Chapter 8

Examples

The main aim of the final chapter is to demonstrate our findings by considering several important and interesting classes of graphs.

8.1 Antitrees

Recall the following definition (see Section 6.1):

Definition 8.1. A connected simple rooted graph $\mathcal{G}_d$ is called an *antitree* if every vertex in the combinatorial sphere S_n, $n \geq 1$,[1] is connected to all vertices in S_{n-1} and S_{n+1} and no vertices in S_k for all $|k - n| \neq 1$.

Notice that combinatorial antitrees admit radial symmetry and every antitree is uniquely determined by its sphere numbers $s_n = \#S_n$, $n \in \mathbb{Z}_{\geq 0}$ (see Figure 6.1, where the antitree with sphere numbers $s_n = n + 1$, $n \in \mathbb{Z}_{\geq 0}$ is depicted).

8.1.1 Radially symmetric antitrees

Both weighted graph Laplacians and Kirchhoff Laplacians on weighted antitrees admit a very detailed analysis in the situation when their coefficients respect the radial symmetry of the underlying combinatorial antitree. In this subsection we focus on radially symmetric weighted metric antitrees and follow [149] in our exposition. More specifically, we assume that the weighted metric antitree $(\mathcal{A}, \mu, \nu)$ is *radially symmetric*, that is, for each $n \geq 0$, all edges connecting the combinatorial spheres S_n and S_{n+1} have the same length, say $\ell_n > 0$, and the same weights μ and ν, say $\mu_n > 0$ and $\nu_n > 0$.

The next result plays a crucial role in further analysis, however, to state it, we first need to introduce the following objects. Let

$$x_n := \sum_{k=0}^{n-1} \ell_k, \quad \mathcal{L} := \sum_{n \geq 0} \ell_n \in (0, \infty],$$

and then set

$$\mu_{\mathcal{A}}(x) = \sum_{n \geq 0} \mu_n s_n s_{n+1} \mathbb{1}_{[x_n, x_{n+1})}(x), \quad \nu_{\mathcal{A}}(x) = \sum_{n \geq 0} \nu_n s_n s_{n+1} \mathbb{1}_{[x_n, x_{n+1})}(x),$$

[1] By definition, the root o is connected to all vertices in S_1 and no vertices in S_k, $k \geq 2$.

for all $x \in [0, \mathcal{L})$. Notice that $\mathcal{L}$ can be interpreted as the height of a metric antitree. Next, we define three different types of operators associated with the differential expression

$$\tau_{\mathcal{A}} = -\frac{1}{\mu_{\mathcal{A}}(x)}\frac{\mathrm{d}}{\mathrm{d}x}\nu_{\mathcal{A}}(x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{8.1}$$

- The operator $\mathrm{H}_{\mathcal{A}}$ is associated with $\tau_{\mathcal{A}}$ in the Hilbert space $L^2([0, \mathcal{L}); \mu_{\mathcal{A}})$ and acts on the maximal domain subject to the Neumann boundary condition at $x = 0$, see (5.5).
- For each integer $n \geq 1$, the operator H_n^1 is associated with $\tau_{\mathcal{A}}$ in the Hilbert space $L^2([x_n, x_{n+1}); \mu_{\mathcal{A}})$ and with Dirichlet boundary conditions at the endpoints,

$$\mathrm{dom}(\mathrm{H}_n^1) = \{f \in H^2([x_n, x_{n+1})) : f(x_n) = f(x_{n+1}) = 0\}.$$

- For each integer $n \geq 1$, the operator H_n^2 is associated with $\tau_{\mathcal{A}}$ in the Hilbert space $L^2([x_{n-1}, x_{n+1}); \mu_{\mathcal{A}})$ and with Dirichlet boundary conditions at the endpoints,

$$\mathrm{dom}(\mathrm{H}_n^2) = \{f \in H_0^1([x_{n-1}, x_{n+1})) : \nu_{\mathcal{A}} f' \in H^1([x_{n-1}, x_{n+1}))\}.$$

With these definitions at hand, we are in a position to state the key result.

Theorem 8.2. *Let $(\mathcal{A}, \mu, \nu)$ be a radially symmetric antitree. Then the corresponding maximal Kirchhoff Laplacian* $\mathbf{H}$ *is unitarily equivalent to the orthogonal sum*

$$\mathrm{H}_{\mathcal{A}} \oplus \bigoplus_{n\geq 1} (\mathrm{I}_{(s_n-1)(s_{n+1}-1)} \otimes \mathrm{H}_n^1) \oplus \bigoplus_{n\geq 1} (\mathrm{I}_{s_n-1} \otimes \mathrm{H}_n^2). \tag{8.2}$$

Here $s_n = \#S_n$, $n \geq 0$ are the sphere numbers of $\mathcal{A}$ and I_k is the identity operator in $\mathbb{C}^k$, $k \in \mathbb{Z}_{\geq 0}$.

Proof. Follows line by line the proof of [149, Theorem 3.5] (see also [31]), where the case $\mu = \nu \equiv \mathbb{1}$ is considered, and we omit it. Let us only mention that the operator $\mathrm{H}_{\mathcal{A}}$ is nothing but the restriction of $\mathbf{H}$ onto the subspace $\mathcal{F}_{\mathrm{sym}}$ of radially symmetric functions

$$\mathcal{F}_{\mathrm{sym}} = \{f \in L^2(\mathcal{A}; \mu) : f(x) = f(y) \text{ if } \varrho_0(x, o) = \varrho_0(y, o)\},$$

which follows easily by comparing the corresponding quadratic forms. Here $\varrho_0(x, o)$ denotes the distance from the point $x \in \mathcal{A}$ to the root o of $\mathcal{A}$ with respect to the length metric ϱ_0. ■

Thus, Theorem 8.2 reduces the analysis of the Kirchhoff Laplacian $\mathbf{H}$ on $(\mathcal{A}, \mu, \nu)$ to the analysis of Sturm–Liouville operators (8.1). In particular, since both H_n^1 and H_n^2 are self-adjoint and have purely discrete simple spectra for each $n \geq 1$, the operator $\mathrm{H}_{\mathcal{A}}$ acting in $L^2([0, \mathcal{L}); \mu_{\mathcal{A}})$ encodes the main spectral and parabolic properties of $\mathbf{H}$. Moreover, take into account that $\mathrm{H}_{\mathcal{A}}$ allows a rather detailed treatment (see Chapter 5). First of all, we easily obtain the following characterization of the self-adjoint and Markovian uniqueness.

Theorem 8.3. *Let $(\mathcal{A}, \mu, \nu)$ be a radially symmetric antitree.*

(i) *The Kirchhoff Laplacian* $\mathbf{H}$ *is self-adjoint if and only if the series*

$$\sum_{n\geq 0} s_n s_{n+1}\mu_n \ell_n \Big(\sum_{k\leq n} \frac{\ell_k}{s_k s_{k+1}\nu_k}\Big)^2 \tag{8.3}$$

diverges. If the series converges, then the deficiency indices of the minimal Kirchhoff Laplacian $\mathbf{H}^0 = \mathbf{H}^*$ *equal* 1.

(ii) *The Kirchhoff Laplacian* $\mathbf{H}$ *admits a unique Markovian restriction if and only if either it is self-adjoint or the series*

$$\mathcal{L}_\nu^{\mathcal{A}} := \sum_{n\geq 0} \frac{\ell_n}{s_n s_{n+1}\nu_n} \tag{8.4}$$

diverges.

Proof. Taking into account decomposition (8.2) and the self-adjointness of the second and the third summands, the self-adjoint uniqueness (resp., Markovian uniqueness) for $\mathbf{H}$ is equivalent to the self-adjoint uniqueness (resp., Markovian uniqueness) for $\mathrm{H}_{\mathcal{A}}$. Applying Lemma 5.2 and Lemma 5.11, we prove (i) and, respectively, (ii). ■

Remark 8.4. It might be useful to compare the self-adjointness criterion obtained in Theorem 8.3 with the Gaffney-type results from Section 7.1.1. Taking into account that by the Hopf–Rinow theorem (see Section 6.4.5), completeness is equivalent to the geodesic completeness, we conclude:

(i) $(\mathcal{A}, \varrho_\eta)$ is complete exactly when (cf. Theorem 7.1)

$$\sum_{n\geq 0} \ell_n \sqrt{\frac{\mu_n}{\nu_n}} = \infty.$$

(ii) if, for simplicity,[2]

$$\sup_n \ell_n \sqrt{\frac{\mu_n}{\nu_n}} < \infty,$$

then $(\mathcal{V}, \varrho_m)$ is complete exactly when (cf. Theorem 7.7)

$$\sum_{n\geq 0} (s_n + s_{n+1})\ell_n \mu_n = \infty.$$

[2] Here we need to take into account the definition of the vertex weight in Section 3.1

On the one hand, the last condition is equivalent to (8.3) only under the restrictive assumptions that (a) $\mathcal{L}_\nu^{\mathcal{A}} < \infty$, and (b) $s_n s_{n+1} \lesssim s_n + s_{n+1}$ for all n. On the other hand, its main drawback that it does not take ν into account.

The next immediate corollary is of some interest when one looks at the self-adjointness and Markovian uniqueness by using graph ends (cf. Section 7.2.1).

Corollary 8.5. *Let $(\mathcal{A}, \mu, \nu)$ be a radially symmetric antitree.*

(i) *If*

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} \mu(\mathrm{d}x) = \sum_{n\geq 0} s_n s_{n+1} \mu_n \ell_n = \infty, \tag{8.5}$$

then the Kirchhoff Laplacian $\mathbf{H}$ *is self-adjoint. Moreover,* (8.5) *is also necessary for the self-adjointness if* $\mathcal{L}_\nu^{\mathcal{A}} < \infty$.

(ii) *If* $\mathcal{L}_\nu^{\mathcal{A}} < \infty$, *then the Kirchhoff Laplacian* $\mathbf{H}$ *admits a unique Markovian restriction if and only if* $\mu(\mathcal{A}) = \infty$.

Remark 8.6. Every infinite antitree has exactly one graph end. By Definition 7.23, this graph end has finite volume if and only if the total volume of a given antitree is finite, $\mu(\mathcal{A}) < \infty$. By Corollary 8.5, the absence of finite volume ends is equivalent to both self-adjoint and Markovian uniqueness exactly when $\mathcal{L}_\nu^{\mathcal{A}} < \infty$, that is, when the series in (8.4) converges.

Remark 8.7. If $\mathbf{H}$ is not self-adjoint, then one can describe its self-adjoint restrictions in the following way. First of all, the decomposition (8.2) implies that it suffices to restrict to the subspace of spherically symmetric functions: for each $f \in \mathrm{dom}(\mathbf{H})$, define the function $f_{\mathrm{sym}}\colon [0, \mathcal{L}) \to \mathbb{C}$ by setting

$$f_{\mathrm{sym}}(x) = \frac{1}{s(x)} \sum_{y\in\mathcal{A}:\varrho_0(o,y)=x} f(y),$$
$$s(x) = \sum_{n\geq 0} s_n s_{n+1} \mathbb{1}_{[x_n,x_{n+1})}(x).$$

It is straightforward to check that $f_{\mathrm{sym}} \in \mathrm{dom}(\mathrm{H}_{\mathcal{A}})$ (cf. [149, Lemma 3.2]). Next, define

$$f_{\mathrm{sym}}(\mathcal{L}) := \lim_{x\to\mathcal{L}} \left(f_{\mathrm{sym}}(x) - \nu_{\mathcal{A}}(x) f'_{\mathrm{sym}}(x) \int_0^x \frac{\mathrm{d}s}{\nu_{\mathcal{A}}(s)} \right),$$
$$f'_{\mathrm{sym}}(\mathcal{L}) := \lim_{x\to\mathcal{L}} \nu_{\mathcal{A}}(x) f'_{\mathrm{sym}}(x).$$

By Lemma 5.5, both limits exist for each $f \in \mathrm{dom}(\mathbf{H})$ and applying (5.9), we conclude that the one-parameter family $\mathbf{H}_\theta$, $\theta \in [0, \pi)$ of self-adjoint restrictions of $\mathbf{H}$ is explicitly given by

$$\mathrm{dom}(\mathbf{H}_\theta) = \{f \in \mathrm{dom}(\mathbf{H}) : \cos(\theta) f_{\mathrm{sym}}(\mathcal{L}) + \sin(\theta) f'_{\mathrm{sym}}(\mathcal{L}) = 0\}. \tag{8.6}$$

Corollary 8.8. *Let* $\mathbf{H}$ *be non-self-adjoint. If* $\mathcal{L}_\nu^{\mathcal{A}} < \infty$*, then the corresponding Dirichlet Laplacian is given by*

$$\operatorname{dom}(\mathbf{H}_D) = \Big\{ f \in \operatorname{dom}(\mathbf{H}) : \lim_{x\to\mathcal{L}} f_{\mathrm{sym}}(x) = 0 \Big\}.$$

Otherwise, the Dirichlet Laplacian coincides with the Neumann Laplacian

$$\operatorname{dom}(\mathbf{H}_N) = \operatorname{dom}(\mathbf{H}_{\pi/2}) = \Big\{ f \in \operatorname{dom}(\mathbf{H}) : \lim_{x\to\mathcal{L}} \nu_{\mathcal{A}}(x) f'_{\mathrm{sym}}(x) = 0 \Big\}.$$

Proof. If $\mathcal{L}_\nu^{\mathcal{A}} = \int_0^{\mathcal{L}} \frac{ds}{\nu_{\mathcal{A}}(s)} < \infty$, then boundary conditions can be written in a standard way since in this case

$$f_{\mathrm{sym}}(\mathcal{L}) = \lim_{x\to\mathcal{L}} f_{\mathrm{sym}}(x) - \mathcal{L}_\nu^{\mathcal{A}} f'_{\mathrm{sym}}(\mathcal{L}),$$

which implies that the limit on the right-hand side exists for all $f \in \operatorname{dom}(\mathbf{H})$. Hence we can replace $f_{\mathrm{sym}}(\mathcal{L})$ in (8.6) by $\widetilde{f}_{\mathrm{sym}}(\mathcal{L}) := \lim_{x\to\mathcal{L}} f_{\mathrm{sym}}(x)$. Taking into account the definition of the Dirichlet Laplacian, this implies the first claim. The second one follows from Theorem 8.3 (ii). ∎

If $\mathbf{H}$ is not self-adjoint, then the spectral analysis is reduced to that of $\mathrm{H}_{\mathcal{A}}$ and Lemma 5.5. Therefore, in the following results we restrict to the case when $\mathbf{H}$ is self-adjoint, that is, the series (8.3) diverges. Using Lemma 5.7, we arrive at the next result.

