $\mathcal{G}$ and $\inf_{\mathcal{E}} |e| > 0$, $\inf_{\mathcal{E}} \mu(e) > 0$. Clearly, in both cases (7.2) holds true and Corollary 7.14 applies.

If the weighted degree Deg is unbounded on $\mathcal{V}$, then one needs to proceed more carefully.

Lemma 7.16. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. Assume that at least one of the following conditions is satisfied:*

- *$(\mathcal{G}, \varrho_\eta)$ is complete as a metric space,*
- *$(\mathcal{V}, \varrho_m)$ is complete as a metric space, where ϱ_m is the star path metric.*

If $\alpha \colon \mathcal{V} \to \mathbb{R}$ satisfies (7.1), then $\mathbf{H}^0_\alpha$ is self-adjoint and bounded from below.

Proof. If $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space, then according to Theorem 7.9 it suffices to show that $\mathbf{H}^0_\alpha$ is bounded from below. However, this easily follows from Theorem 3.22 (i) (take into account also Remark 3.24), since (7.1) implies that $\mathbf{h}^0_\alpha$ is lower semibounded.

If $(\mathcal{V}, \varrho_m)$ is complete as a metric space, combining (7.1) with [132, Proposition 3.1] implies that $\mathbf{h}^0_\alpha$ is self-adjoint and lower semibounded. By Theorem 3.1, $\mathbf{H}^0_\alpha$ is self-adjoint and lower semibounded as well. ■

7.1.3 Jacobi matrices on graphs

Of course, the results from the previous two subsections immediately apply to Jacobi matrices on graphs – Schrödinger-type operators in $\ell^2(\mathcal{V})$ (that is, the vertex weight m is constant). Let us quickly recall the setup (see Section 6.6). For a given $\beta\colon \mathcal{V} \to \mathbb{R}$ and a connected graph q over $\mathcal{V}$ satisfying properties (i), (ii) and (iv) of Section 2.2, consider a second order symmetric difference expression

$$(\tau f)(v) = \beta(v) f(v) - \sum_{u \in \mathcal{V}} q(u,v) f(u), \quad v \in \mathcal{V}. \tag{7.3}$$

As described in Section 2.2, we can associate in $\ell^2(\mathcal{V})$ the minimal $J^0 = J^0_{q,\beta}$ and maximal operator $J = J_{q,\beta}$ with the difference expression (7.3).

Theorem 7.17. *If at least one of the following conditions is satisfied:*

(i) *There is $C \geq 0$ such that*

$$\beta(v) - \sum_{u \in \mathcal{V}} q(u,v) \geq -C \tag{7.4}$$

for all $v \in \mathcal{V}$,

(ii) *The minimal operator J^0 is bounded from below and $(\mathcal{V}, \varrho_p)$ is complete as a metric space, where ϱ_p is the path metric with the edge weights*

$$p(u,v) = \frac{1}{\sqrt{q(u,v)\ \max(\deg(u), \deg(v))}} \tag{7.5}$$

whenever $q(u,v) > 0$ and 0 otherwise,

then the operator J is self-adjoint and $J^0 = J$.

Proof. (i) If $m \equiv \mathbb{1}_{\mathcal{V}}$, then the corresponding star path metric ϱ_m is nothing but the combinatorial distance on $\mathcal{V}$. Taking into account that $(\mathcal{V}, \varrho_{\mathrm{comb}})$ is complete (this can be either verified directly or by using the Hopf–Rinow Theorem 6.39), it remains to apply Lemma 7.16 since $\alpha(v)$ in this case coincides with the left-hand side of (7.4).

(ii) This is a straightforward application of the Glazman–Povzner–Wienholtz theorem on graphs. Indeed, choosing $m \equiv \mathbb{1}_{\mathcal{V}}$, $b = q$ and $\alpha(v) = $ LHS of (7.4), we get that $J^0_{q,\beta} = \mathbf{h}^0_\alpha$ in $\ell^2(\mathcal{V}) = \ell^2(\mathcal{V}; m)$. It remains to notice that the weight (7.5) is intrinsic:

$$\begin{aligned}\sum_{u \sim v} q(u,v) p(u,v)^2 &= \sum_{u \sim v} \frac{1}{\max(\deg(u), \deg(v))} \\ &\leq \sum_{u \sim v} \frac{1}{\deg(v)} = 1\end{aligned}$$

for all $v \in \mathcal{V}$. It remains to apply Theorem 7.11. ■

Remark 7.18. A few remarks are in order.

(i) Theorem 7.17 can be seen as an extension of Wouk's tests for Jacobi matrices to the graph setting (compare (i) and (ii) with [215, Theorem 3 (c) and Theorem 3 (d)], see also [2, Problems I.3 and I.4]). On the other hand, Wouk's test [215, Theorem 3(d)] can be seen as the analog of a one-dimensional predecessor of the Glazman–Povzner–Wienholtz theorem proved by P. Hartman (1948) and F. Rellich (1951) (see [145, Remark 6.5] for further details).

(ii) It is well known that even for Jacobi matrices (5.23) one cannot replace (7.4) by the semiboundedness of the minimal operator J^0. This, in particular, implies that one cannot replace the intrinsic path metric by the star path metric ϱ_m in the completeness assumption of Glazman–Povzner-Wienholtz theorems.

(iii) Under the additional bounded degree assumption, $\sup_{\mathcal{V}} \deg(v) < \infty$, the above result was established in [202, Theorem 6.1] and [167, Theorem 1.3].

Let us give one more sufficient condition for self-adjointness. Recall that, according to Theorem 6.61, for any locally finite graph q over $\mathcal{V}$ one can find edge lengths $|\cdot|$ and weights ν satisfying (6.22) and (6.23). For a given $|\cdot|\colon \mathcal{E}_q \to (0,\infty)$, define the vertex weight $m\colon \mathcal{V} \to (0,\infty)$ by setting

$$m_q(v) = \sum_{u\sim v} |e_{u,v}|, \qquad v \in \mathcal{V}.$$

Taking into account (6.23), let us also introduce the graph $b = b_q$ over $\mathcal{V}$ by setting

$$b_q(u,v) = \begin{cases} \dfrac{\nu(e_{u,v})}{|e_{u,v}|}, & q(u,v) > 0, \\ 0, & q(u,v) = 0. \end{cases}$$

Theorem 7.19. *Let q be a locally finite graph over $\mathcal{V}$ and let $\beta\colon \mathcal{V} \to \mathbb{R}$. Suppose that $|\cdot|\colon \mathcal{E}_q \to (0,\infty)$ and $\nu\colon \mathcal{E}_q \to (0,\infty)$ are edge lengths and weights satisfying (6.22) and such that q admits the representation (6.23). If at least one of the following conditions is satisfied, then the operator J is self-adjoint and $J^0 = J$:*

(i) *The space $(\mathcal{V}, \varrho_m)$ is complete, where ϱ_m is the star path metric (see Example 6.21 (iii)) with $m = m_q$, and there is $M \ge 0$ such that*

$$\beta(v) - \sum_{u\in\mathcal{V}} q(u,v)\sqrt{\frac{m(u)}{m(v)}} \ge -M$$

for all $v \in \mathcal{V}$,

(ii) *The minimal operator J^0 is bounded from below and $(\mathcal{V}, \varrho_b)$ is complete, where ϱ_b is the natural path metric (see Example 6.21 (iii)) with $b = b_q$.*

Proof. Notice that the minimal operator J^0 is unitarily equivalent to the operator $\mathbf{h}^0_\alpha$ acting in $\ell^2(\mathcal{V}; m)$ and associated with the graph $(\mathcal{V}, m; b)$ whose coefficients are defined via (6.24), that is,

$$b(u,v) = b_q(u,v) = q(u,v)\sqrt{m(u)m(v)},$$
$$\alpha(v) = \beta(v)m_q(v) - \sum_{u\in\mathcal{V}} b_q(u,v).$$

If condition (i) is satisfied, then we simply need to apply Lemma 7.16 to $\mathbf{h}^0_\alpha$.

Assume now that (ii) holds true. Observe that the natural path metric ϱ_b is intrinsic with respect to $(\mathcal{V}, m; b)$:

$$\sum_{u\sim v} b(u,v)p_b(u,v)^2 = \sum_{u\sim v} \frac{\nu(e_{u,v})}{|e_{u,v}|}\frac{|e_{u,v}|^2}{\nu(e_{u,v})} = \sum_{u\sim v} |e_{u,v}| = m(v), \quad v \in \mathcal{V}.$$

It remains to apply Theorem 7.11. ∎

7.1.4 Semiboundedness and criticality theory on graphs

Condition (7.1) means that the semiboundedness is preserved if the strength $\alpha\colon \mathcal{V} \to \mathbb{R}$ is not too negative. In fact, (7.1) can be improved by using the concept of relatively bounded perturbations (see, e.g., [126, 184]). Assume for a moment that the strength $\alpha\colon \mathcal{V} \to (-\infty, 0]$ is non-positive. Then α is called *form bounded* with respect to $\mathbf{h}^0$ if there are $\varepsilon \geq 0$ and $\gamma \geq 0$ such that

$$\sum_{v\in\mathcal{V}} |\alpha(v)||f(v)|^2 \leq \frac{\varepsilon}{2}\sum_{u,v\in\mathcal{V}} b(u,v)|f(u)-f(v)|^2 + \gamma\sum_{v\in\mathcal{V}} m(v)|f(v)|^2 \tag{7.6}$$

for all $f \in C_c(\mathcal{V})$. If (7.6) holds with some $\varepsilon < 1$, then α is called *strongly form bounded*. Notice that (7.6) is nothing but

$$\left\langle \frac{|\alpha|}{m} f, f\right\rangle_{\ell^2(\mathcal{V};m)} \leq \varepsilon\,\mathfrak{q}[f] + \gamma\|f\|^2_{\ell^2(\mathcal{V};m)}.$$

Clearly, if α satisfies (7.1), then we can take $\varepsilon = 0$ in (7.6), which further means that the multiplication operator α is bounded in $\ell^2(\mathcal{V}; m)$. The importance of this concept stems from the KLMN theorem (see, e.g., [184]): *if* $\alpha\colon \mathcal{V} \to (-\infty, 0]$ *is strongly form bounded, then the form* $\mathfrak{q}_\alpha = \mathfrak{q} + \alpha$ *defined as a form sum with* $\mathrm{dom}(\mathfrak{q}_\alpha) = \mathrm{dom}(\mathfrak{q})$ *is closed and bounded from below*. Combining this result further with the Glazman–Povzner–Wienholtz theorem for graphs, we would be able to get the self-adjoint uniqueness for Laplacians with δ-couplings once the negative part of α satisfies (7.6) and $(\mathcal{G}, \varrho_\eta)$ is complete.

