Appendix B

Dirichlet forms

In this appendix, we collect necessary definitions and facts about Dirichlet forms. The standard reference is [78]. We stress that most of the literature treats Dirichlet forms on real Hilbert spaces (i.e., restricting to real-valued functions), however the theory easily extends to complex Hilbert spaces (see, e.g., [99, Appendix B]).

B.1 Basic notions

In the following, let X be a locally compact separable metric space and μ a positive Radon measure on X of full support. The associated Hilbert space of complex-valued, square integrable functions is denoted by $\mathscr{H} := L^2(X;\mu)$. For a quadratic form $\mathfrak{t}\colon \mathrm{dom}(\mathfrak{t}) \to \mathbb{C}$, whose domain $\mathrm{dom}(\mathfrak{t})$ is a subspace of $\mathscr{H}$, we denote by $\mathfrak{t}[u,v]$, $u, v \in \mathrm{dom}(\mathfrak{t})$ its corresponding sesquilinear form.

Definition B.1. A *Dirichlet form* in $\mathscr{H}$ is a densely defined, non-negative and closed quadratic form $\mathfrak{t}$ satisfying the *Markovian condition*: for all $f \in \mathrm{dom}(\mathfrak{t})$ and any normal contraction[1] φ, $\varphi \circ f \in \mathrm{dom}(\mathfrak{t})$, and

$$\mathfrak{t}[\varphi \circ f] \le \mathfrak{t}[f]. \tag{B.1}$$

A *Dirichlet form in the wide sense* is a quadratic form $\mathfrak{t}$ satisfying all the above conditions, except that $\mathrm{dom}(\mathfrak{t}) \subseteq \mathscr{H}$ is (possibly) not dense.

By the first representation theorem (see [126, Chapter VI.2.1]), to each Dirichlet form we can associate a non-negative, self-adjoint operator $A\colon \mathrm{dom}(A) \to \mathscr{H}$. The corresponding *heat semigroup* $T_t := \mathrm{e}^{-tA}$, $t \ge 0$ is then *Markovian*, that is, all operators T_t satisfy $0 \le T_t f \le 1$ for functions f with $0 \le f \le 1$. The latter means that e^{-tA} is *positivity preserving* (i.e., maps non-negative functions to non-negative functions) and *contractive* (i.e., it is a contraction in L^∞) . Moreover, the heat semigroup has a canonical extension from $L^1(X;\mu) \cap L^\infty(X;\mu)$ to a positive contraction semigroup on $L^p(X;\mu)$ for all $p \in [1,\infty]$ (see, e.g., [51, Theorem 1.4.1] and also [78, p. 56] for details).

Definition B.2. A Dirichlet form $\mathfrak{t}$ is *strongly local* if $\mathfrak{t}[f,g] = 0$ for any functions $f, g \in \mathrm{dom}(\mathfrak{t})$ with compact support[2] and such that f is constant in a neighborhood

[1] A function $\varphi\colon \mathbb{C} \to \mathbb{C}$ is called a *normal contraction* if $|\varphi(x) - \varphi(y)| \le |x - y|$ for all $x, y \in \mathbb{C}$ and $\varphi(0) = 0$.

[2] The support of a measurable function f is defined as the support of the measure $f d\mu$. If f is continuous, this coincides with the closure of $\{x \in X : f(x) \ne 0\}$.

of supp(g). Moreover, a Dirichlet form $\mathfrak{t}$ is *regular* if the set $\operatorname{dom}(\mathfrak{t}) \cap C_c(X)$ is

(i) dense in $C_c(X)$ with respect to the uniform norm $\|\cdot\|_\infty$, and

(ii) dense in $(\operatorname{dom}(\mathfrak{t}), \|\cdot\|_\mathfrak{t})$ with respect to the graph norm $\|\cdot\|_\mathfrak{t}^2 = \mathfrak{t}[\cdot] + \|\cdot\|_{L^2}^2$.