Lemma 8.9. *Suppose that the Kirchhoff Laplacian* $\mathbf{H}$ *is self-adjoint. Then:*

(i) *We have* $\lambda_0(\mathbf{H}) > 0$ *if and only if*

$$\mathcal{L}_\nu^{\mathcal{A}} < \infty \quad \text{and} \quad \sup_{n\ge 0} \sum_{k\le n} s_k s_{k+1} \mu_k \ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1} \nu_k} < \infty. \tag{8.7}$$

(ii) *We have* $\lambda_0^{\mathrm{ess}}(\mathbf{H}) > 0$ *if and only if either* (8.7) *holds true or*

$$\mathcal{L}_\nu^{\mathcal{A}} = \infty \quad \text{and} \quad \sup_{n\ge 0} \sum_{k\le n} \frac{\ell_k}{s_k s_{k+1} \nu_k} \sum_{k\ge n} s_k s_{k+1} \mu_k \ell_k < \infty. \tag{8.8}$$

(iii) *The spectrum of* $\mathbf{H}$ *is purely discrete if and only if*

- *either* $\mathcal{L}_\nu^{\mathcal{A}} < \infty$ *and*

$$\lim_{n\to\infty} \sum_{k\le n} s_k s_{k+1} \mu_k \ell_k \sum_{k\ge n} \frac{\ell_k}{s_k s_{k+1} \nu_k} = 0,$$

- *or* $\mu(\mathcal{A}) < \infty$ *and*

$$\lim_{n\to\infty} \sum_{k\le n} \frac{\ell_k}{s_k s_{k+1} \nu_k} \sum_{k\ge n} s_k s_{k+1} \mu_k \ell_k = 0.$$

Proof. Taking into account the decomposition (8.2), observe that

$$\lambda_0(\mathbf{H}) = \lambda_0(\mathrm{H}_{\mathcal{A}}), \quad \lambda_0^{\mathrm{ess}}(\mathbf{H}) = \lambda_0^{\mathrm{ess}}(\mathrm{H}_{\mathcal{A}})$$

since $\lambda_0(\mathrm{H}_{\mathcal{A}}) \le \lambda_0(\mathrm{H}_n^j)$ for all $n \ge 1$, as well as $\lambda_0^{\mathrm{ess}}(\mathrm{H}_{\mathcal{A}}) \le \liminf_{n\to 0} \lambda_0(\mathrm{H}_n^j)$, $j \in \{1,2\}$, which follows by using the variational characterization of λ_0 provided by the Rayleigh quotient. Thus, applying Lemma 5.7, we complete the proof. ■

Remark 8.10. A few remarks are in order.

(i) If $\mathbf{H}$ is not self-adjoint, then one can conclude that the spectrum of each self-adjoint restriction $\mathbf{H}_\theta$ (see (8.6)) is purely discrete. Furthermore, taking into account that

$$\sigma(\mathrm{H}_n^1) = \left\{\frac{\pi^2 k^2}{\eta_n^2}\right\}_{k\in\mathbb{Z}_{\ge 1}},$$

where $\eta_n = \ell_n\sqrt{\mu_n/\nu_n}$, $n \ge 0$ are the intrinsic edge lengths, the Weyl law (5.10) for $\mathrm{H}_{\mathcal{A}}$ together with the standard Dirichlet–Neumann bracketing argument applied to H_n^2 (see the proof of [149, Corollary 5.1]), one arrives at the Weyl law for self-adjoint restrictions of $\mathbf{H}$:[3]

$$\lim_{\lambda\to\infty} \frac{N(\lambda;\mathbf{H}_\theta)}{\sqrt{\lambda}} = \frac{1}{\pi} \times \text{intrinsic volume of } \mathcal{A}, \tag{8.9}$$

and the *intrinsic volume* of $\mathcal{A}$ is

$$\eta(\mathcal{A}) = \int_{\mathcal{A}} \eta(\mathrm{d}x) = \sum_{n\ge 0} s_n s_{n+1}\eta_n = \sum_{n\ge 0} s_n s_{n+1}\ell_n\sqrt{\frac{\mu_n}{\nu_n}}.$$

(ii) If $\mathbf{H}$ is self-adjoint, however, has purely discrete spectrum, then Weyl's law (8.9) still takes place. If $\eta(\mathcal{A}) = \infty$, then one can prove criteria for the inclusion $(\mathbf{H} + \mathrm{I})^{-1} \in \mathfrak{S}_p$, $p \in (\frac{1}{2}, \infty)$ (see Remark 5.8 and [149, Theorem 5.6 and Remark 5.7]).

The following result provides an explicit form of the isoperimetric constant for $(\mathcal{A}, \mu, \nu)$ in the radially symmetric case.

Proposition 8.11. *The isoperimetric constant of a radially symmetric metric antitree* $(\mathcal{A}, \mu, \nu)$ *is*

$$\mathrm{Ch}(\mathcal{A}) = \inf_{n\ge 0} \frac{s_n s_{n+1}\sqrt{\mu_n\nu_n}}{\sum_{k=0}^{n} s_k s_{k+1}\mu_k\ell_k}. \tag{8.10}$$

In particular, the following estimate holds true:

$$\lambda_0(\mathbf{H}_D) \ge \frac{1}{4}\mathrm{Ch}(\mathcal{A})^2.$$

[3] Here $N(\lambda; A)$ is the eigenvalue counting function of a (bounded from below) self-adjoint operator A with purely discrete spectrum: $N(\lambda; A) = \#\{k : \lambda_k(A) \le \lambda\}$, where $\{\lambda_k(A)\}_{k\ge 0}$ are the eigenvalues of A (counting multiplicities) in increasing order.

Proof. The decomposition (8.2) as well as the proof of Lemma 8.9 suggests taking the infimum in (7.15) only over radially symmetric subgraphs. Thus, evaluating (7.15) over subantitrees $\mathcal{A}_n$, where one cuts out the part of $\mathcal{A}$ above the combinatorial sphere S_n, the inequality "$\le$" in (8.10) is trivial. The proof of the converse inequality "$\ge$" follows line by line the proof of [149, Theorem 7.1] and we leave it to the reader. ■

Applying the volume growth estimates from Section 7.3.3, we arrive at the following upper bounds.

Proposition 8.12. *Suppose that the radially symmetric antitree* $(\mathcal{A}, \mu, \nu)$ *has infinite intrinsic height (i.e.,* $(\mathcal{A}, \varrho_\eta)$ *is complete),*

$$\sum_{n\ge 0} \eta_n = \sum_{n\ge 0} \ell_n \sqrt{\frac{\mu_n}{\nu_n}} = \infty.$$

Then $\mathbf{H}$ *is self-adjoint and*

$$\lambda_0(\mathbf{H}) \le \frac{1}{4}\mathbf{v}(\mathcal{A})^2, \quad \mathbf{v}(\mathcal{A}) = \liminf_{n\to\infty} \frac{1}{\sum_{k\le n} \eta_k} \log\Big(\sum_{k\le n} s_k s_{k+1} \mu_k \ell_k\Big).$$

Remark 8.13. It might be useful to compare the isoperimetric and volume growth bounds with the positive spectral gap criterion obtained in Lemma 8.9 (i)–(ii). It is rather curious that the volume of the sub-antitrees $\mathcal{A}_n$ (defined in the proof of Proposition 8.11),

$$\sum_{k\le n} s_k s_{k+1} \mu_k \ell_k$$

enters all the estimates and criteria. However, it appears there in rather different ways. The meaning of the quantity

$$\sum_k \frac{\ell_k}{s_k s_{k+1} \nu_k}$$

in both (8.7) and (8.8) remains unclear to us, however, it plays crucial role in understanding both spectral and parabolic properties of the Kirchhoff Laplacian.

Let us finish this subsection by quickly discussing basic parabolic properties.

Lemma 8.14. *Let* $\mathbf{H}_G$ *be the Gaffney Laplacian on a radially symmetric antitree* $(\mathcal{A}, \mu, \nu)$*. If* $\mathbf{H}_G$ *is self-adjoint, then it is recurrent if and only if* $\mathcal{L}_\nu^{\mathcal{A}} = \infty$*. If* $\mathbf{H}_G$ *is not self-adjoint, then* $\mathbf{H}_\theta$ *is recurrent if and only if* $\theta = \frac{\pi}{2}$*.*

Proof. By Lemma B.5, recurrence is equivalent to the fact that there is a sequence approximating (in a suitable sense) the constant function $\mathbb{1}$. However, $\mathbb{1}$ is radially symmetric and thus belongs to the reducing subspace $\mathcal{F}_{\mathrm{sym}}$ of all radially symmetric functions. Thus, $\mathbf{H}_G$ is recurrent exactly when so is its radial part $\mathrm{H}_{\mathcal{A}}$. It remains to apply Lemma 5.13. ■

Lemma 8.15. *Let* $\mathbf{H}_G$ *be the Gaffney Laplacian on a radially symmetric antitree* $\mathcal{A}$*. If* $\mathbf{H}_G$ *is self-adjoint, then it is stochastically incomplete if and only if*

$$\mathcal{L}_\nu^{\mathcal{A}} < \infty \quad \textit{and} \quad \frac{1}{\nu_{\mathcal{A}}(x)} \int_0^x \mu_{\mathcal{A}}(s)\,\mathrm{d}s \in L^1([0, \mathcal{L})).$$

Proof. By the very definition of stochastic completeness (B.2), decomposition (8.2) clearly reduces the problem to the stochastic completeness of the operator $\mathrm{H}_{\mathcal{A}}$ since $\mathbb{1}_{\mathcal{A}} \in \mathcal{F}_{\mathrm{sym}}$. It remains to apply Lemma 5.14. ∎

8.1.2 General case

Removing the symmetry assumption, that is, if at least one of the weights μ or ν or the lengths $|\cdot|$ are no longer radially symmetric, the analysis of the Kirchhoff Laplacian becomes much more complicated. The very first problem – the self-adjoint uniqueness – remains open and, as the next example from [146, Section 7] demonstrates, far from being trivial.

Example 8.16 (Antitrees with arbitrary deficiency indices). We shall assume that the metric antitree is unweighted, that is, $\mu = \nu = \mathbb{1}$ on $\mathcal{A}$ (notice that both weights are radially symmetric). Fix $N \in \mathbb{Z}_{\ge 1}$ and consider the antitree $\mathcal{A}_N$ with sphere numbers $s_n = n + N$, $n \in \mathbb{Z}_{\ge 1}$ (for $N = 1$ this antitree is depicted on Figure 6.1). To assign lengths, let us enumerate the vertices in every combinatorial sphere S_n by $(v_i^n)_{i=1}^{s_n}$ and then denote the edge connecting v_i^n with v_j^{n+1} by e_{ij}^n, $1 \le i \le s_n$, $1 \le j \le s_{n+1}$ and $n \ge 0$. For a sequence of positive real numbers $(\ell_n)_{n\ge 0}$, we first assign edge lengths

$$|e_{ij}^n| = \begin{cases} 2\ell_n, & \text{if } 1 \le i = j \le N, \\ \ell_n, & \text{otherwise}, \end{cases}$$

for all $n \in \mathbb{Z}_{\ge 0}$. It turns out that for the corresponding metric antitree $\mathcal{A}_N$ the space of harmonic functions has dimension $N + 1$ (see Lemma 7.4 in [146]). Choosing lengths such that $\mathrm{vol}(\mathcal{A}_N) \asymp \sum_{n\ge 1} n^2 \ell_n < \infty$, the deficiency indices of the minimal Kirchhoff Laplacian $\mathbf{H}^0$ are equal to the dimension of the space of harmonic functions belonging to $L^2(\mathcal{A})$. By [146, Proposition 7.5], if we choose lengths such that

$$\ell_n = \mathcal{O}\left(\frac{1}{(36N)^n((n + N + 3)!)^2}\right), \quad n \to \infty,$$

then all harmonic functions belong to $L^2(\mathcal{A})$ and hence $\mathrm{n}_\pm(\mathbf{H}^0) = N + 1$.