To proceed further, let us recall the following notion from [140]. For convenience reasons, for each real-valued function $\omega\colon \mathcal{V} \to \mathbb{R}$, we shall denote the corresponding quadratic form by the same letter, that is,

$$\omega[f] := \sum_{v \in \mathcal{V}} \omega(v)|f(v)|^2, \quad f \in C_c(\mathcal{V}).$$

Definition 7.20 ([140]). Let b be a connected, locally finite graph over $(\mathcal{V}, m)$ and let $\mathfrak{q} = \mathfrak{q}_{b,0}$ be the corresponding energy form in $\ell^2(\mathcal{V}, m)$. For $\lambda \geq 0$, the weight $\omega: \mathcal{V} \to [0, \infty)$ is called λ-*critical* with respect to $(\mathcal{V}, m; b)$ (for $\lambda = 0$, it is simply called *critical*) if

- the form $\mathfrak{q} + \lambda m - \omega$ is non-negative on $C_c(\mathcal{V})$, that is, $(\mathfrak{q} + \lambda m)[f] \geq \omega[f]$ for all $f \in C_c(\mathcal{V})$,
- for each weight $\widetilde{\omega}: \mathcal{V} \to [0, \infty)$ satisfying $\widetilde{\omega} \geq \omega$, the form $\mathfrak{q} + \lambda m - \widetilde{\omega}$ is not non-negative on $C_c(\mathcal{V})$.

If the last property does not hold true (i.e., there is $\widetilde{\omega}$ such that $0 \not\equiv \widetilde{\omega} - \omega \geq 0$ and the form $\mathfrak{q} + \lambda m - \widetilde{\omega}$ is non-negative), then the weight ω is called λ-subcritical.

Combining the notion of criticality with the Glazman–Povzner–Wienholtz theorem for graphs, we arrive at the following extension of Lemma 7.16.

Lemma 7.21. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model and let $(\mathcal{V}, m; b)$ be the corresponding weighted graph (3.5)–(3.6). If $(\mathcal{G}, \varrho_\eta)$ is complete and $\alpha: \mathcal{V} \to \mathbb{R}$ is such that $\alpha_- := \frac{1}{2}(|\alpha| - \alpha)$ is λ-subcritical for some $\lambda \geq 0$, then the operator $\mathbf{H}_\alpha$ is self-adjoint and bounded from below.*

Conversely, if $\alpha: \mathcal{V} \to (-\infty, 0]$ is such that $\mathbf{H}^0_\alpha$ is bounded from below, then there is $\lambda \geq 0$ such that the weight $-\alpha$ is λ-subcritical for $\mathbf{h}^0$.

Proof. If $\alpha: \mathcal{V} \to \mathbb{R}$ satisfies the assumptions of Lemma 7.21, then taking into account that the form $\mathfrak{q} + \lambda m - \alpha_-$ is non-negative on $C_c(\mathcal{V})$, we conclude that

$$\mathfrak{q}_\alpha[f] \geq \mathfrak{q}_{-\alpha_-}[f] := \mathfrak{q}[f] - \alpha_-[f] \geq -\lambda \|f\|^2_{\ell^2(\mathcal{V};m)}$$

for all $f \in C_c(\mathcal{V})$. Therefore, the form $\mathfrak{q}_\alpha$ is bounded from below on $C_c(\mathcal{V})$ and hence so is the operator $\mathbf{h}^0_\alpha$. By Theorem 3.1 (ii) (see also Theorem 3.22 and Remark 3.24), the operator $\mathbf{H}^0_\alpha$ is bounded from below. It remains to apply Theorem 7.9.

To prove the last claim it suffices to notice that the semiboundedness of $\mathbf{h}^0_\alpha$, which is equivalent to the semiboundedness of $\mathbf{H}^0_\alpha$, means that there exists $\lambda > 0$ such that $\mathbf{h}^0_\alpha + \lambda \geq 0$, where the inequality is understood in the sense of forms. It is straightforward to see that $-\alpha$ is $(\lambda + 1)$-subcritical for $\mathbf{h}_0$ ∎

Remark 7.22. A few remarks are in order.

(i) The notion of criticality is closely connected with the notion of recurrence (see, e.g., [140, Remark 5.8]). In particular, for $\lambda = 0$, $\mathbf{h}_0$ is critical exactly when it is recurrent.

(ii) A characterization of criticality is presented in [140, Theorem 5.3]. However, for a concrete graph b over $\mathcal{V}$ it is a highly non-trivial task to find critical and (especially) λ-critical weights. One of the approaches is to employ positive λ-harmonic/superharmonic functions, which also leads to *optimal Hardy weights*, however, this requires an explicit form or at least a rather qualified knowledge of their asymptotic behavior (see [139]).

(iii) Let us stress that the Glazman–Povzner–Wienholtz theorem enables us to avoid the use of the KLMN theorem, however, the price to pay is the completeness assumption on $(\mathcal{G}, \varrho_\eta)$.

7.2 Markovian uniqueness and finite energy extensions

In this section we briefly address the question of uniqueness of Markovian extension for the minimal Kirchhoff Laplacian $\mathbf{H}_0$. Notice that by Lemma 4.1 the latter is equivalent to the self-adjointness of the Gaffney Laplacian $\mathbf{H}_G$. We also stress that the self-adjoint uniqueness implies Markovian uniqueness, and hence the results obtained in the previous section provide various sufficient conditions for the Markovian uniqueness as well. In particular, completeness of $\mathcal{G}$ (with respect to particular choices of path metrics) is sufficient for the Markovian uniqueness.

7.2.1 Markovian uniqueness and graph ends

Surprisingly enough, in some cases of interest it is possible to provide a complete characterization of the Markovian uniqueness in purely geometric terms. Intuitively, this problem (as well as the self-adjoint uniqueness) is closely related to finding appropriate boundary notions for infinite graphs. For unweighted metric graphs, that is, with $\mu = \nu \equiv \mathbb{1}$, the question was studied in [146, 148] using graph ends, a graph boundary notion going back to H. Freudenthal and R. Halin (see Section 2.1.3). For this purpose recall the following notion introduced in [146].

Definition 7.23. A topological end $\gamma \in \mathfrak{C}(\mathcal{G})$ of a metric graph $\mathcal{G}$ equipped with the edge weight μ has *finite volume* (with respect to μ) if there is a sequence $\mathcal{U} = (U_n)$ representing γ such that

$$\mu(U_n) = \int_{U_n} \mu(\mathrm{d}x) < \infty$$

for some n. Otherwise γ has *infinite volume*. We denote the set of all finite volume ends by $\mathfrak{C}_0(\mathcal{G}; \mu)$ and equip it with the induced topology from the end space $\mathfrak{C}(\mathcal{G})$.

The above notion leads to a complete characterization of the Markovian uniqueness in the unweighted setting $\mu = \nu \equiv \mathbb{1}$ (see [146, Corollary 3.12]): *All ends of the* *metric graph have infinite volume*. In the present section, we briefly recall the results of [146, 148] and also extend them to the following simple situation.

Theorem 7.24. *Let* $(\mathcal{G}, \mu, \nu)$ *be a weighted metric graph whose weight functions* $\mu, \nu\colon \mathcal{G} \to (0, \infty)$ *are uniformly positive, that is,*

$$\frac{1}{\mu},\ \frac{1}{\nu} \in L^\infty(\mathcal{G}). \tag{7.7}$$

Then the deficiency indices of the minimal Gaffney Laplacian $\mathbf{H}_{G,\min}$ *are equal to the number of finite volume graph ends,*

$$\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = \#\mathfrak{C}_0(\mathcal{G}; \mu). \tag{7.8}$$

Moreover, the following statements are equivalent:

(i) $\mathbf{H}^0$ *admits a unique Markovian extension,*

(ii) $\mathbf{H}_D = \mathbf{H}_N$,

(iii) *the Gaffney Laplacian* $\mathbf{H}_G$ *is self-adjoint,*

(iv) $H^1_0(\mathcal{G}, \mu, \nu) = H^1(\mathcal{G}, \mu, \nu)$,

(v) *all graph ends have infinite volume (with respect to* μ*), i.e.,* $\mathfrak{C}_0(\mathcal{G}; \mu) = \varnothing$.

Before giving the proof of Theorem 7.24, we recall a few standard facts on Sobolev spaces in dimension one. First of all, for every $\mathcal{I} = [0, a)$, $a \in (0, \infty]$ the embedding of $H^1(\mathcal{I})$ into $C_b(\mathcal{I}) = C(\mathcal{I}) \cap L^\infty(\mathcal{I})$ is bounded and

$$\sup_{x\in\mathcal{I}} |f(x)|^2 \le C_a \int_{\mathcal{I}} |f(x)|^2 + |f'(x)|^2\, \mathrm{d}x \tag{7.9}$$

holds for all $f \in H^1(\mathcal{I})$, where $C_a = \sqrt{\coth(a)}$ (see [161]). Moreover, the limit $\lim_{x\to a} f(x)$ exists for every function $f \in H^1(\mathcal{I})$ (see, e.g., [32, Theorem 8.2] for bounded intervals and [32, Corollary 8.9] in the unbounded case).

Returning to our setting, assume that $(\mathcal{G}, \mu, \nu)$ is a weighted metric graph. Suppose further that $\mathcal{P}$ is a path in $\mathcal{G}$. Notice that we can first identify $\mathcal{P}$ with a subset of $\mathcal{G}$, and then further with an interval $\mathcal{I}_{\mathcal{P}} = [0, |\mathcal{P}|)$ of length

$$|\mathcal{P}| := \int_{\mathcal{P}} \mathrm{d}x,$$

where the integral is taken over the subset $\mathcal{P} \subseteq \mathcal{G}$ with respect to the (unweighted) Lebesgue measure on $\mathcal{G}$ (cf. (6.5)). The restriction $f|_{\mathcal{P}}$ of a function $f \in H^1(\mathcal{G}, \mu, \nu)$ to $\mathcal{P} \subseteq \mathcal{G}$ can be identified with a function on $\mathcal{I}_{\mathcal{P}} = [0, |\mathcal{P}|)$. Notice that, in case that (7.7) is satisfied, $f|_{\mathcal{P}}$ belongs to the (unweighted) Sobolev space $H^1(\mathcal{I}_{\mathcal{P}})$. In particular, (7.7) implies the following crucial property of H^1-functions: for every ray $\mathcal{R} = (e_{v_n,v_{n+1}})_{n\ge 0}$, the following limit

$$f(\gamma_{\mathcal{R}}) := \lim_{n\to\infty} f(v_n) \tag{7.10}$$

exists. Moreover, for each topological end $\gamma \in \mathfrak{C}(\mathcal{G})$ this limit is independent of the choice of the ray $\mathcal{R}$ in the corresponding graph end ω_γ. Indeed, for any two equivalent rays $\mathcal{R}$ and $\mathcal{R}'$ there exists a third ray $\mathcal{R}''$ containing infinitely many vertices of both $\mathcal{R}$ and $\mathcal{R}'$, which immediately implies that

$$f(\gamma_{\mathcal{R}}) = f(\gamma_{\mathcal{R}''}) = f(\gamma_{\mathcal{R}'}).$$

Taking into account the relationship between topological ends and graph ends (see Section 2.1.3), this enables us to introduce the following notion.

Definition 7.25. Assume the weights μ, ν satisfy (7.7). Then for $f \in H^1(\mathcal{G}, \mu, \nu)$ and a (topological) end $\gamma \in \mathfrak{C}(\mathcal{G})$, we define

$$f(\gamma) := f(\gamma_{\mathcal{R}}),$$

where $\mathcal{R}$ is any ray belonging to the corresponding graph end ω_γ.