Remark B.3. Let us remark that a regular Dirichlet form $\mathfrak{t}$ has an additional stochastic interpretation: there is an associated (unique up to equivalence) Hunt process $\mathcal{M} = ((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in X})$ on X such that for $t \geq 0$ and $E \subseteq X$ measurable,

$$T_t \mathbb{1}_E(x) = \mathbb{P}_x(X_t \in E) \quad \mu\text{-a.e.}$$

For details on Hunt processes and their relationship to Dirichlet forms we refer to [78, Appendix A.2, Theorem 4.2.8 and Theorem 7.2.1].

B.2 Transience, recurrence and stochastic completeness

Let $\mathfrak{t}$ be a Dirichlet form in $\mathcal{H}$ and let $T_t := \mathrm{e}^{-tA}$, $t > 0$, be the corresponding heat semigroup. For a non-negative function $f \in L^1(X;\mu)$, we define its *potential* $Gf : X \to [0,\infty]$ by

$$Gf(x) = \lim_{N\to\infty} \int_0^N (T_s f)(x)\,\mathrm{d}s,$$

where the limit exists for μ-a.e. $x \in X$. We call the Dirichlet form (or heat semigroup $(T_t)_{t>0}$ associated to it) *transient* if

$$Gf(x) < \infty \quad \mu\text{-a.e.} \quad \text{for all } 0 \leq f \in L^1(X;\mu),$$

and *recurrent* if

$$Gf(x) = 0 \quad \mu\text{-a.e.} \quad \text{or} \quad Gf(x) = \infty \quad \mu\text{-a.e.} \quad \text{for all } 0 \leq f \in L^1(X;\mu).$$

Note that an arbitrary Dirichlet form might be neither recurrent nor transient. However, the dichotomy holds for *irreducible* Dirichlet forms,[3] that is, every irreducible Dirichlet form is either transient or recurrent (but not both!).

Remark B.4. One can reformulate transience/recurrence by means of quadratic forms. For instance (see [78, Theorem 1.5.1]), the Dirichlet form $\mathfrak{t}$ in $\mathcal{H}$ is *transient* exactly when there exists $0 < g \in L^1(X;\mu) \cap L^\infty(X;\mu)$ such that

$$\int_X |f(x)| g(x)\,\mu(\mathrm{d}x) \leq \sqrt{\mathfrak{t}[f]}$$

for all $f \in \operatorname{dom}(\mathfrak{t})$.

[3] A measurable set $Y \subseteq X$ is called $\mathfrak{t}$-invariant if $\mathbb{1}_Y f, \mathbb{1}_{X\setminus Y} f \in \operatorname{dom}(\mathfrak{t})$ and $\mathfrak{t}(f) = \mathfrak{t}(\mathbb{1}_Y f) + \mathfrak{t}(\mathbb{1}_{X\setminus Y} f)$ for any $f \in \operatorname{dom}(\mathfrak{t})$. This is equivalent to the equality $T_t \mathbb{1}_Y f = \mathbb{1}_Y T_t f$ for all $f \in \mathcal{H}$. The form $\mathfrak{t}$ is *irreducible* if $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$ for each $\mathfrak{t}$-invariant set Y.

We also need the following convenient characterization of recurrence (see, e.g., [78, Theorem 1.6.3]).

Lemma B.5. *Let* $\mathfrak{t}$ *be a Dirichlet form in* $\mathcal{H}$. *Then the following are equivalent:*

(i) $\mathfrak{t}$ *is recurrent.*

(ii) *There exists a sequence* (f_n) *in* $\operatorname{dom}(\mathfrak{t})$ *such that* $\lim_{n\to\infty} f_n = \mathbb{1}$ μ*-a.e. on* X *and* $\lim_{n\to\infty} \mathfrak{t}[f_n] = 0$.