Remark 8.17. A few concluding remarks are in order.

(i) Slightly modifying the antitree in Example 8.16 one can construct an example of a metric antitree such that the corresponding minimal Kirchhoff Laplacian has infinite deficiency indices (see [146, Section 7.4]). The above example also demonstrates that the space of harmonic functions, even in the unweighted case, depends in a complicated way on the choice of edge lengths (notice that in the radially symmetric case constants are the only harmonic functions). Thus, the self-adjoint uniqueness becomes a highly non-trivial problem already in the case $\mu = \nu = \mathbb{1}$.

(ii) In contrast to the self-adjoint uniqueness in the case of no radial symmetry, the Markovian uniqueness problem can be answered in several situations of interest. For example, in the case $\mu = \nu = \mathbb{1}$ it was observed in [146] that the Markovian uniqueness is equivalent to the infinite total volume of $\mathcal{A}$ (and the latter is independent of whether the antitree is radially symmetric or not). Moreover, the results of Section 7.2 extend this claim to a much wider setting: if at least one the two conditions

(a) $\frac{1}{\mu}, \frac{1}{\nu} \in L^\infty(\mathcal{A})$, or

(b) $\mathcal{A}$ has finite ν-diameter $D_{1/\nu}(\mathcal{A}) < \infty$, see (7.12),

is satisfied, then the minimal Kirchhoff Laplacian admits a unique Markovian uniqueness if and only if $\mu(\mathcal{A}) = \infty$. If $\mu(\mathcal{A}) < \infty$, then $\mathbf{H}$ admits a one-parameter family of Markovian extensions and their description is very much similar to the one in the radial case. Let us also stress that in the radially symmetric case the condition relating Markovian uniqueness with infinite total volume is $\mathcal{L}_\nu^{\mathcal{A}} < \infty$ (see (8.4)), and this condition is much weaker than both (a) and (b).

8.1.3 Historical remarks and further references

Antitrees also appear in the literature under the name *neural networks* and to a certain extent the corresponding graph Laplacians can be seen a generalization of Jacobi matrices (one may interpret the recurrence relations as "*the values on S_n depend only on the values on S_{n-1} and S_{n+1}*"). Seems, exactly this fact allows to perform a rather detailed analysis of Laplacians (both weighted graph and Kirchhoff) on antitrees. Below we collect some further information.

8.1.3.1 Spectral analysis in the radially symmetric case. The decomposition (8.2) of the maximal Kirchhoff Laplacian in the radially symmetric case reduces the spectral analysis to the study of a Sturm–Liouville operator $\mathrm{H}_{\mathcal{A}}$. One may employ a number of results and techniques available in the one-dimensional setting. In particular, we briefly listed the very basic results (self-adjointness, positive spectral gap, discreteness, etc.). However, one can prove a number of results characterizing the structure of the spectrum of $\mathbf{H}$ in the self-adjoint case. In particular, [149, Section 8] shows that the occurrence of absolutely continuous spectrum is a rather rare event. Antitrees with zero-measure spectrum can be found in [49]. However, using Lemma 5.9, one can construct a rather large and non-trivial class of antitrees whose absolutely continuous spectrum fills the positive semi-axis $[0, \infty)$ (see [149, Section 9]).

8.1.3.2 Family preserving graphs. An antitree is just a particular example of an infinite graph having a lot of symmetry. Actually, antitrees belong to the wider class of *family preserving graphs* (see [30] for definitions), which, in particular, includes rooted radially symmetric trees. The decomposition (8.2) is motivated by a similar decomposition for Laplacians on radially symmetric metric trees observed by K. Naimark and M. Solomyak [169, 170, 196]. For this very reason Laplacians on radially symmetric trees form the most studied class of operators on metric graphs. The literature is enormous and we refer for further references to [31].

Notice that the analog of the decomposition (8.2) for family preserving metric graphs was obtained in [31], however, in contrast to graph Laplacians [30], the setting of [31] excludes graphs with horizontal edges.

8.1.3.3 Historical remarks. Antitrees appear in the study of *discrete Laplacians* on graphs at least since the 1980s [60] (see [48, Section 2] for a historical overview). They played an important role in context with the notion of intrinsic metrics on graphs (see Section 6.4). More precisely, in [213] (see also [135, Section 6] and [92]) R. K. Wojciechowski constructed antitrees of polynomial volume growth (with respect to the combinatorial metric ϱ_{comb}, which is in general not intrinsic) for which the (discrete) combinatorial Laplacian L_{comb} (see Example 6.7) is stochastically incomplete and the bottom of the essential spectrum is strictly positive. At first, these examples presented a sharp contrast to the manifold setting (cf. [34, 90]), but the discrepancies were resolved later by the notion of intrinsic metrics. In this context, antitrees appear as key examples for certain thresholds (see [100, 129]). During the recent years, antitrees were also actively studied from other perspectives and we only refer to a brief selection of articles [30, 31, 48, 149], where further references can be found.

8.2 Cayley graphs

Let G be a countable finitely generated group and let S be a generating set of G. We shall always assume that

- G is countably infinite,
- S is symmetric, $S = S^{-1}$ and finite, $\#S < \infty$,
- the identity element of G does not belong to S (this excludes loops).

The *Cayley graph* $\mathcal{G}_C = \mathscr{C}(\mathsf{G}, S)$ of G with respect to S is the simple graph whose vertex set coincides with G and two vertices $x, y \in \mathcal{G}_C$ are neighbors $x \sim y$ if and only if $xy^{-1} \in S$.

The main aim of this section is to demonstrate some of our findings as well as their relationships with large scale properties of groups. Notice that Cayley graphs corresponding to two different generating sets are quasi-isometric as metric spaces when equipped with the combinatorial distance (word metric), which in particular indicates that many properties of interest are independent of the choice of S (see, for instance, [54, 175, 187] for further details). To simplify our considerations we shall restrict throughout most of Section 8.2 to weighted metric graphs with $\mu = \nu$, that is, the edge weights μ and ν are assumed to coincide.

8.2.1 Markovian uniqueness

The self-adjointness for Kirchhoff Laplacians is a very complicated problem already for abelian groups $(\mathbb{Z}^N, +)$ with $N \geq 2$ (it does not seem to us that a complete answer even in this "simplest" situation is feasible, see also Remark 8.25 below). One can obtain various sufficient conditions by directly applying the results of Section 7.1 (e.g., Gaffney-type theorems) and we leave this to the interested reader. Our first goal is to investigate the Markovian uniqueness on metric Cayley graphs, which is equivalent to the self-adjointness of the corresponding Gaffney Laplacian $\mathbf{H}_G$.

Proposition 8.18. *Let $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ be a Cayley graph.*[4] *Suppose $(\mathcal{G}_C, \mu, \mu)$ is a weighted metric graph whose edge weight μ satisfies $\frac{1}{\mu} \in L^\infty(\mathcal{G})$. Then the deficiency indices of the corresponding minimal Gaffney Laplacian $\mathbf{H}_{G,\min} = \mathbf{H}_G^*$ coincide with the number of finite volume graph ends of $(\mathcal{G}_C, \mu, \mu)$.*

Proof. This immediately follows from Theorem 7.24. ∎

Remark 8.19 (Ends of Cayley graphs). Graph ends of countable finitely generated groups are rather well understood (see [82]). It is not difficult to see that the graphs depicted in Figure 8.1 have, respectively, 2, 1 and infinitely many ends. However, by the Freudenthal–Hopf theorem, only these three options are possible: *a Cayley graph of an infinite finitely generated group has* 1*,* 2 *or infinitely many ends*. Moreover, the end space (equipped with the topology of the end compactification) of $\mathcal{C}(\mathsf{G}, S)$ is independent of the choice of the finite generating set S and hence we shall denote the set of ends by $\mathfrak{C}(\mathsf{G})$. By Hopf's theorem, $\#\mathfrak{C}(\mathsf{G}) = 2$ if and only if G is *virtually infinite cyclic*[5] (equivalently, G has a finite normal subgroup Γ such that the quotient group G/Γ is either infinite cyclic or infinite dihedral). The classification of finitely generated groups with infinitely many ends (equivalently, with exactly 1 end) is due to J.R. Stallings (see, e.g., [82, Chapter 13]). In particular, if G is amenable, then it has finitely many ends (actually, either 1 or 2).

[4] If it is not explicitly stated otherwise, we shall denote by $\mathcal{G}_C$ both a Cayley graph and a metric graph $\mathcal{G}_C$ equipped with some edge lengths.

[5] If a finite index subgroup of G has property "P", then G is called *virtually "P"*.

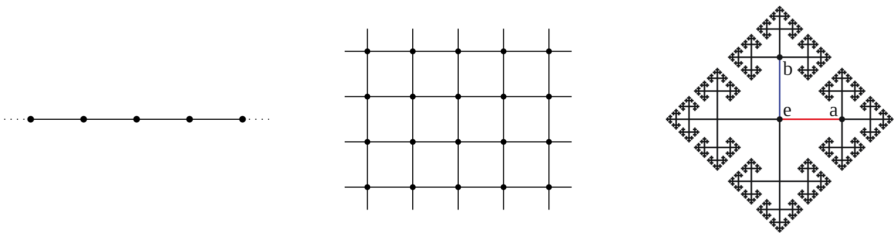


Figure 8.1. Cayley graphs of the abelian groups $\mathbb{Z}$, $\mathbb{Z}^2$ and the free nonabelian group $\mathbb{F}_2$ (the Bethe lattice or infinite Cayley tree).

Thus, we arrive at the following result.

Corollary 8.20. *Assume the conditions of Proposition 8.18. Let also $\mathbf{H}_G$ be the corresponding Gaffney Laplacian.*

(i) *If $\#\mathfrak{C}(\mathsf{G}) = 1$, then $\mathbf{H}_G$ is self-adjoint if and only if $\mu(\mathcal{G}) = \infty$. Otherwise,* $\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = 1$.

(ii) *If $\#\mathfrak{C}(\mathsf{G}) = 2$ (i.e., G is virtually infinite cyclic), then $\mathrm{n}_\pm(\mathbf{H}_{G,\min}) \le 2$. In particular, $\mathbf{H}_G$ is self-adjoint if and only if both ends have infinite volume.*

(iii) *If $\#\mathfrak{C}(\mathsf{G}) > 2$ and at least one of its ends has finite volume, then $\mathbf{H}_{G,\min}$ has infinite deficiency indices.*

(iv) *If $\mu(\mathcal{G}) < \infty$, then the deficiency indices of $\mathbf{H}_{G,\min}$ are equal to the number of ends of G, $\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = \#\mathfrak{C}(\mathsf{G})$.*

Proof. Note that (i), (ii) and (iv) are immediate consequences of Proposition 8.18.

(iii) By the Freudenthal–Hopf theorem, we have

$$\#\mathfrak{C}(\mathsf{G}) = \infty \quad \text{if } \#\mathfrak{C}(\mathsf{G}) > 2$$

(see Remark 8.19). Moreover, the end space is known to be homeomorphic to the Cantor set (see, e.g., [82, Addendum 13.5.8]), and hence there are no free graph ends. Thus, having 1 finite volume end would immediately imply the presence of infinitely many finite volume graph ends. It remains to apply Proposition 8.18. ■

Taking into account that the self-adjointness of $\mathbf{H}_G$ is equivalent to the Markovian uniqueness for the minimal Kirchhoff Laplacian, we arrive at the following characterization in the case of amenable groups.

Corollary 8.21. *Assume the conditions of Proposition 8.18. If G is amenable and not virtually infinite cyclic, then the minimal Kirchhoff Laplacian admits a unique Markovian extension if and only if*

$$\mu(\mathcal{G}_C) = \int_{\mathcal{G}_C} \mu = \sum_{e \in \mathcal{E}} \mu(e)|e| = \infty.$$

Remark 8.22. For Cayley graphs of infinite groups with finitely many ends one can describe the sets of Markovian and finite energy extensions of the minimal Kirchhoff Laplacian in a rather transparent way (see, e.g., Section 7.2.2 and [146, Section 6], [148]). If G has infinitely many ends and the Gaffney Laplacian is not self-adjoint, then it is not closed (see [148, Corollary 3.14]) and the description of its closure is an open problem (even if $\mu \equiv \mathbb{1}$). Moreover, in some cases its closure may coincide with the maximal Kirchhoff Laplacian (for instance, if $\mathcal{G}_C$ is a Cayley graph of the free group $\mathbb{F}_2$ and $\mu(\mathcal{G}_C) < \infty$, see [148, Lemma 4.6]). In our opinion, the description of finite energy extensions (via boundary conditions) in the general case is a highly non-trivial problem (see Sections 7.2.2–7.2.3). On the other hand, Markovian extensions can still be described in terms of Dirichlet forms (in the wide sense) on the Royden boundary [133], however this correspondence is in general not bijective (see Section 7.2.2 for a detailed discussion).