As is easily verified, the values $f(\gamma)$, $\gamma \in \mathfrak{C}(\mathcal{G})$ are independent of the choice of the model of $(\mathcal{G}, \mu, \nu)$. It turns out that we obtain a continuous extension of f to the end compactification $\widehat{\mathcal{G}} = \mathcal{G} \cup \mathfrak{C}(\mathcal{G})$.

Proposition 7.26. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph satisfying (7.7). Then for every function $f \in H^1(\mathcal{G}, \mu, \nu)$, its extension $f : \widehat{\mathcal{G}} \to \mathbb{C}$ is continuous.*

Proof. Let $\gamma \in \mathfrak{C}(\mathcal{G})$ be a topological end represented by a sequence of open subsets $\mathcal{U} = (U_n)$. To prove that $f : \widehat{\mathcal{G}} \to \mathbb{C}$ is continuous in γ, we have to show that (see Section 2.1.3 for the definition of the topology on $\widehat{\mathcal{G}}$)

$$\lim_{n\to\infty} \sup_{x \in U_n} |f(x) - f(\gamma)| = 0.$$

As is readily verified (for instance, we can always refine the fixed model of $(\mathcal{G}, \mu, \nu)$), it suffices to prove this statement for vertices $v \in \mathcal{V}$, that is, to establish that

$$\lim_{n\to\infty} \sup_{v \in \mathcal{V} \cap U_n} |f(v) - f(\gamma)| = 0. \tag{7.11}$$

In order to obtain (7.11), we distinguish two cases. Assume first that each of the open sets U_n contains a ray $\mathcal{R}_n \subseteq U_n$ with length $|\mathcal{R}_n| > 1$. As is easily verified, then each vertex $v \in U_n$ is contained in a path without self-intersections $\mathcal{P}_v \subseteq U_n$ of length $|\mathcal{P}_v| \geq \frac{1}{2}$. Since $\bigcap_n U_n = \varnothing$, it follows from (7.9) and assumption (7.7) that

$$\lim_{n\to\infty} \sup_{v \in \mathcal{V} \cap U} |f(v)|^2 \leq \lim_{n\to\infty} C_{1/2} \int_{U_n} |f(x)|^2 + |\nabla f(x)|^2 \, dx = 0.$$

Clearly, this also implies $f(\gamma) = 0$ and hence proves (7.11) in the first case.

On the other hand, suppose that there exists a set U_N such that all rays $\mathcal{R} \subseteq U_N$ have length $|\mathcal{R}| \leq 1$. Since every vertex $v \in U_n$, $n \geq N$, is contained in some ray

$\mathcal{R}_v \subseteq U_n$ with $\mathcal{R}_v \in \omega_\gamma$, we have

$$\sup_{v \in \mathcal{V} \cap U_n} |f(v) - f(\gamma)| \le \sup_{v \in \mathcal{V} \cap U_n} \int_{\mathcal{R}_v} |\nabla f(x)|\, \mathrm{d}x \le \left(\int_{U_n} |\nabla f(x)|^2\, \mathrm{d}x \right)^{1/2}$$

and assumption (7.7) again implies that (7.11) holds true. ■

This leads to a description of $H_0^1(\mathcal{G}, \mu, \nu) = \overline{H_c^1(\mathcal{G})}^{\|\cdot\|_{H^1(\mathcal{G},\mu,\nu)}}$ as the space of H^1-functions with vanishing boundary values.

Theorem 7.27. *Assume that (7.7) holds true. Then*

$$H_0^1(\mathcal{G}, \mu, \nu) = \{f \in H^1(\mathcal{G}, \mu, \nu) : f(\gamma) = 0 \, \textit{for all } \gamma \in \mathfrak{C}(\mathcal{G})\}.$$

Proof. First of all, notice that $\sup_{x \in \widehat{\mathcal{G}}} |f(x)| \le C \|f\|_{H^1(\mathcal{G},\mu,\nu)}$ for every function $f \in H^1(\mathcal{G}, \mu, \nu)$ and some uniform constant $C > 0$ (this follows, e.g., from the closed graph theorem). On the other hand, if $f \in H^1(\mathcal{G}, \mu, \nu)$ has compact support, then $f(\gamma) = 0$ for all graph ends $\gamma \in \mathfrak{C}(\mathcal{G})$. This proves the first inclusion "$\subseteq$".

The proof of the converse inclusion "$\supseteq$" follows line to line the proof of [146, Theorem 3.12] (see also the proof of [83, Theorem 4.14]). First of all, we may assume that $f \in H^1(\mathcal{G}, \mu, \nu)$ is non-negative and vanishes on $\mathfrak{C}(\mathcal{G})$. Then for every $s > 0$, the set

$$A_s = \{x \in \mathcal{G} : f(x) \ge s\}$$

is a compact subset of $\mathcal{G}$. In particular, defining $\phi_n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$\phi_n(s) = \begin{cases} s - \dfrac{1}{n} & \text{if } s \ge \dfrac{1}{n}, \\ 0 & \text{if } s < \dfrac{1}{n}, \end{cases}$$

the composition $f_n := \phi_n \circ f$ has compact support in $\mathcal{G}$. Moreover, $|\phi_n(s)| \le |s|$ and $|\phi_n(s) - \phi_n(t)| \le |s - t|$ for all $s, t \ge 0$ and hence f_n belongs to $H_0^1(\mathcal{G}, \mu, \nu)$ for all n. As is easily verified, $\lim_{n\to\infty} f_n = f$ in $H^1(\mathcal{G}, \mu, \nu)$, which finishes the proof. ■

To prove the main results of this section, we also need the following lemma.

Lemma 7.28. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph satisfying (7.7). Then for any finite collection of distinct finite volume ends $(\gamma_i)_{i=1}^N$, there exists a function $g \in \operatorname{dom}(\mathbf{H}_N)$ with $g(\gamma_1) = 1$ and $g(\gamma_2) = \cdots = g(\gamma_N) = 0$.*

Proof. Fix a representing sequence of open subsets $\mathcal{U}^i = (U_n^i)$ for each of the topological ends γ_i, $i = 1, \dots, N$. Without loss of generality, we may suppose that $U := U_0^1$ has measure $\mu(U) < \infty$ and $U \cap U_0^i = \varnothing$ for all $i = 2, \dots, N$. Moreover, since ∂U is compact, the edge set $\mathcal{E}_0 = \{e \in \mathcal{E} : e \cap \partial U \ne \varnothing\}$ is finite and hence its union $K := \bigcup_{e\in\mathcal{E}_0} e$ is a compact subset of $\mathcal{G}$. Clearly, we can easily construct a function $g \in H^1(\mathcal{G}, \mu, \nu) \cap \mathrm{dom}(\mathbf{H})$ which satisfies $g \equiv 1$ on U, $\{x : \nabla g(x) \neq 0\} \subseteq K$ and $g \equiv 0$ on $\mathcal{G} \setminus (U \cup K)$. Notice that in this step we need the finite volume property of γ_1 to ensure that $g \in L^2(\mathcal{G}; \mu)$. Taking into account that $U \cap U_0^i = \varnothing$ for $i = 2, \dots, N$, it is easily verified that g has the claimed boundary values in the graph ends γ_i, $i = 1, \dots, N$.

It remains to prove that g belongs to $\mathrm{dom}(\mathbf{H}_N)$. However, since g satisfies the Kirchhoff conditions on $\mathcal{G}$ and is (componentwise) constant on $\mathcal{G} \setminus K$, integration by parts gives

$$\begin{aligned}\mathfrak{Q}_N[g,h] &= \int_{\mathcal{G}} \nabla g(x)\nabla h(x)^*\,\nu(\mathrm{d}x) = \int_K \nabla g(x)\nabla h(x)^*\,\nu(\mathrm{d}x)\\ &= -\int_K \Delta g(x) h^*(x)\,\mu(\mathrm{d}x) = -\int_{\mathcal{G}} \Delta g(x) h^*(x)\,\mu(\mathrm{d}x)\end{aligned}$$

for every function $h \in H^1(\mathcal{G}, \mu, \nu)$. In particular, g belongs to $\mathrm{dom}(\mathbf{H}_N)$ by the first representation theorem (see, e.g., [126, Chapter 6]). ■

After these preparations we proceed with the proof of Theorem 7.24.

Proof of Theorem 7.24. First of all, since $\mathbf{H}_N$ is a self-adjoint extension of $\mathbf{H}_{G,\min}$, the second von Neumann formula (cf. [191, Theorem 13.10]) implies

$$\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = \dim\big(\mathrm{dom}(\mathbf{H}_N)/\mathrm{dom}(\mathbf{H}_{G,\min})\big).$$

The lower estimate "$\geq$" in (7.8) then follows immediately from Lemma 7.28, Theorem 7.27 and the fact that $\mathrm{dom}(\mathbf{H}_{G,\min}) \subseteq H_0^1(\mathcal{G}, \mu, \nu)$. This, in particular, implies the equality if $\#\mathfrak{C}_0(\mathcal{G};\mu) = \infty$. Hence we only need to prove (7.8) if $\#\mathfrak{C}_0(\mathcal{G};\mu) < \infty$.

In this case, by Lemma 7.28, for every finite volume end $\gamma \in \mathfrak{C}_0(\mathcal{G};\mu)$, we can fix a function $g_\gamma \in \mathrm{dom}(\mathbf{H}_N)$ with $g_\gamma(\gamma) = 1$ and $g_\gamma(\gamma') = 0$ for all $\gamma' \in \mathfrak{C}_0(\mathcal{G};\mu)$, $\gamma' \neq \gamma$. Then every function $f \in \mathrm{dom}(\mathbf{H}_N)$ can be written as

$$f = f - \sum_{\gamma\in\mathfrak{C}_0(\mathcal{G};\mu)} f(\gamma) g_\gamma + \sum_{\gamma\in\mathfrak{C}_0(\mathcal{G};\mu)} f(\gamma) g_\gamma =: f_0 + f_{\mathfrak{C}_0}.$$

Clearly, f_0 belongs to $\mathrm{dom}(\mathbf{H}_N)$ and $f_0(\gamma) = 0$ at all finite volume graph ends $\gamma \in \mathfrak{C}(\mathcal{G})$. In fact, $f_0(\gamma) = 0$ for all graph ends (including ends of infinite volume) since f_0 extends continuously to the end compactification (see Proposition 7.26) and belongs to $L^2(\mathcal{G};\mu)$. Therefore, by Theorem 7.27, f_0 belongs to $H_0^1(\mathcal{G}, \mu, \nu)$ and, comparing (2.22) with (2.23), this implies that $f_0 \in \mathrm{dom}(\mathbf{H}_{G,\min})$ and, moreover, that $\mathrm{dom}(\mathbf{H}_N)$ admits the following decomposition:

$$\mathrm{dom}(\mathbf{H}_N) = \mathrm{dom}(\mathbf{H}_{G,\min}) \dotplus \mathrm{span}\{g_\gamma : \gamma \in \mathfrak{C}_0(\mathcal{G};\mu)\}.$$

In particular, we conclude that

$$\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = \dim\big(\mathrm{dom}(\mathbf{H}_N)/\mathrm{dom}(\mathbf{H}_{G,\min})\big) = \#\mathfrak{C}_0(\mathcal{G};\mu).$$

The remaining equivalences follow from Lemma 4.1 (see also Lemma 2.15 and (2.23)). ■

Let us stress that finite volume graph ends do not provide a characterization of Markovian uniqueness for general weighted graphs $(\mathcal{G}, \mu, \nu)$. This was already observed in the simple case of weighted path graphs in Section 5.1 (see in particular Lemma 5.11). Notice that $\mathbb{Z}_{\geq 0}$ has only one graph end γ and it has finite volume exactly when the sum in (5.7) converges. Hence, by Lemma 5.11, the Gaffney Laplacian $\mathbf{H}_G$ is self-adjoint if either the quantity $\mathcal{L}_\nu = \infty$ (and in this case the volume of the graph end is irrelevant) or $\mathcal{L}_\nu < \infty$ and γ has infinite volume.

On the other hand, the result for path graphs suggests that finite volume ends can be used under a suitable generalization of the condition $\mathcal{L}_\nu < \infty$ from Lemma 5.11. It turns out that this guess is indeed correct and we outline the idea in the following. For any path $\mathcal{P}$ in $\mathcal{G}$, we define its ν-length as (cf. (6.5))

$$|\mathcal{P}|_{1/\nu} = \int_{\mathcal{P}} \frac{\mathrm{d}s}{\nu(s)},$$

where the integral is taken over the corresponding subset $\mathcal{P} \subseteq \mathcal{G}$. Moreover, for any subset $U \subseteq \mathcal{G}$, its ν-*diameter at infinity* is defined as

$$D_{1/\nu}(U) := \sup_{\mathcal{P} \subseteq U} |\mathcal{P}|_{1/\nu},$$

where the supremum is taken over all paths $\mathcal{P}$ without self-intersection in U. Suppose $\gamma \in \mathfrak{C}(\mathcal{G})$ is a topological end represented by a sequence of open subsets $\mathcal{U} = (U_n)$. Then we define its ν-diameter[1] by

$$D_{1/\nu}(\gamma) = \inf_n D_{1/\nu}(U_n) = \lim_{n \to \infty} D_{1/\nu}(U_n).$$

Remark 7.29. As is readily verified, the value of $D_{1/\nu}(\gamma)$ is independent of the choice of the representing sequence $\mathcal{U} = (U_n)$.

It turns out that the conclusions of Theorem 7.7 are also valid under the assumption

$$D_{1/\nu}(\gamma) < \infty \quad \text{for all graph ends } \gamma \in \mathfrak{C}(\mathcal{G}) \tag{7.12}$$

instead of (7.7). For instance, it is easy to see that for each $f \in H^1(\mathcal{G}, \mu, \nu)$ and a ray $\mathcal{R}$,

$$\int_{\mathcal{R}} |\nabla f| \, \mathrm{d}s < \infty$$

and in particular, the limits in (7.10) exist. A careful analysis of the rest of the proof for Theorem 7.7 shows that it can be carried over as well and we omit the details.

[1]Let us stress that $D_{1/\nu}(U)$ does not coincide with the standard definition of the diameter for metric spaces.

Remark 7.30. Assumption (7.12) can be seen as a generalization of the condition $\mathcal{L}_\nu < \infty$ in Lemma 5.11. On the other hand, neither of the conditions (7.7) and (7.12) implies the other one.

7.2.2 Markovian and finite energy extensions

Let us now briefly comment on the problem of describing the self-adjoint restrictions of the Gaffney Laplacian $\mathbf{H}_G$. This class of extensions is called *finite energy extensions* in [146] and by Lemma 2.18, these are exactly the self-adjoint extensions $\widetilde{\mathbf{H}}$ of the minimal operator $\mathbf{H}_0$ satisfying $\mathrm{dom}(\widetilde{\mathbf{H}}) \subset H^1(\mathcal{G}, \mu, \nu)$. Their importance stems from the fact that they contain all Markovian extensions (see Lemma 4.1). Moreover, the kernels of their heat semigroups and resolvents are well behaved (the results of [146, Section 5] extend verbatim if at least one of the assumptions (7.7) or (7.12) is satisfied).

The preceding sections suggest to describe finite energy extensions in terms of finite volume graph ends. It turns out that, if (7.7) or (7.12) holds true and in addition $\#\mathfrak{C}_0(\mathcal{G}; \mu) < \infty$, this is indeed possible. Namely, in this case the maximal Gaffney Laplacian $\mathbf{H}_G$ is closed (this can be proved analogous to [148, Theorem 3.12 (i)]). Moreover, these assumptions allow to introduce a suitable notion of a normal derivative for finite volume graph ends $\gamma \in \mathfrak{C}_0(\mathcal{G}; \mu)$ (modifying the notions in [146, Section 6] using the weights). This leads to a complete description of all Markovian extensions of the minimal Laplacian $\mathbf{H}_0$ and all self-adjoint restrictions of the maximal Gaffney Laplacian $\mathbf{H}_G$ in terms of certain boundary conditions on finite volume graph ends (analogous to [146, Section 6.3] and [148, Remark 3.13 (ii)]). The proofs of these claims can easily be carried over from [146, 148], however, the full exposition reads a bit technical and hence we do not develop it here.

If $\#\mathfrak{C}_0(\mathcal{G}; \mu) = \infty$, that is, the deficiency indices of the minimal Gaffney Laplacian are infinite, then even in the unweighted case $\mu = \nu \equiv \mathbb{1}$ the above methods are not sufficient for a description of finite energy extensions. We stress that in this case the Gaffney Laplacian $\mathbf{H}_G$ is not closed in general (see [148, Section 4]) and, moreover, in many interesting cases (see [148, Section 4]), its closure equals the maximal Laplacian $\mathbf{H}$, $\overline{\mathbf{H}}_G = \mathbf{H}$ (which is further equivalent to the equality of the minimal Kirchhoff and Gaffney Laplacians), and hence the problem is essentially as difficult as the description of self-adjoint extensions of the minimal Laplacian $\mathbf{H}^0$.

We would also like to stress that, by Theorem 4.12 and Theorem 6.16, the problem of describing Markovian extensions is equivalent for weighted metric and discrete graphs. Moreover, for weighted graph Laplacians, a description of Markovian extensions was obtained in [133] in terms of Dirichlet forms (in the wide sense) on the corresponding Royden boundary (see, e.g., [83, 134, 195] for details and definitions) equipped with a harmonic measure (in fact, on the so-called harmonic boundary, which is a subset). It should also be stressed that there is no finiteness assumption on the deficiency indices of the Gaffney Laplacian in [133]. However, let us emphasize that this description is by means of quadratic forms and not via boundary conditions. Moreover, the correspondence between Markovian extensions and Dirichlet forms (in the wide sense) on the boundary is in general not one-to-one and hence also does not lead to a complete characterization of the Markovian uniqueness. On the other hand, if the weighted graph $(\mathcal{V}, m; b)$ has finite total mass, $m(\mathcal{V}) < \infty$, it becomes a bijection and in this case the Royden boundary should be the correct concept to study Markovian extensions.

In general, the Royden boundary of a graph $(\mathcal{V}, m; b)$ can be rather big and hard to describe (see [216] for the toy model $\mathcal{G}_d = \mathbb{Z}$). Its relationship to the standard one-point compactification is closely connected to the Liouville property for finite energy harmonic functions [134, Theorem 6.2]. However, in the special case that $\sum_{u,v} 1/b(u,v) < \infty$, the Royden boundary coincides with the space of graph ends and several other graph boundaries (see [83, Section 4.6] for details). Hence, under the additional assumption that $m(\mathcal{V}) < \infty$, we recover precisely the space of finite volume ends (in the discrete setting). Moreover, one can show that under either of the assumptions (7.12) and (7.7), the space of finite volume ends $\mathfrak{C}_0(\mathcal{G}; \mu)$ of a weighted metric graph $(\mathcal{G}, \mu, \nu)$ can be embedded into the Royden boundary of the discrete graph $(\mathcal{V}, m; b)$ for any model (the weights are defined by (3.5) and (3.6)). However, in general it seems that these two boundaries do not compare.

7.2.3 A few more comments

Let us point out that, by Theorem 4.12 and Theorem 6.16, the problem of characterizing Markovian uniqueness is equivalent for Laplacians on weighted metric graphs and graph Laplacians. Moreover, for weighted metric graphs $(\mathcal{G}, \mu, \nu)$ this question was studied in [97, Chapter 2] using metric completions (with respect to several different metrics). In the parallel settings of discrete graphs and manifolds, results were obtained in terms of polarity of metric boundaries in [115] and [93, 162, 163]. These techniques obviously apply to weighted metric graphs as well (alternatively, the results from [115] can also be transferred using the correspondence between H^1-spaces and intrinsic metrics, see Section 4.3 and Section 6.4). However, none of these approaches leads to a complete description of the uniqueness of Markovian extensions (e.g., the characterization in [115, Theorem 3] requires finite capacity of the metric boundary).

An important concept in context with graphs is the construction of boundaries by employing C^*-algebra techniques (this includes both Royden and Kuramochi boundaries, see [83, 125, 134, 168, 195] for further details and references). Under the assumptions (7.7) or (7.12), finite volume graph ends can also be constructed by using this method. Indeed, $\mathcal{A} := H^1(\mathcal{G}, \mu, \nu) \subset C_b(\mathcal{G})$ is a subalgebra by Proposition 7.26 and hence its $\|\cdot\|_\infty$-closure $\widetilde{\mathcal{A}} := \overline{\mathcal{A}}^{\|\cdot\|_\infty}$ is isomorphic to $C_0(\widetilde{X})$, where

$\widetilde{X}$ is the space of characters equipped with the weak*-topology with respect to $\widetilde{\mathcal{A}}$. In general, describing $\widetilde{X}$ for some concrete C^*-algebra is a rather complicated task. However, it turns out that in our situation $\widetilde{X}$ coincides with $\widetilde{\mathcal{G}} := \mathcal{G} \cup \mathfrak{C}_0(\mathcal{G};\mu)$. Indeed, $\widetilde{\mathcal{G}} = \mathcal{G} \cup \mathfrak{C}_0(\mathcal{G};\mu)$ equipped with the induced topology of the end compactification $\widehat{\mathcal{G}}$ is a locally compact Hausdorff space. Proposition 7.26 together with Theorem 7.27 shows that each function $f \in H^1(\mathcal{G},\mu,\nu)$ has a unique continuous extension to $\widetilde{\mathcal{G}}$ and this extension belongs to $C_0(\widetilde{\mathcal{G}})$. Moreover, by Lemma 7.28, $H^1(\mathcal{G},\mu,\nu)$ is point-separating and nowhere vanishing on $\widetilde{\mathcal{G}}$ and hence $\widetilde{\mathcal{A}} = C_0(\widetilde{\mathcal{G}})$ by the Stone–Weierstrass theorem. Thus the resulting boundary notion is precisely the space of finite volume graph ends.