Since the deficiency indices of the minimal Kirchhoff Laplacian are not smaller than the deficiency indices of the Gaffney Laplacian, Corollary 8.20 immediately provides us with the following result.

Corollary 8.23. *Assume the conditions of Proposition 8.18. Let also* $\mathbf{H}^0$ *be the corresponding minimal Kirchhoff Laplacian. If* $\#\mathfrak{C}(\mathsf{G}) > 2$ *and at least one of its ends has finite volume, then* $\mathrm{n}_{\pm}(\mathbf{H}^0) = \infty$.

Let us consider the simplest example.

Example 8.24 (Infinite cyclic group). Let $\mathsf{G} = (\mathbb{Z}, +)$ be the infinite cyclic group and $S = \{-1, 1\}$ the standard set of generators. Then $\mathcal{C}(\mathbb{Z}, S)$ is nothing but the infinite path graph (see the first graph on Figure 8.1). In this case the study of self-adjoint and Markovian extensions of the weighted Kirchhoff Laplacian is reduced to the analysis in Section 5.1. Lemma 5.2 and Lemma 5.11 provide a complete characterization of self-adjoint and Markovian uniqueness, however, now one needs to deal with two ends and hence one has to replace one series (5.6) by two series with summations to $-\infty$ and ∞, respectively.

Remark 8.25. A few remarks are in order.

(i) Unfortunately, the above example seems to be the only case when a complete answer to the self-adjoint uniqueness for Kirchhoff Laplacians on weighted metric graphs can be obtained. Moreover, this characterization employs Weyl's limit point/limit circle alternative for Sturm–Liouville operators (see the proof of Lemma 5.2 and also [208]). Thus, upon changing either the generating set S in the above example or by considering a Cayley graph of an arbitrary virtually infinite cyclic group (e.g., $\Gamma \times \mathbb{Z}$ with a finite group Γ, see Figure 8.2), the problem of finding deficiency indices of the minimal Kirchhoff Laplacian on the corresponding weighted metric graph

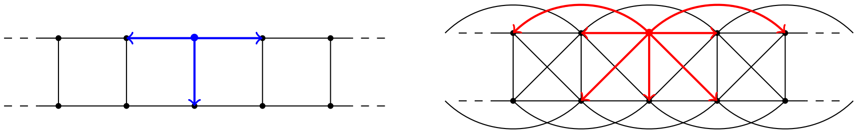


Figure 8.2. Cayley graphs of $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}$ (with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2) for two different generating sets.

seems rather non-trivial. In particular, the answer clearly depends on both the generating set S and the group Γ.

(ii) The free abelian group $(\mathbb{Z}^n, +)$, $n \in \mathbb{Z}_{\geq 2}$ and the free non-abelian group $\mathbb{F}_n$, $n \in \mathbb{Z}_{\geq 2}$ are the most natural candidates if one wishes to study the case of groups with 1 and, respectively, infinitely many ends (see Figure 8.1). The Gaffney-type theorems (Theorem 7.1 and Theorem 7.7) provide rather transparent sufficient conditions guaranteeing the self-adjoint uniqueness (for instance, one can employ the Hopf–Rinow theorem to verify the completeness assumption, see Section 6.4.5). Imposing the radial symmetry assumption for Cayley graphs of $\mathbb{F}_n$, one would be able to reduce the analysis to the one in Section 8.1.1 (see also Section 8.1.3.2), and the self-adjointness in this case can be characterized analogously to Theorem 8.3 (see [196]).

8.2.2 Spectral gap

For a finitely generated group G and a generating set S, the *isoperimetric constant* of its Cayley graph $\mathscr{G}_C = \mathscr{C}(\mathsf{G}, S)$ is defined by

$$\mathrm{Ch}_S(\mathsf{G}) = \inf_{X \subset \mathsf{G}} \frac{\#\partial X}{\#X}, \quad \partial X = \{(u, v) \in X \times (\mathsf{G} \setminus X) : uv^{-1} \in S\}, \tag{8.11}$$

where the infimum is taken over all finite subsets.[6]

Remark 8.26. Notice that the discrete isoperimetric constant defined in Section 7.3.2 for a weighted graph $(\mathcal{V}, m; b)$ looks very much similar to (8.11). In fact, upon choosing b and m as in Example 6.24 (i), that is, the corresponding graph Laplacian is the normalized graph Laplacian, the combinatorial distance is intrinsic. Taking into account that $\mathscr{C}(\mathsf{G}, S)$ is a regular graph and each vertex has degree equal to the cardinality of S, we get $|\partial X| = \#\partial X$, $m(X) = \#S \cdot \#X$ for any $X \subset \mathsf{G}$ and hence (7.22) implies

$$\mathrm{Ch}_S(\mathsf{G}) = \#S \cdot \mathrm{Ch}_d(\mathscr{G}_C).$$

[6]This definition extends to all connected graphs in an obvious way. A graph $\mathscr{G}_d$ has the *strong isoperimetric property* if its isoperimetric constant is positive (see [212]).

Let us recall the following notion (see, e.g., [175, Chapter 3], [212, Section 12.A]). A group is called *amenable* if it admits a left-invariant mean. For discrete groups one can define amenability in a more transparent way: a countable group G is amenable if it admits a *Følner sequence*, that is, there is a sequence (X_n) of non-empty finite subsets $X_n \subset \mathsf{G}$ which exhausts G, $\bigcup_{n\geq 0} X_n = \mathsf{G}$ and for each group element $g \in \mathsf{G}$,

$$\lim_{n\to\infty} \frac{\#(gX_n \cap X_n)}{\#X_n} = 1,$$

where $gX = \{gx : x \in X\}$ is the left translation of a set $X \subset \mathsf{G}$ by g.

Remark 8.27. Amenability was introduced by J. von Neumann in 1929 and now it is one of the most important concepts in analytic group theory. Amenability is known for many important classes of groups. For instance, all abelian or more generally all (virtually) nilpotent groups as well as all (virtually) solvable groups are amenable. The free non-abelian groups $\mathbb{F}_n$, $n \geq 2$, as well as any group containing $\mathbb{F}_2$ as a subgroup (e.g., the modular group $\mathrm{PSL}(2,\mathbb{Z})$) are not amenable (however, there are non-amenable groups without free subgroups). Moreover, amenability is invariant under quasi-isometries.

The analysis of spectral gaps of both weighted graph Laplacians and Kirchhoff Laplacians heavily relies on Kesten's amenability criterion [141], which can be seen as another instance of Følner's amenability criterion (see [212, Proposition 12.4]):

Theorem 8.28 (H. Kesten [141]). *Let $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ be a Cayley graph of a finitely generated group G. Then the isoperimetric constant $\mathrm{Ch}_S(\mathsf{G})$ equals zero if and only if G is amenable.*[7]

Remark 8.29. Notice that for amenable groups the isoperimetric constant is independent of the choice of S since it always equals 0. For non-amenable groups, $\mathrm{Ch}_S(\mathsf{G})$ depends on S, however, it always stays strictly positive. Thus, we can say that a group G has the *strong isoperimetric property* if one (and hence all) of its Cayley graphs satisfies $\mathrm{Ch}_S(\mathsf{G}) > 0$. By Kesten's theorem, the strong isoperimetric property for finitely generated groups is equivalent to non-amenability.

Using connections between discrete isoperimetric constants and isoperimetric constants for weighted metric graphs, we arrive at the following result.

Proposition 8.30. *Assume that $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ is a Cayley graph of a finitely generated group G. Also, let $(\mathcal{G}_C, \mu, \mu)$ be a weighted metric graph having finite intrinsic size and $\mathbf{H}_D$ the corresponding Dirichlet Laplacian.*

[7]The original statement is slightly different and it states that amenability is equivalent to the zero spectral gap for the generator of the simple random walk on $\mathcal{G}_C$. However, it is not difficult to see that both statements are equivalent (cf., e.g., (7.23)) and for convenience reasons we decided to state Kesten's criterion in the above form.

(i) *If* G *is non-amenable and the weight* μ *satisfies*

$$\frac{1}{\mu} \in L^\infty(\mathcal{G}) \quad \textit{and} \quad \sup_{e\in\mathcal{E}} \mu(e)|e| < \infty,$$

then $\lambda_0(\mathbf{H}_D) > 0$.

(ii) *If* G *is amenable, then* $\lambda_0(\mathbf{H}_D) = \lambda_0^{\mathrm{ess}}(\mathbf{H}_D) = 0$ *whenever*

$$\mu \in L^\infty(\mathcal{G}) \quad \textit{and} \quad \inf_{e\in\mathcal{E}} \mu(e)|e| > 0.$$

Proof. (i) By assumption, $(\mathcal{G}_C, \mu, \mu)$ has finite intrinsic size. Moreover, the intrinsic length coincides with the edge length and hence the corresponding discrete isoperimetric constant is given by (see (7.22))

$$\mathrm{Ch}_d(\mathcal{G}_C) = \inf_{X\subset\mathsf{G}} \frac{|\partial X|}{m(X)},$$

where

$$|\partial X| = \sum_{e\in\partial X} \mu(e), \quad m(X) = \sum_{v\in X}\sum_{e\in\mathcal{E}_v} \mu(e)|e|.$$

Therefore, we get the estimate

$$\frac{|\partial X|}{m(X)} \geq \frac{\inf_{e\in\mathcal{E}} \mu(e)}{\sup_{e\in\mathcal{E}} \mu(e)|e|} \frac{\#\partial X}{\#S\cdot\#X}$$

for all finite subsets $X \subset \mathsf{G}$. This immediately implies that $\mathrm{Ch}_d(\mathcal{G}_C) \geq C\,\mathrm{Ch}_S(\mathsf{G})$ with some positive $C > 0$. Hence, by Theorem 8.28, $\mathrm{Ch}_d(\mathcal{G}_C) > 0$. Therefore, the estimate (7.28) together with the Cheeger-type bound (7.16) imply the claim.

(ii) Combining Theorem 8.28 with the straightforward estimate

$$\frac{|\partial X|}{m(X)} \leq \frac{\sup_{e\in\mathcal{E}} \mu(e)}{\inf_{e\in\mathcal{E}} \mu(e)|e|} \frac{\#\partial X}{\#S\cdot\#X},$$

we conclude that $\mathrm{Ch}_d(\mathcal{G}_C) = 0$ if G is amenable. Since

$$\inf_{e\in\mathcal{E}} |e| \geq \frac{\inf_{e\in\mathcal{E}} \mu(e)|e|}{\sup_{e\in\mathcal{E}} \mu(e)} > 0,$$

we can apply Proposition 7.38 and the Buser-type bound (7.16) to conclude that $\lambda_0(\mathbf{H}_D) = 0$. Finally, if $\lambda_0^{\mathrm{ess}}(\mathbf{H}_D) > 0$, then $\lambda = 0$ is an eigenvalue of $\mathbf{H}_D$ with eigenfunction $f \equiv \mathbb{1}_{\mathcal{G}}$. However, our assumptions imply that $\mathcal{G}$ has infinite total volume and hence $\mathbb{1}_{\mathcal{G}} \notin L^2(\mathcal{G},\mu)$. This contradiction completes the proof. ■

As an immediate corollary we arrive at the following metric graph analog of Kesten's amenability criterion.

Corollary 8.31. *Let* $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ *be a Cayley graph. The following assertions are equivalent:*

(i) G *is non-amenable.*

(ii) $\mathrm{Ch}(\mathcal{G}_C) > 0$ *for all* $(\mathcal{G}_C, \mu, \mu)$ *having finite intrinsic size with the edge weight satisfying* $\mu, \frac{1}{\mu} \in L^\infty(\mathcal{G})$.

(iii) $\lambda_0(\mathbf{H}_D) > 0$ *for all* $(\mathcal{G}_C, \mu, \mu)$ *having finite intrinsic size with the edge weight satisfying* $\mu, \frac{1}{\mu} \in L^\infty(\mathcal{G})$.

Remark 8.32. If G is an amenable group, then the analysis of $\lambda_0(\mathbf{H}_D)$ and $\lambda_0^{\mathrm{ess}}(\mathbf{H}_D)$ in the case $\inf_{e\in\mathcal{E}} \mu(e)|e| = 0$ remains an open (and, in our opinion, rather complicated) problem. On the other hand, volume growth estimates (see Section 7.3.3 and the follow-up section) can be used to establish the equality $\lambda_0(\mathbf{H}_D) = 0$ for Cayley graphs of amenable groups in the case $\inf_{e\in\mathcal{E}} \mu(e)|e| = 0$. In particular, for polynomially growing groups or for groups of intermediate growth (see Section 8.2.3 for definitions) one may clearly allow a certain qualitative decay of edge lengths and weights at "infinity" in order to ensure the zero spectral gap.