7.3 Spectral estimates

The aim of this section is to obtain spectral estimates for Laplacians on a weighted metric graph $(\mathcal{G},\mu,\nu)$. For simplicity, we restrict to the Dirichlet Laplacian $\mathbf{H}_D$ and present estimates for the bottom of its spectrum,

$$\lambda_0(\mathbf{H}_D) := \inf \sigma(\mathbf{H}_D).$$

We also recall from Theorem 4.27 that if $(\mathcal{G},\mu,\nu)$ has infinite intrinsic size, that is, there is a model with $\eta^*(\mathcal{E}) = \infty$, then $\lambda_0(\mathbf{H}_D) = 0$ (in fact, this holds true for all Markovian and all non-negative extensions of the minimal Kirchhoff Laplacian). Therefore, without loss of generality we can restrict our considerations in this section to the case when

$$(\mathcal{G},\mu,\nu) \textit{ has finite intrinsic size.}$$

7.3.1 Isoperimetric estimates

We begin with estimates for $\lambda_0(\mathbf{H}_D)$ in terms of *isoperimetric constants*. Our exposition follows closely [147], where the special case of unweighted metric graphs (i.e., $\mu = \nu \equiv \mathbb{1}$) was considered.

Assume that we have fixed a model of $(\mathcal{G},\mu,\nu)$ with underlying combinatorial graph $\mathcal{G}_d = (\mathcal{V},\mathcal{E})$. Then clearly every finite subgraph $\mathcal{K} = (\mathcal{V}(\mathcal{K}),\mathcal{E}(\mathcal{K}))$ of $\mathcal{G}_d$ can be identified with a compact subset of $\mathcal{G}$. Moreover, its volume with respect to μ and its topological boundary are given by

$$\mu(\mathcal{K}) = \sum_{e\in\mathcal{E}(\mathcal{K})} |e|\mu(e), \quad \partial\mathcal{K} = \{v \in \mathcal{V}(\mathcal{K}) : \deg_{\mathcal{K}}(v) < \deg_{\mathcal{G}}(v)\}. \tag{7.13}$$

We introduce the *boundary area* of $\mathcal{K}$ as

$$\operatorname{area}(\partial\mathcal{K}) = \operatorname{area}(\partial\mathcal{K},\mu,\nu) = \sum_{v\in\partial\mathcal{K}} \sum_{\vec{e}\in\vec{\mathcal{E}}_v(\mathcal{K})} \sqrt{\mu\nu}(e). \tag{7.14}$$

Definition 7.31. The *isoperimetric constant* of a weighted metric graph $(\mathcal{G}, \mu, \nu)$ is defined as

$$\mathrm{Ch}(\mathcal{G}) = \mathrm{Ch}(\mathcal{G}, \mu, \nu) := \inf_{\mathcal{K}} \frac{\mathrm{area}(\partial \mathcal{K})}{\mu(\mathcal{K})}, \tag{7.15}$$

where the infimum is taken over all finite, connected subgraphs $\mathcal{K} = (\mathcal{V}(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ of a fixed model of $(\mathcal{G}, \mu, \nu)$.

The above definition of $\mathrm{Ch}(\mathcal{G})$ is given in terms of a fixed model of $(\mathcal{G}, \mu, \nu)$, however, we have the following simple fact.

Lemma 7.32. *The isoperimetric constant* $\mathrm{Ch}(\mathcal{G})$ *does not depend on the choice of the model.*

Proof. First of all, it is not difficult to see that (7.15) remains unchanged under refinement of the model (see Section 2.4.3). Namely, any subgraph in a refined model can be completed to a subgraph in a coarser model by adding the "remaining parts" of edges. It is also clear that this procedure decreases the quotient in (7.15). Hence for two given models of $(\mathcal{G}, \mu, \nu)$, we can take their common refinement (take all the vertices of both models as the vertex set) and hence the claim follows. ■

The next result provides Cheeger- and Buser-type estimates on weighted metric graphs.

Theorem 7.33. *For a weighted metric graph* $(\mathcal{G}, \mu, \nu)$*,*

$$\frac{1}{4}\mathrm{Ch}(\mathcal{G})^2 \le \lambda_0(\mathbf{H}_D) \le \frac{\pi^2}{2 \sup \eta_*}\mathrm{Ch}(\mathcal{G}), \tag{7.16}$$

where $\eta_*(\mathcal{G})$ *is defined by* (7.2) *and the supremum is taken over all models of the weighted metric graph* $(\mathcal{G}, \mu, \nu)$*.*[2]

Proof. (i) *Cheeger's estimate*. First of all, recall that $\lambda_0(\mathbf{H}_D)$ is given by the variational characterization

$$\lambda_0(\mathbf{H}_D) = \inf_{0 \ne f \in H_0^1(\mathcal{G})} \frac{\|\nabla f\|^2_{L^2(\mathcal{G};\nu)}}{\|f\|^2_{L^2(\mathcal{G};\mu)}}. \tag{7.17}$$

Hence the lower estimate in (7.16) will follow from the inequality

$$\mathrm{Ch}(\mathcal{G})\|f\|_{L^2(\mathcal{G};\mu)} \le 2\|\nabla f\|_{L^2(\mathcal{G};\nu)}, \quad f \in H_0^1(\mathcal{G}).$$

[2]Notice that in practice $\sup \eta_*(\mathcal{G})$ can be computed by first removing all inessential vertices from a fixed model of $(\mathcal{G}, \mu, \nu)$ and then finding the shortest intrinsic length among the obtained edges. Here we apply the convention that an infinite ray of inessential vertices becomes transformed into a non-compact edge (sometimes called a lead, leg or half-edge), whose intrinsic length is the total intrinsic length of the ray.

Without loss of generality we can assume that f is real-valued, compactly supported and smooth on all edges $e \in \mathcal{E}$. Recall also that for any compactly supported, continuous and edgewise $\mathcal{C}^1$-function $h \colon \mathcal{G} \to [0,\infty)$, the following co-area formulas hold true (see, e.g., [147, Lemma 3.6]):

$$\int_{\mathcal{G}} h(x)\mu(\mathrm{d}x) = \int_0^\infty \mu(\Omega_h(t))\,\mathrm{d}t.$$

$$\int_{\mathcal{G}} |\nabla h(x)|\,\omega(\mathrm{d}x) = \int_0^\infty \operatorname{area}(\partial\Omega_h(t))\,\mathrm{d}t,$$

where $\Omega_h(t) := \{x \in \mathcal{G} : h(x) > t\}$ for all $t \geq 0$,

$$\omega := \sqrt{\mu\nu}, \quad \omega(\mathrm{d}x) := \sqrt{\mu\nu(x)}\,\mathrm{d}x,$$

and

$$\operatorname{area}(\partial\Omega_h(t)) := \sum_{x \in \partial\Omega_h(t)} \omega(x).$$

Notice that for almost every $t > 0$, the boundary $\partial\Omega_h(t)$ contains no vertices and hence the above integral is well defined. Indeed, every $x \in \partial\Omega_h(t)$ satisfies $h(x) = t$ and hence the claim follows from the countability of the vertex set.

Moreover, if $\partial\Omega_h(t) \cap \mathcal{V} = \varnothing$, then we can associate with $\Omega_h(t)$ the subgraph $\mathcal{K}_t \subseteq \mathcal{G}_d$ consisting of all edges $e \in \mathcal{E}$ with $\Omega_h(t) \cap e \neq \varnothing$ and their endpoints. It is then easily verified that (see also [147, proof of Lemma 3.7])

$$\frac{\operatorname{area}(\partial\Omega_h(t))}{\mu(\Omega_h(t))} \geq \frac{\operatorname{area}(\mathcal{K}_t)}{\mu(\mathcal{K}_t)} \geq \operatorname{Ch}(\mathcal{G}). \tag{7.18}$$

By choosing $h = f^2$, we conclude from the co-area formulas that

$$\operatorname{Ch}(\mathcal{G})\|f\|^2_{L^2(\mathcal{G};\mu)} \leq 2\int_{\mathcal{G}} |\nabla f(x) f(x)|\,\omega(\mathrm{d}x) \leq 2\|f\|_{L^2(\mathcal{G};\mu)}\|\nabla f\|_{L^2(\mathcal{G};\nu)},$$

where the last inequality follows from the Cauchy–Schwarz inequality.

(ii) *Buser's estimate*. Fix a model of $(\mathcal{G},\mu,\nu)$. The edge set of a finite connected subgraph $\mathcal{K} = (\mathcal{V}(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ can be split into

$$\mathcal{E}(\mathcal{K}) = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2,$$

where $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the mutually disjoint sets of edges of $\mathcal{E}(\mathcal{K})$ with, respectively, all endpoints in $\mathcal{V}(\mathcal{K}) \setminus \partial\mathcal{K}$, exactly one endpoint in $\partial\mathcal{K}$ (and hence exactly one endpoint in $\mathcal{V}(\mathcal{K}) \setminus \partial\mathcal{K}$), and all endpoints in $\partial\mathcal{K}$.[3] Notice in particular that

$$\operatorname{area}(\partial\mathcal{K}) = \sum_{e\in\mathcal{E}_1} \sqrt{\mu\nu}(e) + 2\sum_{e\in\mathcal{E}_2} \sqrt{\mu\nu}(e).$$

[3] Loop edges in $\mathcal{E}(\mathcal{K})$ are considered either as elements of $\mathcal{E}_2$ or $\mathcal{E}_0$, depending on their vertex belonging to $\partial\mathcal{K}$ or not.