8.2.3 Interlude: Growth in groups

The growth of a group is one of the most important quasi-isometric invariants (see [54, 159, 175]). Considering the identity element of G as the root o of its Cayley graph $\mathcal{C}(\mathsf{G}, S)$, one defines the growth function $\gamma_{\mathsf{G}}\colon \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{>0}$ by setting

$$\gamma_{\mathsf{G}}(n) = \#\{g \in \mathsf{G} : \varrho_{\mathrm{comb}}(g, o) \le n\},$$

where ϱ_{comb} is the combinatorial distance (a.k.a. word metric) on $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ (see Example 6.21 (i)). Behavior of γ_{G} for large n is independent of a choice of a generating set, that is, if $\widetilde{\gamma}_{\mathsf{G}}$ is the growth function of G corresponding to another generating set $\widetilde{S}$, then there is $C > 0$ such that $C^{-1}\gamma_{\mathsf{G}}(n) \le \widetilde{\gamma}_{\mathsf{G}}(n) \le C\gamma_{\mathsf{G}}(n)$ for all $n \in \mathbb{Z}_{\ge 0}$.

Clearly, $\gamma_{\mathsf{G}}(n) \le \exp(C\,n)$ for all $n \in \mathbb{Z}_{\ge 0}$. A group G has *subexponential growth* if $\log \gamma_{\mathsf{G}}(n) = o(n)$ as $n \to \infty$; otherwise, G is of *exponential growth*. Notice that non-amenable groups have exponential growth. If

$$d_{\mathsf{G}} := \limsup_{n\to\infty} \frac{\log \gamma_{\mathsf{G}}(n)}{\log n}$$

is finite, then G has *polynomial growth* and in this case d_{G} is its *degree*.

For large classes of groups the behavior of γ_{G} is well understood (e.g., Gromov's characterization of groups of polynomial growth, the Milnor–Wolf theorem for solvable groups, the Tits alternative for linear groups, etc. The subject is enormous and we only refer to [159] for further details and references). For instance, if G is virtually nilpotent, then the degree of growth d_{G} of γ_{G} is a natural number and it can be efficiently computed by the *Bass–Guivarc'h formula* (see, e.g., [159, Theorem 4.2], [212, f-la (3.15)]). For example, for the Heisenberg group over the integers $\mathrm{U}(3,\mathbb{Z})$, $\gamma(n) \asymp n^4$ as $n \to \infty$. The celebrated Gromov's polynomial growth theorem states that *only virtually nilpotent groups have polynomial growth.*

There are also groups of *intermediate growth*: those are groups of subexponential growth with $d_{\mathsf{G}} = \infty$, that is, γ_{G} grows faster than any polynomial, however, slower than any exponential function. Let us stress, however, that for groups of intermediate growth finding the precise rate of growth is a subtle issue. For instance, for the first Grigorchuk group this question was settled in the very recent work of A. Erschler and T. Zheng [65]: in this case

$$\frac{\log\log\gamma(n)}{\log n} = \frac{\log 2}{\log s_0} + o(1)$$

as $n \to \infty$, where s_0 is the positive root of $s^3 - s^2 - 2s = 4$.

8.2.4 Transience and recurrence

As before, $\mathcal{G}_C = \mathscr{C}(\mathsf{G}, S)$ is a Cayley graph of a finitely generated group G. Also, let $(\mathcal{G}_C, \mu, \nu)$ be a weighted metric graph (notice that in this subsection we allow $\mu \neq \nu$!) and let $\mathbf{H}_D$ be the corresponding Dirichlet Laplacian. Define

$$b_\nu(u,v) = \begin{cases} \dfrac{\nu(e_{u,v})}{|e_{u,v}|}, & u^{-1}v \in S, \\ 0, & u^{-1}v \notin S, \end{cases} \qquad (u,v) \in \mathsf{G}.$$

We begin with the following straightforward application of Theorem 7.49:

Corollary 8.33. *The heat semigroup* $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ *is recurrent if and only if the discrete time random walk on* G *with transition probabilities* $\mathcal{P}_\nu = (p_\nu(u,v))_{u,v\in\mathsf{G}}$ *defined by*

$$p_\nu(u,v) = P(X_{n+1} = v : X_n = u) = \frac{b_\nu(u,v)}{\sum_{g\in S} b_\nu(u,ug)} \tag{8.12}$$

is recurrent.

The above result reduces the problem of recurrence on weighted metric graphs to a thoroughly studied field – recurrence of random walks on groups. The literature on the subject is enormous and we only refer to the classic text [212]. Recall that a group G is called *recurrent* if the simple random walk on its Cayley graph $\mathscr{C}(\mathsf{G}, S)$ is recurrent for some (and hence for all) S. The classification of recurrent groups was accomplished in the 1980s and it is a combination of two seminal theorems – relationship between decay of return probabilities and growth in groups established by N. Th. Varopoulos [206] and M. Gromov's characterization of groups of polynomial growth (see, e.g., [206, Chapter VI.6], [212, Theorem 3.24]).

Theorem 8.34 (N.Th. Varopoulos). *The following assertions are equivalent:*

(i) G *is recurrent.*

(ii) *The growth function* γ_{G} *has polynomial growth of degree at most two, i.e.,* $\gamma_{\mathsf{G}}(n) \le C(1+n^2)$ *for all* $n \in \mathbb{Z}_{\ge 0}$.

(iii) G *contains a finite index subgroup isomorphic either to* $\mathbb{Z}$ *or to* $\mathbb{Z}^2$.

Remark 8.35. In fact, the original statement is much stronger. Suppose $\mathfrak{p}$ is a symmetric probability measure on G which generates G. It defines a random walk on G by setting

$$P(X_{n+1} = v : X_n = u) = \mathfrak{p}(\{u^{-1}v\}), \quad u, v \in \mathsf{G}.$$

The problem to characterize groups admitting a recurrent random walk was formulated by H. Kesten in 1967. It turns out that only recurrent groups admit recurrent random walks. Moreover, if G is recurrent, then every random walk generated by a symmetric probability measure $\mathfrak{p}$ with finite second moment is recurrent (we refer to [212, Chapter I.3] for further details and information).

Therefore, we arrive at the following result.

Theorem 8.36. *Let* $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ *be a Cayley graph,* $(\mathcal{G}_C, \mu, \nu)$ *a weighted metric graph,* $\mathbf{H}_D$ *the corresponding Dirichlet Laplacian.*

(i) *If* G *is recurrent, i.e.,* G *contains a finite index subgroup isomorphic either to* $\mathbb{Z}$ *or to* $\mathbb{Z}^2$*, and the edge weight* ν *satisfies*

$$\sup_{e \in \mathcal{E}} \frac{\nu(e)}{|e|} < \infty, \tag{8.13}$$

then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ *is recurrent.*

(ii) *If* G *is transient (i.e.,* G *does not contain a finite index subgroup isomorphic either to* $\mathbb{Z}$ *or to* $\mathbb{Z}^2$*) and the edge weight* ν *satisfies*

$$\inf_{e \in \mathcal{E}} \frac{\nu(e)}{|e|} > 0,$$

then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ *is transient.*

Proof. The proof is a straightforward application of Corollary 8.33 and Theorem 8.34. Namely, Corollary 8.33 reduces the study of recurrence/transience for $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ to the study of recurrence/transience of the discrete time random walk (8.12) on G. On the other hand, the energy form of the simple random walk on $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ is given by

$$\mathfrak{q}_{\mathsf{G},S}[f] = \frac{1}{2} \sum_{v \in \mathsf{G}} \sum_{u \in S} |f(v) - f(u^{-1}v)|^2.$$

By definition, G is recurrent/transient if and only if the energy form $\mathfrak{q}_{\mathsf{G},S}$ is recurrent/transient. Taking into account that the energy form associated with the random walk (8.12) is given by

$$\mathfrak{q}_\nu[f] = \frac{1}{2}\sum_{v\in \mathsf{G}}\sum_{u\in S} \frac{\nu(e_{u,v})}{|e_{u,v}|}|f(v) - f(u^{-1}v)|^2,$$

it remains to use Lemma B.7 to complete the proof of both claims. ∎

Let us finish this subsection with one immediate corollary.

Corollary 8.37. *Let $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ be a Cayley graph and let $(\mathcal{G}_C, |\cdot|)$ be an unweighted metric graph, $\mu = \nu \equiv \mathbb{1}$.*

(i) *If G contains a finite index subgroup isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}^2$ and $\inf_{e\in\mathcal{E}} |e| > 0$, then $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is recurrent.*

(ii) *If G does not contain a finite index subgroup isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}^2$ and $\sup_{e\in\mathcal{E}} |e| < \infty$, then the heat semigroup $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is transient.*

Remark 8.38. A few remarks are in order.

(i) If $\mathsf{G} = (\mathbb{Z}, +)$ and $\mathcal{C}$ is the Cayley graph of G with the standard set of generators $S = \{-1, 1\}$, one can show (cf. Lemma 5.13) that $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ is recurrent if and only if

$$\sum_{n\in\mathbb{Z}_{<0}} \frac{|e_n|}{\nu_n} = \infty \quad \text{and} \quad \sum_{n\in\mathbb{Z}_{>0}} \frac{|e_n|}{\nu_n} = \infty.$$

(ii) Using the volume test, one can slightly improve both Theorem 8.36 (i) and Corollary 8.37 (i) in the case when G contains a finite index subgroup isomorphic to $\mathbb{Z}^2$.

(iii) Applying the volume test (Section 7.4), one may obtain some sufficient conditions for recurrence for groups which grow faster than quadratic polynomials, however, in this case one needs to know the qualitative behavior of the corresponding growth function.

8.2.5 Ultracontractivity and eigenvalue estimates

In fact, the results in the previous section have a number of further and much stronger consequences. However, to simplify the exposition we restrict to unweighted metric graphs, that is, we shall assume throughout this subsection that $\mu = \nu \equiv \mathbb{1}$ on $\mathcal{G}$.

We begin with the following result.

Theorem 8.39. *Let $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ be a Cayley graph, $(\mathcal{G}_C, |\cdot|)$ a (unweighted) metric graph, and $\mathbf{H}_D$ the corresponding Dirichlet Laplacian. Assume also that G is not recurrent (i.e., it does not contain a finite index subgroup isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}^2$) and the edge lengths satisfy*

$$\sup_{e\in\mathcal{E}} |e| < \infty. \tag{8.14}$$

Then $(\mathrm{e}^{-t\mathbf{H}_D})_{t>0}$ *is ultracontractive and, moreover:*

(i) *If* $\gamma_{\mathsf{G}}(n) \approx n^N$ *as* $n \to \infty$ *with some* $N \in \mathbb{Z}_{\ge 3}$*, then*

$$\|\mathrm{e}^{-t\mathbf{H}_D}\|_{1\to\infty} \le C_N t^{-\frac{N}{2}}, \quad t > 0. \tag{8.15}$$

(ii) *If* G *is not virtually nilpotent (i.e.,* γ_{G} *has superpolynomial growth)*[8]*, then* (8.15) *holds true for all* $N > 2$.

Proof. Notice that we only need to prove (8.15) since ultracontractivity is its immediate consequence. By Theorem 4.30, (8.15) is equivalent to the analogous ultracontractivity bound for the associated weighted graph Laplacian $\mathbf{h}_D$:

$$\|\mathrm{e}^{-t\mathbf{h}_D}\|_{1\to\infty} \le C t^{-\frac{N}{2}}, \quad t > 0.$$

However, by Theorem C.2 the latter is equivalent to the Sobolev-type inequality (4.38)

$$\Big(\sum_{v\in\mathsf{G}} |f(v)|^{\frac{2N}{N-2}} m(v)\Big)^{\frac{N-2}{N}} \le C \sum_{v\in\mathsf{G}}\sum_{u\in S} \frac{1}{|e_{u,v}|} |f(v) - f(u^{-1}v)|^2 \tag{8.16}$$

for all $f \in \mathrm{dom}(\mathfrak{q}_D)$. Here the vertex weight m is given by (take into account that the model has finite size by assumption and $\mu \equiv \mathbb{1}$)

$$m(v) = \sum_{u\in S} |e_{v,uv}|. \tag{8.17}$$

However, (8.14) implies that (8.16) would follow from the inequality

$$\Big(\sum_{v\in\mathsf{G}} |f(v)|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}} \le C \sum_{v\in\mathsf{G}}\sum_{u\in S} |f(v) - f(u^{-1}v)|^2. \tag{8.18}$$

Now it remains to notice that the latter inequality is a consequence of our growth assumptions on G. If γ_{G} grows polynomially and $\gamma_{\mathsf{G}}(n) \approx n^N$ for some $N \ge 3$ as $n \to \infty$, then (8.18) holds true by [206, Theorem VI.5.2]). If G is not virtually nilpotent, then, by the Gromov theorem, γ_{G} has superpolynomial growth and it remains to apply [206, Theorem VI.3.2]. ■

Remark 8.40. Let us stress that (8.14) is necessary for the validity of (8.15) with $N > 2$ (see Lemma 4.32).

For groups having at most quadratic growth, the next result is an immediate consequence of recurrence:

Corollary 8.41. *Let* G *be recurrent (i.e.,* G *contains a finite index subgroup isomorphic either to* $\mathbb{Z}$ *or to* $\mathbb{Z}^2$*). Let also* $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ *be its Cayley graph and* $(\mathcal{G}_C, |\cdot|)$

[8] This means that for each $N > 0$ there is $c > 0$ such that $\gamma_{\mathsf{G}}(n) \ge c n^N$ for all large n.

an unweighted metric graph. If $\inf_{e\in\mathcal{E}} |e| > 0$*, then*

$$\limsup_{t>0} t\,\|\mathrm{e}^{-t\mathbf{H}_D}\|_{1\to\infty} \in (0,\infty].$$

Let us mention that removing the assumption $\inf_{e\in\mathcal{E}} |e| > 0$ in the above corollary, one may construct metric graphs such that the corresponding Dirichlet Laplacian satisfies (8.15) with some $N > 2$.