Consider the test function $f\colon\mathcal{G}\to\mathbb{R}$ defined by

$$f|_e = \begin{cases} 1, & e\in\mathcal{E}_0,\\ \sin\left(\frac{\pi}{|e|}|\cdot - e_\iota|\right), & e\in\mathcal{E}_2,\\ \sin\left(\frac{\pi}{2|e|}|\cdot - u|\right), & e=e_{u,v}\in\mathcal{E}_1 \text{ with } u\in\partial\mathcal{K},\\ 0, & e\notin\mathcal{E}(\mathcal{K}). \end{cases}$$

By construction, f belongs to $H^1_c(\mathcal{G})$ and its support coincides with $\mathcal{K}$. Moreover,

$$\|f\|^2_{L^2(\mathcal{G};\mu)} = \sum_{e\in\mathcal{E}_0}\mu(e)|e| + \sum_{e\in\mathcal{E}_1\cup\mathcal{E}_2}\frac{\mu(e)|e|}{2} \ge \frac{\mu(\mathcal{K})}{2},$$

$$\begin{aligned}\|\nabla f\|^2_{L^2(\mathcal{G};\nu)} &= \frac{\pi^2}{8}\sum_{e\in\mathcal{E}_1}\frac{\nu(e)}{|e|} + \frac{\pi^2}{2}\sum_{e\in\mathcal{E}_2}\frac{\nu(e)}{|e|}\\ &= \frac{\pi^2}{8}\sum_{e\in\mathcal{E}_1}\frac{\sqrt{\mu\nu}(e)}{\eta(e)} + \frac{\pi^2}{2}\sum_{e\in\mathcal{E}_2}\frac{\sqrt{\mu\nu}(e)}{\eta(e)} \le \frac{\pi^2}{4\eta_*(\mathcal{G})}\mathrm{area}(\partial\mathcal{K}),\end{aligned}$$

and then using (7.17) and taking the supremum over all models, we arrive at the second inequality in (7.16). ■

In a similar way, one can obtain isoperimetric estimates for $\lambda_0^{\mathrm{ess}}(\mathbf{H}_D)$, the bottom of the essential spectrum of $\mathbf{H}_D$. More precisely, for any finite, connected subgraph $\mathcal{K}=(\mathcal{V}(\mathcal{K}),\mathcal{E}(\mathcal{K}))$ of our fixed model, define

$$H^1_0(\mathcal{G}\setminus\mathcal{K}) := \{f\in H^1_0(\mathcal{G}) : \mathrm{supp}(f)\subseteq\mathcal{G}\setminus\mathcal{K}\}.$$

Then a standard Persson-type argument (a.k.a. Glazman's decomposition principle, see [84]) implies that

$$\lambda_0^{\mathrm{ess}}(\mathbf{H}_D) = \sup_{\mathcal{K}}\inf_{f\in H^1_0(\mathcal{G}\setminus\mathcal{K})}\frac{\|\nabla f\|^2_{L^2(\mathcal{G};\nu)}}{\|f\|^2_{L^2(\mathcal{G};\mu)}}, \tag{7.19}$$

where the supremum is taken over all finite, connected subgraphs $\mathcal{K}$ of $\mathcal{G}$. Setting $\mathcal{K}_1\le\mathcal{K}_2$ exactly when $\mathcal{K}_1$ is a subgraph of $\mathcal{K}_2$, we can see the set of all finite, connected subgraphs of $\mathcal{G}$ as a directed set. Moreover, if $\mathcal{K}_1\le\mathcal{K}_2$, then

$$H^1_0(\mathcal{G}\setminus\mathcal{K}_2)\subseteq H^1_0(\mathcal{G}\setminus\mathcal{K}_1),$$

and hence (7.19) can be rewritten as

$$\lambda_0^{\mathrm{ess}}(\mathbf{H}_D) = \lim_{\mathcal{K}}\inf_{f\in H^1_0(\mathcal{G}\setminus\mathcal{K})}\frac{\|\nabla f\|^2_{L^2(\mathcal{G};\nu)}}{\|f\|^2_{L^2(\mathcal{G};\mu)}}, \tag{7.20}$$

where the limit is taken over all finite, connected subgraphs $\mathcal{K}$ of $\mathcal{G}$ in the sense of nets. Thus, Theorem 7.33 together with (7.20) suggest that, roughly speaking, $\lambda_0^{\mathrm{ess}}(\mathcal{G})$ is related to the isoperimetric behavior of $(\mathcal{G}, \mu, \nu)$ "at infinity". This leads to the following definition:

Definition 7.34. Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph together with a fixed model. For any finite, connected subgraph $\mathcal{K} = (\mathcal{V}(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ of $\mathcal{G}$, define

$$\mathrm{Ch}_{\mathcal{K}}(\mathcal{G}) = \inf_{\mathcal{K}' \subseteq \mathcal{G} \setminus \mathcal{K}} \frac{\mathrm{area}(\partial \mathcal{K}')}{\mu(\mathcal{K}')},$$

where the infimum is over all finite, connected subgraphs $\mathcal{K}'$ of $\mathcal{G}$ with $\mathcal{K}' \subseteq \mathcal{G} \setminus \mathcal{K}$. The *isoperimetric constant at infinity* of $(\mathcal{G}, \mu, \nu)$ is given by

$$\mathrm{Ch}^{\mathrm{ess}}(\mathcal{G}) := \sup_{\mathcal{K}} \mathrm{Ch}_{\mathcal{K}}(\mathcal{G}) = \lim_{\mathcal{K}} \mathrm{Ch}_{\mathcal{K}}(\mathcal{G}), \tag{7.21}$$

where both the supremum and the net limit are taken over all finite, connected subgraphs $\mathcal{K}$ of $\mathcal{G}$.

It turns out that (e.g., by an argument as in Lemma 7.32) the definition of $\mathrm{Ch}^{\mathrm{ess}}(\mathcal{G})$ does not depend on the choice of the model of $(\mathcal{G}, \mu, \nu)$. Moreover, we obtain the following estimates:

Theorem 7.35. *Let $\mathcal{E}$ be the edge set of a fixed model of $(\mathcal{G}, \mu, \nu)$ and set*

$$\eta_*^{\mathrm{ess}}(\mathcal{G}) := \sup_{\widetilde{\mathcal{E}} \text{ finite}} \inf_{e \in \mathcal{E} \setminus \widetilde{\mathcal{E}}} \eta(e).$$

Then

$$\frac{1}{4}\mathrm{Ch}^{\mathrm{ess}}(\mathcal{G})^2 \le \lambda_0^{\mathrm{ess}}(\mathbf{H}_D) \le \frac{\pi^2}{2 \sup \eta_*^{\mathrm{ess}}(\mathcal{G})} \mathrm{Ch}^{\mathrm{ess}}(\mathcal{G}).$$

Here the supremum is taken over all models of $(\mathcal{G}, \mu, \nu)$.

In particular, $\sigma(\mathbf{H}_D)$ is purely discrete if $\mathrm{Ch}^{\mathrm{ess}}(\mathcal{G}) = \infty$.

Proof. Following the proof of Theorem 7.33, we get

$$\frac{1}{4}\mathrm{Ch}_{\mathcal{K}}(\mathcal{G})^2 \le \inf_{f \in H_0^1(\mathcal{G} \setminus \mathcal{K})} \frac{\|\nabla f\|^2_{L^2(\mathcal{G};\nu)}}{\|f\|^2_{L^2(\mathcal{G};\mu)}} \le \frac{\pi^2}{2} \frac{\mathrm{Ch}_{\mathcal{K}}(\mathcal{G})}{\eta_*^{\mathcal{K}}(\mathcal{E})}$$

for any finite, connected subgraph $\mathcal{K}$ of $\mathcal{G}$ (with $\eta_*^{\mathcal{K}}(\mathcal{E}) := \inf_{e \in \mathcal{E} \setminus \mathcal{E}(\mathcal{K})} \eta(e)$). For instance, if f belongs to $H_0^1(\mathcal{G} \setminus \mathcal{K})$, then the set $\Omega_{f^2}(t)$ is contained in $\mathcal{G} \setminus \mathcal{K}$ for all $t > 0$. In particular, this means that the subgraph $\mathcal{K}_t$ in (7.18) is contained in $\mathcal{G} \setminus \mathcal{K}$. The claim then follows from (7.19) together with (7.21). ∎

Remark 7.36. Going back to Cheeger's inequality for manifolds [42], isoperimetric constants are known to provide spectral estimates for both manifolds and graphs, see, e.g., [5, 6, 18, 38, 42, 59, 61, 147, 172]. For unweighted discrete graphs, the first works on this topic include [5, 6, 59, 61]. Employing the notion of an intrinsic metric, an isoperimetric constant and the corresponding estimate for weighted graphs $(\mathcal{V}, m; b)$ were introduced in [18] (see Section 7.3.2 for more details). For unweighted metric graphs, $\mu = \nu \equiv \mathbb{1}$, Cheeger's inequality was proven in [172] for finite metric graphs and in [147] for infinite metric graphs.

7.3.2 Connection with discrete isoperimetric constants

The combinatorial structure of $\mathrm{Ch}(\mathcal{G})$ enables us to investigate it by combinatorial methods. More precisely, in the case of unweighted metric graphs ($\mu \equiv \nu \equiv \mathbb{1}$), $\mathrm{Ch}(\mathcal{G})$ was studied using discrete, curvature-like quantities in [147, Section 6] and [173]. These methods can be extended to the setting of weighted metric graphs as well and this will be done elsewhere (see also Section 8.3.2 for the special case of tilings). Our main goal in this section is to discuss connections with discrete isoperimetric constants of the corresponding weighted graphs.

Let $(\mathcal{V}, m; b)$ be a locally finite connected graph and let $p\colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ be an intrinsic weight function (see Section 6.4.2). Following [18] (see also [147, Appendix]), we define an isoperimetric constant $\mathrm{Ch}_d(\mathcal{V})$ for $(\mathcal{V}, m; b)$ by

$$\mathrm{Ch}_d(\mathcal{V}) = \mathrm{Ch}_d(\mathcal{V}, m; b) := \inf_X \frac{|\partial X|}{m(X)}, \tag{7.22}$$

where the infimum is over all finite, connected subsets $X \subseteq \mathcal{V}$ and

$$\partial X = \{(u, v) \in X \times (\mathcal{V} \setminus X) : b(u, v) > 0\},$$
$$|\partial X| = \sum_{(u,v)\in\partial X} b(u, v)p(u, v), \quad m(X) = \sum_{v\in X} m(v).$$

We recall that, by [18, Theorem 3.2 and Theorem 3.6] (see also [147, Appendix]), the Dirichlet Laplacian $\mathbf{h}_D$ on $(\mathcal{V}, m; b)$ satisfies the following spectral estimate:

$$\frac{1}{2}\mathrm{Ch}_d(\mathcal{V})^2 \le \lambda_0(\mathbf{h}_D) \le \frac{\mathrm{Ch}_d(\mathcal{V})}{p_*(\mathcal{V})}, \tag{7.23}$$

where $p_*(\mathcal{V}) := \inf_{b(u,v)>0} p(u, v)$.

Remark 7.37. Notice that the isoperimetric constant $\mathrm{Ch}_d(\mathcal{V})$ is defined slightly differently in [18]. Namely, the weight $p(u, v)$ in the definition of $|\partial X|$ is replaced by the distance $\varrho(u, v)$ in an intrinsic metric ϱ. On the other hand, it is straightforward to verify that [18, Theorem 3.2 and Theorem 3.6] remain valid also for our definition (see [147, Appendix] for details).

Recall that we had assumed that the weighted metric graph $(\mathcal{G}, \mu, \nu)$ has finite intrinsic size. Fix a model of $(\mathcal{G}, \mu, \nu)$ (which then also has finite intrinsic size). Consider the locally finite graph $(\mathcal{V}, m; b)$ defined by (3.3)–(3.6) and the corresponding discrete Laplacian $\mathbf{h}$ (see (3.7)). Recall also that we obtain an intrinsic weight $p_\eta \colon \mathcal{V} \times \mathcal{V} \to [0, \infty)$ (see Remark 6.28) given by

$$p_\eta(u, v) = \begin{cases} \min\limits_{e \in \mathcal{E}_{u,v}} \eta(e), & u \sim v \text{ and } u \neq v, \\ 0, & \text{else}, \end{cases} \qquad (u, v) \in \mathcal{V} \times \mathcal{V}. \tag{7.24}$$

In Theorem 4.27 (see also Theorem 3.1 (vii)) we have seen that there is a close connection between $\lambda_0(\mathbf{h}_D)$ and $\lambda_0(\mathbf{H}_D)$. In fact, it is easy to notice also connections between the corresponding isoperimetric constants. Namely, suppose that our fixed model of the weighted metric graph $(\mathcal{G}, \mu, \nu)$ has no multiple edges. Then

$$b(u, v) p_\eta(u, v) = \sqrt{\mu\nu}(e_{uv}) \tag{7.25}$$

for all vertices $u \sim v$, $u \neq v$. On the other hand, we can associate to every finite subset $X \subset \mathcal{V}$ the subgraph $\mathcal{K}_X$ of $\mathcal{G}_d$ consisting of all edges in the stars $\mathcal{E}_v$, $v \in X$ (and all incident vertices). Clearly, we have

$$\mu(\mathcal{K}_X) \le m(X) \le 2\mu(\mathcal{K}_X). \tag{7.26}$$

Taking into account the definitions (7.15) and (7.22), this indicates a connection between $\mathrm{Ch}(\mathcal{G})$ and $\mathrm{Ch}_d(\mathcal{V})$. The following explicit estimates hold:

Proposition 7.38. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph having finite intrinsic size and fix a model with underlying combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ having no multiple edges. Then*

$$\mathrm{Ch}(\mathcal{G}) \le 2\mathrm{Ch}_d(\mathcal{V}), \qquad \frac{2}{\mathrm{Ch}(\mathcal{G})} \le \frac{1}{\mathrm{Ch}_d(\mathcal{V})} + \eta^*(\mathcal{E}), \tag{7.27}$$

where $\mathrm{Ch}_d(\mathcal{V})$ is the isoperimetric constant (7.22) of $(\mathcal{V}, m; b)$ for the intrinsic weight given by (7.24). In particular,

$$\mathrm{Ch}(\mathcal{G}) > 0 \quad \textit{exactly when} \quad \mathrm{Ch}_d(\mathcal{V}) > 0. \tag{7.28}$$

Proof. Let $X \subset \mathcal{V}$ be a finite, connected vertex set. Consider the connected subgraph $\mathcal{K}_X$ of $\mathcal{G}_d$ having the edge set $\mathcal{E}(\mathcal{K}_X) := \bigcup_{v \in X} \mathcal{E}_v$. Using (7.25), it is not hard to see that (see also [147, Lemma 4.2])

$$\mathrm{area}(\partial\mathcal{K}_X) \le |\partial X|.$$

Taking into account (7.26), we arrive at the first inequality in (7.27). The rest of the proof can be carried over line to line from [147, Lemma 4.2] and we omit the details. ∎

Remark 7.39. A few remarks are in order.

(i) The second estimate in Proposition 7.38 is sharp. For example, the equality holds true on every *simple unweighted, equilateral metric graph*, that is, when $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is simple and $\mu = \nu \equiv \mathbb{1}$ with $|e| = 1$ for all edges $e \in \mathcal{E}$ (see [173, equation (4.5)]).

(ii) Surprisingly, Proposition 7.38 and even the equivalence (7.28) can fail for models with multiple edges. The reason is precisely that (7.25) is no longer valid in the presence of multiple edges (see also (7.24)). However, the equivalence (7.28) holds true for models having finite intrinsic size and satisfying the additional condition

$$\inf_{e\in\mathcal{E}} \frac{p_\eta(e_\imath, e_\tau)}{\eta(e)} > 0,$$

which clearly allows to recover an adapted version of (7.25).

7.3.3 Volume growth estimates

Going back to the work of R. Brooks [34], another well-known tool for Laplacians on manifolds and graphs are spectral estimates in terms of volume growth (see, e.g., [34, 71, 100, 198] and the references therein). Moreover, these results can be formulated in the abstract framework of Dirichlet forms (see [198] for the strongly local case and [100] for generalizations). In this form, they directly apply to weighted metric graphs and we shortly discuss this in the following.

Let ϱ_η be the intrinsic metric on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ (see Section 6.4.1). For any $x \in \mathcal{G}$ and $r > 0$, we denote an intrinsic distance ball of radius r by

$$B_r(x) = B_r(x; \varrho_\eta) := \{y \in \mathcal{G} : \varrho_\eta(x, y) < r\}. \tag{7.29}$$

The *exponential volume growth* $\mathbf{v}(\mathcal{G})$ of $\mathcal{G}$ is defined by

$$\mathbf{v}(\mathcal{G}) := \liminf_{r\to\infty} \frac{1}{r} \log \mu(B_r(x_0)), \tag{7.30}$$

where x_0 is any point of $\mathcal{G}$ (since $\mathcal{G}$ is connected, the limit in (7.30) does not depend on x_0). Moreover, we also introduce

$$\mathbf{v}_*(\mathcal{G}) := \liminf_{r\to\infty} \frac{1}{r} \inf_{x\in\mathcal{G}} \log \frac{\mu(B_r(x))}{\mu(B_1(x))},$$

where by notational convention $\frac{\infty}{a} := \infty$ for any $a \in (0, \infty]$. Notice in particular that

$$\mathbf{v}_*(\mathcal{G}) \le \mathbf{v}(\mathcal{G}).$$

Applying the results of [198, Theorem 5] (see also [100, Theorem 1.1]), we arrive at the following estimate:

Theorem 7.40. *Suppose that $(\mathcal{G}, \varrho_\eta)$ is complete. Then*

$$\lambda_0(\mathbf{H}_D) \le \lambda_0^{\mathrm{ess}}(\mathbf{H}_D) \le \frac{1}{4}\mathbf{v}_*(\mathcal{G})^2 \le \frac{1}{4}\mathbf{v}(\mathcal{G})^2. \tag{7.31}$$

Remark 7.41. The assumptions in Theorem 7.40 are not optimal. For instance, by Theorem 7.1, the completeness assumption implies that the maximal Kirchhoff Laplacian $\mathbf{H}$ is self-adjoint and hence

$$\mathbf{H}^0 = \mathbf{H}_D = \mathbf{H}_N = \mathbf{H}.$$

On the other hand, the proof in [198, Theorem 5] shows that the Neumann extension $\mathbf{H}_N$ on any weighted metric graph $(\mathcal{G}, \mu, \nu)$ satisfies

$$\lambda_0(\mathbf{H}_N) \le \frac{1}{4}\mathbf{v}_*(\mathcal{G})^2 \le \frac{1}{4}\mathbf{v}(\mathcal{G})^2.$$

In particular, we obtain (7.31) whenever $\mathbf{H}_D = \mathbf{H}_N$, that is, when $\mathbf{H}^0$ admits a unique Markovian extension. The latter is a much weaker condition than the completeness of $(\mathcal{G}, \varrho_\eta)$ (see Section 7.2 and also Theorem 7.24).

7.4 Recurrence and transience

There are numerous characterizations of recurrence/transience and we refer to [78] for further details. Intuitively one may explain recurrence of a Brownian motion/random walk as insufficiency of volume in the state space. The qualitative form of this heuristic statement in the manifold context has a venerable history (we refer to the excellent exposition of A. Grigor'yan [90] for further details) and in the case of complete Riemannian manifolds the corresponding result (see [90, Theorem 7.3]) was proved in the 1980s independently by L. Karp, N.Th. Varopoulos, and A. Grigor'yan. It was extended to strongly local Dirichlet forms by K.-T. Sturm and in our setting of weighted metric graphs, [198, Theorem 3] reads as follows:

Theorem 7.42 ([198]). *Assume that a weighted metric graph $(\mathcal{G}, \mu, \nu)$ is such that $(\mathcal{G}, \varrho_\eta)$ is complete. Then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}})_{t>0}$ generated by the Kirchhoff Laplacian*[4] $\mathbf{H}$ *is recurrent if for some (and hence for all) $x \in \mathcal{G}$,*

$$\int_1^\infty \frac{r}{\mu(B_r(x))}\,\mathrm{d}r = \infty, \tag{7.32}$$

where $B_r(x)$ is the intrinsic metric ball (7.29). That is, the following equivalent properties hold true:

(i) *Every non-negative superharmonic function is constant.*

(ii) *Every bounded superharmonic function is constant.*

(iii) *Every bounded subharmonic function is constant.*

[4]Recall that by Theorem 7.1 completeness implies that the maximal Kirchhoff Laplacian $\mathbf{H}$ is self-adjoint and hence coincides with both the Dirichlet $\mathbf{H}_D$ and Neumann $\mathbf{H}_N$ Laplacian.

(iv) *Every potential*

$$Gf(x) = \lim_{N\to\infty} \int_0^N (\mathrm{e}^{-s\mathbf{H}} f)(x)\,\mathrm{d}s, \quad x \in \mathcal{G},$$

is identically ∞ *for all nonzero* $0 \le f \in L^1(\mathcal{G};\mu)$.

Remark 7.43. In fact, the above result is an immediate consequence of a Karp-type theorem proved for strongly local regular Dirichlet forms in the same paper. More specifically, by [198, Theorem 1], if $(\mathcal{G}, \varrho_\eta)$ is complete, then every nonzero subharmonic function $u \ge 0$ satisfying

$$\int_1^\infty \frac{r}{\|u\mathbb{1}_{B_r(x)}\|^p_{L^p(\mathcal{G};\mu)}}\,\mathrm{d}r = \infty \tag{7.33}$$

for some $p \in (1,\infty)$ and $x \in \mathcal{G}$, is constant. Thus, if $u \ge 0$ is a bounded subharmonic function, then $\|u\mathbb{1}_{B_r(x)}\|^p_{L^p(\mathcal{G};\mu)} \le C\mu(B_r(x))$ and hence (7.33) follows from (7.32), which further implies that u is constant.

Remark 7.44. It appears that in the setting of weighted metric graphs the completeness assumption in both Theorem 7.42 and Karp's theorem is superfluous. Namely, it seems to us that at least in the setting of Theorem 7.24, one can replace this assumption by the Markovian uniqueness (which, according to Theorem 7.24, is equivalent to the absence of finite volume ends).

We would like to demonstrate two applications of the above theorem. First of all, employing connections between intrinsic metrics on weighted graphs and cable systems, we arrive at the analogs of Karp's theorem and Theorem 7.42 for graphs.

Theorem 7.45 ([113]). *Let b be a locally finite, connected graph over $(\mathcal{V}, m)$. Let also ϱ be an intrinsic metric of finite jump size such that $(\mathcal{V}, \varrho)$ is complete and ϱ generates the discrete topology on $\mathcal{V}$. Then every nonzero subharmonic function $u \ge 0$ satisfying*

$$\int_1^\infty \frac{r}{\|u\mathbb{1}_{B_r(v;\varrho)}\|^p_{\ell^p(\mathcal{V};m)}}\,\mathrm{d}r = \infty$$

for some $p \in (1,\infty)$ and $v \in \mathcal{V}$, is constant. In particular, if for some $v \in \mathcal{V}$

$$\int_1^\infty \frac{r}{m(B_r(v;\varrho))}\,\mathrm{d}r = \infty,$$

then the heat semigroup $(\mathrm{e}^{-t\mathbf{h}})_{t>0}$ generated by the graph Laplacian $\mathbf{h}$ is recurrent.