We would like to finish this subsection with a remark on the so-called Cwikel–Lieb–Rozenblum inequality. Let us consider Laplacians $\mathbf{H}_\alpha$ with δ-couplings on the vertices, that is, $\alpha\colon \mathsf{G} \to \mathbb{R}$ and at each vertex $v \in \mathsf{G}$ we replace the Kirchhoff condition by (2.13). As before, if $\mathbf{H}_\alpha$ is not self-adjoint, we shall consider the Friedrichs extension of the minimal operator (of course, if it is bounded from below) and by abusing the notation we shall denote it by the same letter $\mathbf{H}_\alpha$. Moreover, we shall use the standard notation $\alpha_\pm = \frac{1}{2}(|\alpha| \pm \alpha)$.

Theorem 8.42. *Let* $\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ *be a Cayley graph,* $(\mathcal{G}_C, |\cdot|)$ *a (unweighted) metric graph,* $\alpha\colon \mathsf{G} \to \mathbb{R}$*, and* $\mathbf{H}_\alpha$ *the corresponding Laplacian.*

(i) *If* $\gamma_{\mathsf{G}}(n) \le C(1+n^2)$ *for all n and* $\inf_{e\in\mathcal{E}} |e| > 0$*, then* $\mathbf{H}_\alpha$ *has at least one negative eigenvalue whenever* $0 \not\equiv \alpha = -\alpha_- \in C_c(\mathcal{V})$*.*

(ii) *If* $\gamma_{\mathsf{G}}(n) \asymp n^N$ *as* $n \to \infty$ *with some* $N \in \mathbb{Z}_{\ge 3}$ *and* (8.14) *is satisfied, then the operator* $\mathbf{H}_\alpha$ *is bounded below whenever* $\frac{\alpha_-}{m} \in \ell^{N/2}(\mathsf{G}; m)$*. Moreover, its negative spectrum is discrete and*

$$\kappa_-(\mathbf{H}_\alpha) \le C \sum_{v\in\mathsf{G}} \alpha_-(v)^{\frac{N}{2}} m(v)^{1-\frac{N}{2}}, \qquad (8.19)$$

where m is given by (8.17) *and the constant* $C > 0$ *depends only on the underlying metric graph.*

(iii) *If* G *is not virtually nilpotent,* (8.14) *is satisfied and* $\alpha_- \in \ell^{N/2}(\mathsf{G}; m)$ *for some* $N > 2$*, then the operator* $\mathbf{H}_\alpha$ *is bounded below, its negative spectrum is discrete and the bound* (8.19) *holds true.*

Proof. To simplify the proof, let us assume that $\mathbf{H}_\alpha$ is self-adjoint.[9] First of all, by Theorem 3.1 (iv), $\kappa_-(\mathbf{H}_\alpha) = \kappa_-(\mathbf{h}_\alpha)$ and hence we need to prove the corresponding claims for $\mathbf{h}_\alpha$.

(i) By Corollary 8.37 (i) and Theorem 4.17, the heat semigroup generated by $\mathbf{h}_D$ is recurrent, which immediately implies the claim.

To prove (ii) and (iii), we just need to apply [152, Theorems 1.2 and 1.3], which relate the ultracontractivity estimates established by Theorem 8.39 and Theorem 4.30 for $\mathbf{h}_\alpha$ with Cwikel–Lieb–Rozenblum bounds. ■

[9] One may assume $\mathcal{G}_C$ is complete with respect to the natural path metric, and then by Theorem 7.9, the operator $\mathbf{H}_\alpha$ is self-adjoint once it is bounded from below; see also Lemma 7.16.

Remark 8.43. Notice that applying [152, Theorems 1.2 and 1.3] directly to the Dirichlet Laplacian $\mathbf{H}_D$, we arrive at the Cwikel–Lieb–Rozenblum estimates for additive perturbation, that is, for Schrödinger operators $-\Delta + V(x)$. It is also well known (see [75]) that ultracontractivity estimates and Sobolev-type inequalities lead to Lieb–Thirring bounds ($\mathfrak{S}_p$ estimates on the negative spectra, see also Theorem 3.1 (viii)), however, we are not going to pursue this goal here.

Let us also stress that Theorem 8.42 (iii) makes sense only for amenable G since otherwise $\mathbf{H}_D$ has a positive spectral gap (see Proposition 8.30).

8.2.6 Historical remarks and further references

The theory of random walks on groups was founded by H. Kesten [142] (in fact, in his PhD thesis). The idea to relate growth of groups with recurrence is also due to Kesten (*Kesten's conjecture*). The literature on the subject is enormous and in this respect we only refer to the excellent book by W. Woess [212].

Kesten's amenability criterion has been heavily exploited to study random walks on groups. However, we are also aware of some cases when Kesten's criterion has been used in the "opposite" direction. The most striking, in our opinion, application appears in the solution of the von Neumann–Day problem (widely known as the "von Neumann conjecture"): A. Yu. Olshanskii constructed a Tarski monster group in [178]; S. I. Adyan in [1] proved that a simple random walk on the *free Burnside group* $\mathrm{B}(m,n)$ of rank $m \geq 2$ with odd exponent $n \geq 665$ has a spectral radius < 1, which implies non-amenability of $\mathrm{B}(m,n)$ for this range of m and n.[10] Let us also mention that recently L. Bartholdi and B. Virág [16] proved that the so-called *Basilica group* is amenable by showing that return probabilities of the simple random walk decay at subexponential rates.

Let us mention that one of the motivations to investigate random walks on groups came from manifolds. By the Švarc–Milnor lemma, the fundamental group $\pi_1(M)$ of a compact manifold M and its universal cover $\widetilde{M}$ are quasi-isometric and thus there are close relationships between them. For instance, it was proved independently by R. Brooks [33] and N. Th. Varopoulos [203] that the Laplace–Beltrami operator on $\widetilde{M}$ has a positive spectral gap if and only if $\pi_1(M)$ is not amenable. Moreover, Varopoulos [203] showed that the Brownian motion on $\widetilde{M}$ is recurrent if and only if the group $\pi_1(M)$ is recurrent.

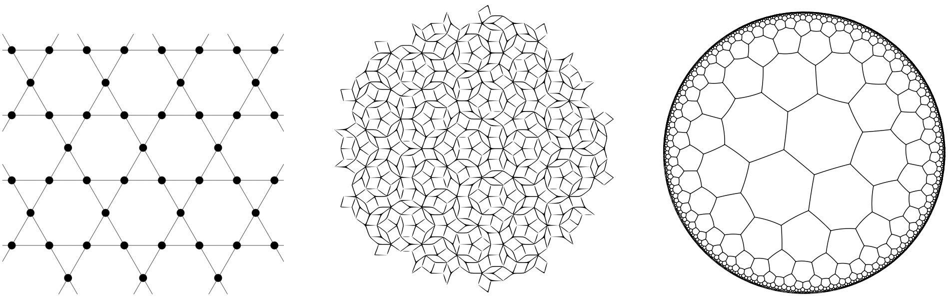
[10] In fact, both A. Yu. Olshanskii and S. I. Adyan used one criterion of R. Grigorchuk [87], who computed the spectral gap for the generator of the simple random walk by means of the so-called *co-growth function*, see [87, Section 4] and also [54, Section VII.D]; notice also that [87, Theorem 7.1] establishes non-amenability for a class of groups for which the problem of identity is solved by Dehn's algorithm.

The importance of Sobolev-type inequalities for ultracontractivity estimates was realized by N.Th. Varopoulos. The subject is enormous and we even did not touch here Nash-type inequalities. We refer for further details and references to [206, 212].

Concluding this section, let us mention recent very active work related to understanding spectra of groups. More specifically, the *spectrum of* G is the spectrum of a generator of a simple random walk on G, i.e., the spectrum of the normalized Laplacian (or, equivalently, combinatorial Laplacian since $\mathcal{C}(\mathsf{G}, S)$ is a regular graph) on a Cayley graph $\mathcal{C}(\mathsf{G}, S)$ of a given group G. The study of a spectral gap is the simplest (and rather widely studied) issue in this topic. In particular, to understand the support of the spectrum as well as its structure are much harder tasks. A complete picture is known only in some specific cases (e.g., abelian groups $(\mathbb{Z}^n, +)$, free group $\mathbb{F}_p$ (see [142]), the Lamplighter group (see [89]); however, this list is by no means complete). In particular, it is not completely clear what kind of spectra groups may have (it is still open whether Cantor spectrum can occur on a Cayley graph, however, it is shown in [36] that the support of the Kesten–von Neumann–Serre spectral measure of the Basilica group is a Cantor set). Another interesting question is how the spectrum depends on the chosen generating set or on the choice of weights on the generators. The subject is rapidly developing and we only refer to a very brief selection of recent articles [36, 50, 63, 88] for further results and information.

8.3 Tessellations

In the present section, we discuss graphs arising from tessellations of $\mathbb{R}^2$ (see Figure 8.3 for examples) and combine their distinctive combinatorial properties with our previous findings.



(a) **(b)** **(c)**

Figure 8.3. (a) The Kagome lattice, (b) a Penrose tiling in $\mathbb{R}^2$ and (c) a tessellation of the Poincaré disc by heptagons.[11]

[11] Image credit for Figure 8.3: (a) WilliamSix, CC BY-SA 2.5, via Wikimedia Commons; (b) xJaM derivative work: Sprak, public domain, via Wikimedia Commons; (c) Theon, CC BY-SA 3.0, via Wikimedia Commons.

In order to formalize this setting, we first need a few definitions. Recall that a *plane graph* is a planar graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ embedded in $\mathbb{R}^2$ by some fixed planar embedding. In particular, any plane graph $\mathcal{G}_d$ can be regarded as a subset of the Euclidean plane $\mathbb{R}^2$, which we always assume to be closed. We denote by $\mathcal{F}$ the set of *faces* of $\mathcal{G}_d$, i.e., the closures of the connected components of $\mathbb{R}^2 \setminus \mathcal{G}_d$. We stress that, since $\mathcal{G}_d$ may be infinite, it may have several unbounded faces and all of them are included in $\mathcal{F}$. We denote by $\mathcal{F}_b$ the set of *bounded faces* of $\mathcal{G}_d$.

In order to avoid technical difficulties, we impose the following assumptions.

Definition 8.44. A plane graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is *tessellating* if the following additional conditions hold:

(i) $\mathcal{F}$ is locally finite, i.e., each compact subset $K \subset \mathbb{R}^2$ intersects only finitely many faces.

(ii) Each bounded face $F \in \mathcal{F}_b$ is a closed topological disc and its boundary ∂F consists of a finite cycle of at least three edges.

(iii) Each unbounded face $F \in \mathcal{F} \setminus \mathcal{F}_b$ is a closed topological half-plane and its boundary ∂F consists of a (countably) infinite chain of edges.

(iv) $\#\mathcal{F}_e = 2$ for all $e \in \mathcal{E}$, where $\mathcal{F}_e := \{F \in \mathcal{F} : e \subset \partial F\}$.

(v) Each vertex $v \in \mathcal{V}$ has degree ≥ 3.

Here a subset $A \subseteq \mathbb{R}^2$ is called a *closed topological disc (half-plane)* if it is an image of the closed unit ball in $\mathbb{R}^2$ (the closed upper half-plane) under a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$. For a face $F \in \mathcal{F}$, we define

$$\mathcal{E}_F := \{e \in \mathcal{E} : e \subseteq \partial F\},$$
$$d_{\mathcal{F}}(F) := \#\mathcal{E}_F,$$

where the latter is called the *degree* of a face $F \in \mathcal{F}$. Notice that according to Definition 8.44, $d_{\mathcal{F}}(F) \geq 3$ for all faces F and $\deg(v) \geq 3$ for all vertices v. In particular, the graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ has no loops and vertices of degree one or two. Moreover, every tessellating graph $\mathcal{G}_d$ is an infinite, locally finite graph.

The above assumptions imply that $\mathcal{F}$ is a *locally finite tessellation* (or *tiling*) of $\mathbb{R}^2$, i.e., a locally finite, countable family $\mathbb{T}$ of closed subsets $T \subset \mathbb{R}^2$ such that the interiors are pairwise disjoint and

$$\bigcup_{T \in \mathbb{T}} T = \mathbb{R}^2.$$

In addition, the original graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ coincides with the *edge graph of the tessellation* $\mathcal{F}$: by calling a connected component of the intersection of at least two sets in $\mathcal{F}$ an $\mathcal{F}$-vertex, if it has only one point and an $\mathcal{F}$-edge otherwise, we recover precisely the vertex and edge sets $\mathcal{V}$ and $\mathcal{E}$. In fact, this connection is the motivation behind our terminology.

Remark 8.45. Tessellating graphs include all infinite trees $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ satisfying $\deg(v) \geq 3$ for each vertex $v \in \mathcal{V}$.