Proof. The proof is analogous to the one of Theorem 6.57. Indeed, assume first that ϱ is an intrinsic path metric for $(\mathcal{V}, m; b)$ having finite jump size. Then by Lemma 6.33 there is a cable system $(\mathcal{G}, \mu, \nu)$ such that ϱ coincides with the restriction $\varrho_\mathcal{V}$ of ϱ_η onto $\mathcal{V} \times \mathcal{V}$ and $(\mathcal{G}, \varrho_\eta)$ is complete.

Take now a non-negative function $\mathbf{f}\colon \mathcal{V} \to \mathbb{R}_{\ge 0}$ which is L-subharmonic. By Lemma 6.52, the corresponding function $f = \imath_{\mathcal{V}}^{-1}(\mathbf{f})$ is non-negative and subharmonic with respect to $(\mathcal{G}, \mu, \nu)$. Taking into account the relationships between the p-norms (see Lemma 4.2) and using the corresponding results for weighted metric graphs, one easily completes the proof of the first claim. The second one follows in a similar way from Theorem 4.17, Lemma 6.41 and Theorem 7.42.

If ϱ is not a path metric, then we proceed as in part (iii) of the proof of Corollary 7.3. Namely, if ϱ has finite jump size, then the construction there gives an intrinsic path metric $\widetilde{\varrho}$ of finite jump size such that $(\mathcal{V}, \widetilde{\varrho})$ is complete and $\varrho \le \widetilde{\varrho}$. It remains to notice that $B_s(x;\widetilde{\varrho}) \subseteq B_s(x;\varrho)$ and then apply the above arguments. ∎

Remark 7.46. Let us mention that Theorem 7.45 was first established in [113] (see Theorem 1.1 and Corollary 1.6 there) by using an absolutely different approach, which, in particular, allows to treat non-locally finite graphs.

To proceed with another application, notice that the characterization of recurrence either in terms of extended Dirichlet spaces (see Lemma B.7) or by means of subharmonic functions indicates that it essentially depends on the energy form only and not on the underlying Hilbert space. In our situation, the energy form depends only on the underlying metric graph $\mathcal{G}$ and the edge weight ν, however, ν enters Theorem 7.42 implicitly as a requirement that $(\mathcal{G}, \varrho_\eta)$ is complete. So, first of all, we arrive at the following result.

Lemma 7.47. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ generated by the Dirichlet Laplacian $\mathbf{H}_D$ is recurrent if $\mathcal{G}$ is complete with respect to the length metric ϱ_0 and for some (and hence for all) $x \in \mathcal{G}$,*

$$\int_1^\infty \frac{r}{\nu(B_r(x;\varrho_0))}\mathrm{d}r = \infty,$$

where $B_r(x;\varrho_0)$ is the metric ball in $(\mathcal{G}, \varrho_0)$.

Proof. As the Dirichlet form of $\mathbf{H}_D$ is regular, recurrence of the corresponding semigroup implies the uniqueness of a Markovian extension for $\mathbf{H}^0$. Moreover, taking into account the regularity of $\mathfrak{Q}_D$ once again, we conclude that $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is recurrent exactly when there is a sequence $(f_n) \subset H^1_c(\mathcal{G})$ which approximates $\mathbb{1}$ and such that $\mathfrak{Q}[f_n] = o(1)$. Next recall that $H^1_c(\mathcal{G})$ is independent of μ. Therefore, if $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is recurrent for some choice of μ, it is automatically recurrent for any other choice of μ. Now it remains to consider the weighted metric graph $(\mathcal{G}, \nu, \nu)$, that is, to replace μ by ν, and apply [198, Theorem 3] (see Theorem 7.42) by taking into account that the length metric ϱ_0 coincides with the intrinsic metric ϱ_η for $(\mathcal{G}, \nu, \nu)$. ∎

Remark 7.48. The above proof indicates that one may come up with a more clever choice of the weight μ (for instance, choosing $\mu(e) = \nu(e)/|e|^2$ for each $e \in \mathcal{E}$, one arrives at Laplacians, which are closely connected with discrete time random walks, see below). However, this of course depends on the concrete situation since, at the same time, one wants to ensure the completeness of $\mathcal{G}$ with respect to the corresponding intrinsic metric ϱ_η, which clearly depends on this choice.

The usefulness of the arguments in the proof of Lemma 7.47 can be demonstrated by the following result. Before stating it, let us associate with the metric graph $\mathcal{G}$ and the edge weight ν the following discrete time random walk: choose a simple model $(\mathcal{V}, \mathcal{E}, |\cdot|, \mu, \nu)$ of $(\mathcal{G}, \mu, \nu)$ and set

$$b_\nu(u,v) = \begin{cases} \dfrac{\nu(e_{u,v})}{|e_{u,v}|}, & u \sim v, \\ 0, & u \not\sim v, \end{cases} \qquad (u,v) \in \mathcal{V}, \tag{7.34}$$

together with

$$m_\nu(v) = \sum_{u \sim v} b_\nu(u,v), \quad v \in \mathcal{V}.$$

Consider the corresponding graph Laplacian (let us denote it by $\mathbf{h}_\nu$). By Lemma 2.9, it is bounded. Moreover, it generates a discrete time random walk on $\mathcal{V}$ (see Remark 2.11). Namely, this random walk on $\mathcal{V}$ is a Markov chain $(X_n)_{n\ge 0}$ with state space $\mathcal{V}$ and transition probabilities $\mathrm{P}_\nu = (p_\nu(u,v))_{u,v\in\mathcal{V}}$ defined by

$$p_\nu(u,v) = P(X_{n+1} = v : X_n = u) = \frac{b_\nu(u,v)}{m_\nu(v)}.$$

Since the graph b over $\mathcal{V}$ is connected by construction, the corresponding Markov chain is *irreducible*. Moreover, it is reversible (again by construction).

Theorem 7.49. *Let $(\mathcal{G}, \mu, \nu)$ be a weighted metric graph. Then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ generated by the Dirichlet Laplacian $\mathbf{H}_D$ is recurrent if and only if for some (and hence for all) simple model of $(\mathcal{G}, \mu, \nu)$ the discrete time random walk on $\mathcal{V}$ with transition probabilities $\mathcal{P}_\nu = (p_\nu(u,v))_{u,v\in\mathcal{V}}$ is recurrent.*

Proof. First, by Theorem 4.17, $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is recurrent if and only if the semigroup $(\mathrm{e}^{-t\mathbf{h}_D})_{t>0}$ is recurrent. Here $\mathbf{h}_D$ is the Dirichlet Laplacian defined by (3.7) (with $\alpha \equiv 0$). Notice that the edge weight b given by (3.6) coincides with b_ν defined by (7.34). Using exactly the same argument as in the proof of Lemma 7.47, however, applied in the discrete graph setting, we conclude that the recurrence of $\mathbf{h}_D$ is independent of the choice of m and hence, in particular, $(\mathrm{e}^{-t\mathbf{h}_D})_{t>0}$ is recurrent if and only if $(\mathrm{e}^{-t\mathbf{h}_\nu})_{t>0}$ is recurrent. However, the latter holds exactly when the corresponding discrete time random walk is recurrent. ■

Remark 7.50. Theorem 7.49 connects the study of recurrence on metric graphs with the study of recurrence for discrete time random walks, which is a classical topic (the standard reference is the book by W. Woess [212]). We shall demonstrate these connections by concrete examples (Cayley graphs and tessellations) in the next chapter. Let us only mention that the idea to relate Brownian motion on a Riemannian manifold with random walks goes back at least to the work of S. Kakutani [121] on the type problem for simply connected Riemann surfaces (see [90] for further details).

7.5 Stochastic completeness

Here we follow the same line of reasoning as in the previous section. Recall the following result of K.-T. Sturm [198, Theorem 4].

Theorem 7.51 ([198]). *Assume that a weighted metric graph $(\mathcal{G}, \mu, \nu)$ is such that $(\mathcal{G}, \varrho_\eta)$ is complete as a metric space. Then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}})_{t>0}$ generated by the Kirchhoff Laplacian $\mathbf{H}$ is stochastically complete if for some (and hence for all) $x \in \mathcal{G}$,*[5]

$$\int_1^\infty \frac{r}{\log \mu(B_r(x))} \mathrm{d}r = \infty,$$

where $B_r(x)$ is the metric ball (7.29).

Remark 7.52. A few remarks are in order.

(i) Let us recall that stochastic completeness means that $\mathrm{e}^{-t\mathbf{H}}\mathbb{1} = \mathbb{1}$ for some (and hence for all) $t > 0$. There are various equivalent characterizations and in terms of λ-harmonic/subharmonic functions stochastic completeness means that:

- for some $\lambda > 0$ every bounded non-negative λ-harmonic function is constant,
- for all $\lambda > 0$ every bounded non-negative λ-subharmonic function is constant.

(ii) In the context of manifolds, the volume test is due to L. Karp and P. Li, and A. Grigor'yan (for a detailed historical account we refer to [90]).

(iii) Similar to the recurrence statement (see Remark 7.44), we are convinced that in the setting of weighted metric graphs the completeness assumption in Theorem 7.51 is superfluous. At least in the setting of Theorem 7.24, one can replace this assumption by the Markovian uniqueness and this will be addressed elsewhere.

Taking into account the relationships between the parabolic properties of Laplacians on metric graphs and weighted graphs (see Section 4.6), we arrive at the following result.

[5] Recall that by Theorem 7.1 completeness implies that the maximal Kirchhoff Laplacian $\mathbf{H}$ is self-adjoint and hence coincides with both the Dirichlet $\mathbf{H}_D$ and Neumann $\mathbf{H}_N$ Laplacian.

Theorem 7.53 ([72, 114]). *Let b be a locally finite, connected graph over $(\mathcal{V}, m)$. Let ϱ be an intrinsic metric of finite jump size such that $(\mathcal{V}, \varrho)$ is complete and ϱ generates the discrete topology on $\mathcal{V}$. If for some (and hence all) $v \in \mathcal{V}$,*

$$\int_1^\infty \frac{r}{\log m(B_r(v;\varrho))} \mathrm{d}r = \infty,$$

where $B_r(v;\varrho)$ is the metric ball in $(\mathcal{V}, \varrho)$, then the semigroup $(\mathrm{e}^{-t\mathbf{h}})_{t>0}$ is stochastically complete.

Proof. For an intrinsic path metric of finite jump size, the proof follows by combining Lemma 6.33 with Theorem 7.51 and Lemma 6.41. Finally, the argument in the proof of Corollary 7.3 allows to reduce to this case. ∎

Remark 7.54. Theorem 7.53 was first proved by M. Folz [72] by using Sturm's theorem 7.51 and also by connecting stochastic completeness on graphs and metric graphs via the corresponding transfer probabilities as described in Section 4.2 (see also [114], where a different proof of the latter connection was given using the weak Omori–Yau maximum principle). A different approach avoiding connections with metric graphs was suggested in [116] and the Grigor'yan volume test is proved under the only assumption that there exists an intrinsic pseudo-metric whose distance balls are finite, that is, there is no finite jump assumption and non-locally finite graphs are allowed as well.