A *plane weighted metric graph* is a weighted metric graph $(\mathcal{G}, \mu, \nu)$ together with a fixed model whose underlying combinatorial graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is planar and embedded into $\mathbb{R}^2$. If the plane graph $\mathcal{G}_d$ is tessellating, then $(\mathcal{G}, \mu, \nu)$ is called a *tessellating weighted metric graph*. Let us also stress that the edge lengths and weights of $(\mathcal{G}, \mu, \nu)$ are in general not related to the Euclidean arc lengths of the corresponding plane graph $\mathcal{G}_d$.

Remark 8.46. Notice that the fixed model in the definition of a tessellating weighted metric graph $(\mathcal{G}, \mu, \nu)$ is unique according to (v) in Definition 8.44, which excludes inessential vertices. Moreover, it is easily seen that the weighted metric graph $(\mathcal{G}, \mu, \nu)$ has *finite intrinsic size* exactly when this particular model has finite intrinsic size. On the other hand, let us emphasize that the embedding of a planar graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ into $\mathbb{R}^2$ is not unique. For instance, the degrees of the faces depend on the embedding (whereas their number is invariant by Euler's formula) and, in general, different embeddings lead to non-isomorphic dual graphs (see, e.g., [73, Chapter 5.5 and Figure 5.7] for further details).

8.3.1 Markovian uniqueness

The combinatorial structure of plane graphs leads to simple criteria for Markovian uniqueness.

Corollary 8.47. *Let $(\mathcal{G}, \mu, \nu)$ be a tessellating weighted metric graph such that all faces are bounded, $\mathcal{F} = \mathcal{F}_b$. Assume that either $\frac{1}{\mu}, \frac{1}{\nu} \in L^\infty(\mathcal{G})$ or that $\mathcal{G}$ has finite ν-diameter (see (7.12)). Then the following are equivalent:*

- (i) $\mathbf{H}^0$ *admits a unique Markovian extension,*
- (ii) $\mathbf{H}_D = \mathbf{H}_N$,
- (iii) *the Gaffney Laplacian* $\mathbf{H}_G$ *is self-adjoint,*
- (iv) $H^1_0(\mathcal{G}, \mu, \nu) = H^1(\mathcal{G}, \mu, \nu)$,
- (v) $\mathcal{G}$ *has infinite volume,* $\mu(\mathcal{G}) = \infty$.

If one (equivalently, all) of the above properties fails, then the deficiency indices of the minimal Gaffney Laplacian $\mathbf{H}_{G,\min}$ *are equal to*

$$\mathrm{n}_\pm(\mathbf{H}_{G,\min}) = 1.$$

Proof. The claims follow immediately from Theorem 7.24 (see also (7.12)) and the fact that $\mathcal{G}$ has exactly one graph end since $\mathcal{F} = \mathcal{F}_b$. ∎

Remark 8.48. If $\mathcal{F}$ contains unbounded faces, then the graph might have more than one end. For instance, every infinite tree $\mathcal{T} = (\mathcal{G}, \mathcal{E})$ with $\deg(v) \geq 3$ for all $v \in \mathcal{V}$

can be embedded in $\mathbb{R}^2$ as a tessellating graph with infinitely many unbounded faces. On the other hand, $\mathcal{T}$ has uncountably many graph ends.

8.3.2 Spectral gap estimates

In this subsection, we discuss lower estimates for the isoperimetric constant of tessellating weighted metric graphs. To simplify our considerations, in this subsection we consider only weighted metric graphs with equal weight functions $(\mathcal{G}, \mu, \mu)$, that is, we assume that $\mu = \nu$. Without loss of generality we shall also assume that $(\mathcal{G}, \mu, \mu)$ has finite intrinsic size since otherwise

$$0 = \mathrm{Ch}(\mathcal{G}) = \lambda_0(\mathbf{H}_D),$$

according to Corollary 3.18 and estimate (7.16). For each edge $e \in \mathcal{E}$ of $\mathcal{G}_d$, we define its *characteristic value* as (see (3.5) for the definition of m)

$$\mathbf{c}(e) := \frac{1}{|e|\mu(e)} - \sum_{v: v\in e} \frac{1}{m(v)} - \sum_{F\in \mathcal{F}_e\cap\mathcal{F}_b} \frac{1}{\mu(\partial F)}, \tag{8.20}$$

and also set

$$\mathbf{c}(\mathcal{E}) := \inf_{e\in\mathcal{E}} \mathbf{c}(e).$$

All summands on the right-hand side (8.20) admit a clear interpretation in terms of the edge weight μ:

- the first summand is the reciprocal of $\int_e \mu = |e|\mu(e)$,
- we have, because of finite intrinsic size,

$$m(v) = \sum_{e\in\mathcal{E}_v} |e|\mu(e) = \mu(\mathcal{E}_v) = \int_{\mathcal{E}_v} \mu,$$

- finally, $\mu(\partial F) = \int_{\partial F} \mu = \sum_{e\in\mathcal{E}_F} |e|\mu(e)$ is the weighted perimeter of F.

Remark 8.49. A few remarks are in order.

(i) Setting $\mu(e) = |e| = 1$ for all $e \in \mathcal{E}$ in (8.20),

$$\mathbf{c}(e) = 1 - \sum_{v: v\in e} \frac{1}{\deg(v)} - \sum_{F\in\mathcal{F}_e\cap\mathcal{F}_b} \frac{1}{d_{\mathcal{F}}(F)},$$

which coincides with the *characteristic number* $\phi(e)$ of the edge e introduced in [211].

(ii) As is easily shown, the characteristic values $\mathbf{c}(e)$, $e \in \mathcal{E}$, depend on the embedding of the planar graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ in $\mathbb{R}^2$. Namely, the definition of $\mathbf{c}(e)$ takes into account all edges $e' \in \mathcal{E}$ which share a face with e, and this edge set depends heavily on the embedding.

(iii) As is discussed below in Section 8.3.3.2, the characteristic values are related to discrete curvature notions for plane graphs. However, our choice of the sign differs from the standard one in the literature and this explains why our results are formulated in terms of positive curvature.

It turns out that, if the weight function $\mu: \mathcal{G} \to (0, \infty)$ is uniformly positive on $\mathcal{G}$, that is, it additionally satisfies

$$\frac{1}{\mu} \in L^\infty(\mathcal{G}),$$

then the characteristic edge values give rise to lower estimates for the isoperimetric constant $\mathrm{Ch}(\mathcal{G})$.

Theorem 8.50. *Let $(\mathcal{G}, \mu, \mu)$ be a tessellating weighted metric graph. Then*

$$\frac{\mathbf{c}(\mathcal{E})}{\|\frac{1}{\mu}\|_\infty} \le \mathrm{Ch}(\mathcal{G}).$$

In particular, if $\mathbf{c}(\mathcal{E}) \ge 0$, the following spectral estimate holds true for the Dirichlet Laplacian $\mathbf{H}_D$:

$$\frac{1}{4}\left(\frac{\mathbf{c}(\mathcal{E})}{\|\frac{1}{\mu}\|_\infty}\right)^2 \le \lambda_0(\mathbf{H}_D).$$

The method of proof follows closely [211] and consists in a rather elegant application of *Euler's identity* for finite plane graphs $\mathcal{K} = (\mathcal{V}(\mathcal{K}), \mathcal{E}(\mathcal{K}))$,

$$\#\mathcal{V}(\mathcal{K}) - \#\mathcal{E}(\mathcal{K}) + \#\mathcal{F}_b(\mathcal{K}) = \#\mathcal{C}(\mathcal{K}), \tag{8.21}$$

where $\mathcal{F}_b(\mathcal{K})$ denotes the set of bounded faces of $\mathcal{K}$ and $\mathcal{C}(\mathcal{K})$ is the set of connected components of $\mathcal{K}$ (see, e.g., [28, Section 1.4]).

Proof of Theorem 8.50. The estimates in Theorem 8.50 are trivial if $\mathbf{c}(\mathcal{E}) \le 0$, thus we can assume without loss of generality that $\mathbf{c}(\mathcal{E})$ is positive. Therefore, taking into account (7.31) and the Cheeger-type bound in Theorem 7.33, it suffices to prove that the estimate

$$\frac{\mathbf{c}(\mathcal{E})}{\|\frac{1}{\mu}\|_\infty} \le \frac{\mathrm{area}(\partial\mathcal{K})}{\mu(\mathcal{K})} \tag{8.22}$$

holds true for all finite subgraphs $\mathcal{K} = (\mathcal{V}(\mathcal{K}), \mathcal{E}(\mathcal{K}))$ of $\mathcal{G}_d$. Here (see (7.13)–(7.14))

$$\mu(\mathcal{K}) = \sum_{e \in \mathcal{E}(\mathcal{K})} |e|\mu(e), \qquad \mathrm{area}(\partial\mathcal{K}) = \mathrm{area}(\partial\mathcal{K}, \mu, \mu) = \sum_{v \in \partial\mathcal{K}} \sum_{e \in \mathcal{E}_v(\mathcal{K})} \mu(e),$$

where $\partial\mathcal{K} = \{v \in \mathcal{V}(\mathcal{K}) : \deg_{\mathcal{K}}(v) < \deg_{\mathcal{G}}(v)\}$. Clearly,

$$\mathbf{c}(\mathcal{E})\mu(\mathcal{K}) = \mathbf{c}(\mathcal{E}) \int_{\mathcal{K}} \mu(\mathrm{d}x) \le \int_{\mathcal{K}} \mathbf{c}(x)\mu(\mathrm{d}x),$$

and hence it is enough to show that

$$\int_{\mathcal{K}} \mathbf{c}(x)\mu(\mathrm{d}x) \leq \left\| \frac{1}{\mu} \right\|_{\infty} \operatorname{area}(\partial\mathcal{K}).$$

By (8.20), the left-hand side in the above inequality is equal to

$$\begin{aligned}\int_{\mathcal{K}} \mathbf{c}(x)\mu(\mathrm{d}x) &= \sum_{e\in\mathcal{E}(\mathcal{K})} \mathbf{c}(e)|e|\mu(e)\\ &= \#\mathcal{E}(\mathcal{K}) - \sum_{v\in\mathcal{V}} \frac{\mu(\mathcal{E}_v \cap \mathcal{E}(\mathcal{K}))}{m(v)} - \sum_{F\in\mathcal{F}_b} \frac{\mu(\mathcal{E}_F \cap \mathcal{E}(\mathcal{K}))}{\mu(\partial F)}.\end{aligned}$$

Notice that for a non-boundary vertex $v \in \mathcal{V}(\mathcal{K}) \setminus \partial\mathcal{K}$, the equality

$$\mu(\mathcal{E}_v \cap \mathcal{E}(\mathcal{K})) = \mu(\mathcal{E}_v) = \sum_{e\in\mathcal{E}_v} \mu(e)|e| = m(v)$$

holds true (recall that our graph has finite intrinsic size and hence we have equality instead of $\geq$ on the right-hand side). Consider the subgraph $\mathcal{K}^\circ = (\mathcal{V}(\mathcal{K}^\circ), \mathcal{E}(\mathcal{K}^\circ))$ of $\mathcal{K}$ which consists of all vertices in $\mathcal{V}(\mathcal{K}^\circ) := \mathcal{V}(\mathcal{K}) \setminus \partial\mathcal{K}$ and all edges between such vertices. Notice also that each face $F \in \mathcal{F}$ whose boundary consists only of edges in $\mathcal{K}^\circ$, that is $\partial F \subseteq \mathcal{E}(\mathcal{K}^\circ)$, defines a bounded face of $\mathcal{K}^\circ$ and satisfies

$$\mu(\mathcal{E}_F \cap \mathcal{E}(\mathcal{K})) = \mu(\mathcal{E}_F \cap \mathcal{E}(\mathcal{K}^\circ)) = \mu(\mathcal{E}_F) = \mu(\partial F).$$

Denoting by $\mathcal{P}(\mathcal{K}^\circ)$ the set of all such faces $F \in \mathcal{F}$, we arrive at the estimate

$$\int_{\mathcal{K}} \mathbf{c}(x)\,\mu(\mathrm{d}x) \leq \#\mathcal{E}(\mathcal{K}) - \#\mathcal{V}(\mathcal{K}^\circ) - \#\mathcal{P}(\mathcal{K}^\circ). \tag{8.23}$$

Clearly, we also have the elementary inequality

$$\#(\mathcal{E}(\mathcal{K}) \setminus \mathcal{E}(\mathcal{K}^\circ)) \leq \left\| \frac{1}{\mu} \right\|_{\infty} \operatorname{area}(\partial\mathcal{K}).$$

Hence, if all bounded faces of $\mathcal{K}^\circ$ are of the above form, that is,

$$\mathcal{F}_b(\mathcal{K}^\circ) = \mathcal{P}(\mathcal{K}^\circ), \tag{8.24}$$

we can apply Euler's formula (8.21) to the subgraph $\mathcal{K}^\circ$ and conclude that

$$\text{RHS of (8.23)} = \#\mathcal{E}(\mathcal{K}) - \#\mathcal{E}(\mathcal{K}^\circ) - \#\mathcal{C}(\mathcal{K}^\circ) \leq \left\| \frac{1}{\mu} \right\|_{\infty} \operatorname{area}(\partial\mathcal{K}).$$

In particular, we have established the estimate (8.22) in this special case.

On the other hand, if (8.24) fails for some finite subgraph $\mathcal{K}$ of the fixed model, we can construct a new subgraph $\widehat{\mathcal{K}}$ by "filling up its holes". That is, we consider all faces $F \in \mathcal{F}$ which are contained in some bounded face F of $\mathcal{K}^\circ$ and add all vertices and edges of such faces to $\mathcal{K}$. It is easily shown that the obtained subgraph $\widehat{\mathcal{K}}$ satisfies the estimates

$$\mu(\mathcal{K}) \le \mu(\widehat{\mathcal{K}}) \quad \text{and} \quad \operatorname{area}(\partial\widehat{\mathcal{K}}) \le \operatorname{area}(\partial\mathcal{K})$$

together with condition (8.24). Hence inequality (8.22) holds in the general case and the proof is complete. ∎

Remark 8.51. The estimate in Theorem 8.50 is not optimal and can be improved using methods similar to [173, Theorem 3.3], where the case $\mu = \nu \equiv \mathbb{1}$ was considered (see also [138, Theorem 1] and [128, Theorem 6]). On the other hand, these results look more technical and, for the sake of a clear exposition, we decided not to include them.

Notice that Theorem 8.50 applies to infinite trees.

Proposition 8.52. *Let* $(\mathcal{T}, \mu, \mu)$ *be a weighted metric tree having a model such that all vertices satisfy* $\deg(v) \ge 3$. *Then*

$$\operatorname{Ch}(\mathcal{G}) \ge \frac{1}{\|\frac{1}{\mu}\|_\infty} \inf_{e\in\mathcal{E}} \left(\frac{1}{\mu(e)|e|} - \sum_{v\in e} \frac{1}{m(v)} \right).$$

Example 8.53. Consider the graphs depicted in Figure 8.3. For simplicity, we consider unweighted, equilateral metric graphs: $\mu = \nu \equiv \mathbb{1}$ and $|e| = 1$ for all $e \in \mathcal{E}$.

(a) *Kagome lattice:* All vertices have degree $\deg(v) = 3$ and each edge is adjacent to a triangle and a hexagon. In particular, the characteristic value of all edges $e \in \mathcal{E}$ is equal to

$$\mathbf{c}(e) = 1 - 2 \cdot \frac{1}{4} - \frac{1}{3} - \frac{1}{6} = 0.$$

(b) *Penrose tiling:* Notice first that each face is a rhombus. However, the characteristic edge value is not constant in this case, since the degrees of the adjacent vertices vary. For instance, there are infinitely many edges $e = e_{uv}$ such that $\deg(u) = 3$ and $\deg(v) = 5$ and in this case

$$\mathbf{c}(e) = 1 - \frac{1}{3} - \frac{1}{5} - 2 \cdot \frac{1}{4} = -\frac{1}{30}.$$

(c) *Hyperbolic tessellation:* As each face is a hyperbolic heptagon, $d_{\mathcal{F}}(F) = 5$ for all $F \in \mathcal{F}$ and all vertices have degree $\deg(v) = 3$. More generally, we can consider (p, q)-regular tessellations (i.e., $\deg(v) = p$ for all vertices v and $d_{\mathcal{F}}(F) = q$ for all faces F) for some $p \in \mathbb{Z}_{\ge 3}$ and $q \in \mathbb{Z}_{\ge 3} \cup \{\infty\}$ (note that $q = \infty$ corresponds to p-regular trees, in which case all faces are unbounded). In this case, the characteristic value $\mathbf{c}(e)$ of all edges $e \in \mathcal{E}$ is equal to

$$\mathbf{c}_{p,q} := 1 - \frac{2}{p} - \frac{2}{q}.$$

It turns out that $\mathbf{c}_{p,q} \geq 0$ for every (p,q)-regular tessellation of $\mathbb{R}^2$ (see, e.g., [57, Theorem 1.7]). Clearly,

$$\mathbf{c}_{p,q} = 0$$

exactly when

$$(p,q) \in \{(4,4),(3,6),(6,3)\},$$

and in these cases $\mathcal{G}_d$ is isomorphic to the square, hexagonal or triangle lattice in $\mathbb{R}^2$. In particular, one easily shows that $\mathrm{Ch}(\mathcal{G}) = 0$ in all three cases. On the other hand, if $\mathbf{c}_{p,q} > 0$, then $\mathcal{G}_d$ is isomorphic to the edge graph of a tessellation of the Poincaré disc $\mathbb{H}^2$ with regular q-gons of interior angle $\frac{2\pi}{p}$ (see [101, Remark 4.2]). Moreover, Theorem 8.50 implies that $\mathrm{Ch}(\mathcal{G}) > 0$. The explicit value is given by (see [173, equation (4.6)])

$$\mathrm{Ch}(\mathcal{G}_{p,q}) = \frac{p-2}{p-1+\frac{p}{2}\left(\sqrt{\frac{(p-2)(q-2)}{pq-2(p+q)}}-1\right)}$$

and can be found from results on isoperimetric constants of discrete graphs (see [101, 105]).

Notice that Theorem 8.50 leads to trivial bounds for the Kagome lattice and the Penrose tiling in Example 8.53. However, one can easily show directly that $\mathrm{Ch}(\mathcal{G}) = 0$ for these examples. It turns out that these graphs actually satisfy a stronger property:

Proposition 8.54. *Let $(\mathcal{G}, \mu, \mu)$ be a tessellating weighted metric graph such that $\inf_{e\in\mathcal{E}} |e| > 0$ and $\sup_{F\in\mathcal{F}} \mu(\partial F) < \infty$. Suppose further that*

$$\inf_{F\in\mathcal{F}} \mathrm{mes}(F) > 0 \quad \text{and} \quad \sup_{F\in\mathcal{F}} \sup_{x,y\in\partial F} \|x-y\|_{\mathbb{R}^2} < \infty,$$

where $\mathrm{mes}(F)$ denotes the Lebesgue measure of the subset $F \subseteq \mathbb{R}^2$ and $\|x-y\|_{\mathbb{R}^2}$ is the Euclidean distance in $\mathbb{R}^2$. Then the Kirchhoff Laplacian $\mathbf{H}$ is self-adjoint and the corresponding heat semigroup $(\mathrm{e}^{-t\mathbf{H}})_{t>0}$ is recurrent. In particular,

$$\lambda_0(\mathbf{H}) = \mathrm{Ch}(\mathcal{G}) = 0.$$

Proof. Under the above assumptions, the intrinsic metric ϱ_η of $(\mathcal{G}, \mu, \mu)$ coincides with the length metric ϱ_0 and $(\mathcal{G}, \varrho_0)$ is complete. Hence, by Theorem 7.1, the Kirchhoff Laplacian $\mathbf{H}$ is self-adjoint. Moreover, by Theorem 7.42, it suffices to prove that

$$\mu(B_r(x)) = O(r^2) \quad \text{as } r \to \infty$$

for some fixed (and hence all) points x on $\mathcal{G}$. Here, $B_r(x) = B_r(x;\varrho_0) \subset \mathcal{G}$ denotes the distance ball of radius r centered at $x \in \mathcal{G}$ with respect to the length metric ϱ_0.

By assumption, the Lebesgue measure of all faces F of $\mathcal{G}$ is uniformly bounded below. Using the condition on the diameter of the faces, it follows that for some uniform constant $b > 0$, each Euclidean ball in $\mathbb{R}^2$ of (large) radius r can intersect at most br^2 faces of $\mathcal{G}$. Moreover, observe that for some $a > 0$,

$$\|u - v\|_{\mathbb{R}^2} \le a\varrho_0(u, v), \quad u, v \in \mathcal{V}.$$

Indeed, by our assumptions, the length $|e|$ of each edge $e \in \mathcal{E}$ is comparable to the distance of its endpoints in $\mathbb{R}^2$ and the estimate immediately follows. Altogether, for every vertex $u \in \mathcal{V}$ and large r,

$$\frac{\mu(B_r(u))}{\sup_{F \in \mathcal{F}} \mu(\partial F)} \le \#\{F \in \mathcal{F} : \partial F \cap \mathcal{V} \cap B_r(u) \ne \varnothing\} \le ba^2r^2$$

and this completes the proof. ■

Remark 8.55. A few remarks are in order.

(i) The recurrence of random walks on edge graphs of tessellations was studied by P. M. Soardi [193] and W. Woess [212]. By [212, Theorem 6.29], the simple random walk on the edge graph of every quasi-regular tessellation of $\mathbb{R}^2$ is recurrent (see [212, Definition 6.28] for definitions and [193] for a preceding result). In fact, [212, Theorem 6.29] can be used to show that Proposition 8.54 holds for weighted metric graphs on quasi-regular tessellations, allowing general edge lengths and weights $\mu \ne \nu$ with the only assumption (8.13) (see the proof of Theorem 8.36). However, the assumptions in Proposition 8.54 allow to give an elegant short proof and we decided to include only this elementary statement.

(ii) The same arguments apply in case when $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is an infinite semiplanar graph with non-negative vertex curvature (see [111, 112] for details and definitions). Again, in this case [112, Theorem 1.3] implies that the simple random walk on $\mathcal{G}_d$ is recurrent, and under the assumption (8.13), the same holds for the semigroup $(e^{-t\mathbf{H}_D})_{t>0}$ on a weighted metric graph $(\mathcal{G}, \mu, \nu)$ over $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$.

8.3.3 Historical remarks and further comments

8.3.3.1 Markovian uniqueness. The strong assumptions on the weights in Corollary 8.47 are indeed necessary. For instance, it was proved in [40] (see also [21, 22] for preceding results) that every locally finite, *vertex-nonamenable*[12] planar graph

[12] This means that there exists some $\varepsilon > 0$ such that for all finite vertex sets $X \subset V$ the inequality $\#\{u \in \mathcal{V} \setminus X : \text{there exists } v \in X \text{ with } u \sim v\} \ge \varepsilon\#X$ holds true.

$\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ admits a non-constant L_{comb}-harmonic function of finite energy, where L_{comb} is the combinatorial Laplacian from Example 6.7. Notice that all graphs $\mathcal{G}_{p,q}$ in Example 8.53 (iii) with $\mathbf{c}_{p,q} > 0$ are vertex-nonamenable and have exactly one graph end if $q < \infty$. Hence, setting $|e| = \nu(e) = 1$ for all edges $e \in \mathcal{E}$, one can obtain a weighted metric graph $(\mathcal{G}_{p,q}, \mu, \nu)$ admitting at least two linearly independent harmonic functions of finite energy. Choosing edge weights μ sufficiently small, these finite energy harmonic functions would also belong to H^1. In particular, this immediately implies that the corresponding (minimal) Gaffney Laplacian has deficiency indices $\mathrm{n}_\pm(\mathbf{H}_{G,\min}) \geq 2$ regardless of the number of ends (for example, one can choose μ sufficiently small in order to ensure a positive spectral gap).

8.3.3.2 Discrete curvature for plane graphs. The results in Section 8.3.2 can also be seen in context with *discrete curvature notions for plane graphs* and their relation to geometric properties. Going back to earlier works such as [95, 118, 197], several notions of curvature have been introduced for plane graphs and they have been used to investigate their geometric properties (see, e.g., the survey [130] and the works [19, 57, 101, 104, 111, 112, 128, 138, 177, 197, 211, 218]). In particular, these curvature notions have been used to investigate isoperimetric constants (see, e.g., [104, 138, 173, 176, 177, 211, 218]) and the obtained spectral estimates resemble an estimate by H. P. McKean in the manifold setting [166]. In the unweighted case $\mu = \nu \equiv \mathbb{1}$, the characteristic edge values (8.20) coincide with the ones introduced in [173, 211] for (unweighted) discrete and metric graphs, respectively (up to the choice of sign). Theorem 8.50 can be seen as the analog of [173, eq. (1.3)] in the weighted setting.

8.3.3.3 Parabolic properties. The above recurrence results (see Proposition 8.54 and Remark 8.55) are also connected to the notion of quasi-isometries between metric spaces (see Remark 6.31). In fact, by [193, Theorem 4.11] the edge graph of every normal tessellation of $\mathbb{R}^2$ is quasi-isometric to $\mathbb{R}^2$ and in this case, the recurrence of the associated discrete Laplacians (and related Kirchhoff Laplacians on metric graphs) follows from the equivalence of recurrence between quasi-isometric spaces, see [47, Théorème 7.2] and also [122, 160]. Clearly, similar considerations apply to (sufficiently well-behaved) tessellations of other two-dimensional Riemannian manifolds (e.g., the Poincaré disc), however, we cannot point to an explicit reference. On the other hand, it should be stressed that the quasi-isometry property breaks down for general quasi-regular tessellations of $\mathbb{R}^2$ (see [193, Section 7]) and hence the results of [193, 212] indeed go beyond this setting.

As for the question of stochastic completeness on weighted tessellating graphs, one can either proceed with the volume tests or by employing various curvature conditions. Notice that, similar to the manifold setting, stochastic incompleteness is related to a very fast decay of curvature to negative infinity (see, e.g., [214, Section 8]).