A Dirichlet form (or heat semigroup $(T_t)_{t>0}$ associated to it) is *stochastically complete* if[4]

$$T_t \mathbb{1} = \mathbb{1} \quad \mu\text{-a.e.} \tag{B.2}$$

for some (equivalently for all) $t > 0$. For a regular Dirichlet form, this means that the associated stochastic process has infinite lifetime almost surely (see [78, p. 187] for details). If A is the generator of the corresponding heat semigroup $(T_t)_{t>0}$, then stochastic completeness is equivalent to the equality

$$\lambda(A+\lambda)^{-1}\mathbb{1} = \mathbb{1} \quad \mu\text{-a.e.}$$

for some (and hence for all) $\lambda > 0$. Similarly to Lemma B.5, one can characterize stochastic completeness in terms of the quadratic form (e.g., [78, Theorem 1.6.6]).

Lemma B.6. *Let* $\mathfrak{t}$ *be a Dirichlet form in* $\mathcal{H}$. *Then the following are equivalent:*

(i) $\mathfrak{t}$ *is stochastically complete.*

(ii) *There is a sequence* (f_n) *in* $\operatorname{dom}(\mathfrak{t})$ *such that* $0 \le f_n \le 1$, $\lim_{n\to\infty} f_n = \mathbb{1}$ μ*-a.e. on* X*, and*

$$\lim_{n\to\infty} \mathfrak{t}[f_n, g] = 0$$

for all $g \in \operatorname{dom}(\mathfrak{t}) \cap L^1(X;\mu)$.

B.3 Extended Dirichlet spaces

Let $\mathfrak{t}\colon \operatorname{dom}(\mathfrak{t}) \to [0,\infty)$ be a Dirichlet form on $\mathcal{H} = L^2(X;\mu)$. A sequence (f_n) in $\operatorname{dom}(\mathfrak{t})$ is said to be an *approximating sequence* for a function $f\colon X \to \mathbb{C}$ if one has $\lim_{n\to\infty} f_n = f$ μ-a.e. on X and $(f_n)_n$ is a $\mathfrak{t}$-Cauchy sequence, that is,

$$\lim_{m,n\to\infty} \mathfrak{t}[f_n - f_m] = 0.$$

[4]Usually in the literature (see, e.g., [78, 198]) the term *conservative* is used in this context and then one says that (X,μ) is stochastically complete with respect to $\mathfrak{t}$ (or with respect to the heat semigroup $(T_t)_{t>0}$ associated to it).

The *extended Dirichlet space* of $\mathfrak{t}$ is the space of all measurable functions on X which admit at least one approximating sequence. It turns out that (see [78, Theorem 1.5.2]) for a function $f \in \operatorname{dom}(\mathfrak{t}_e)$, where $\operatorname{dom}(\mathfrak{t}_e)$ is the extended Dirichlet space of $\mathfrak{t}$, the limit

$$\mathfrak{t}_e[f] := \lim_{n\to\infty} \mathfrak{t}[f_n]$$

exists and is independent of the approximating sequence (f_n). In particular, this extends the Dirichlet form $\mathfrak{t}$ to a non-negative quadratic form $\mathfrak{t}_e$ on $\operatorname{dom}(\mathfrak{t}_e)$:

$$\begin{aligned} \mathfrak{t}_e\colon \operatorname{dom}(\mathfrak{t}_e) &\to [0,\infty), \\ f &\mapsto \mathfrak{t}_e[f]. \end{aligned}$$

The obtained form $\mathfrak{t}_e$ is called the *extended Dirichlet form* of $\mathfrak{t}$.

The *Markovian condition* also carries over from $\mathfrak{t}$ to $\mathfrak{t}_e$: for each normal contraction $\varphi\colon \mathbb{C} \to \mathbb{C}$ and $f \in \operatorname{dom}(\mathfrak{t}_e)$, $\varphi \circ f$ belongs to $\operatorname{dom}(\mathfrak{t}_e)$ and (B.1) holds (see, e.g., [78, Corollary 1.6.3]). Moreover, the form domain of $\mathfrak{t}$ (see [78, Theorem 1.5.2]) can be recovered from $\mathfrak{t}_e$ by the relation

$$\operatorname{dom}(\mathfrak{t}) = \operatorname{dom}(\mathfrak{t}_e) \cap L^2(X;\mu). \tag{B.3}$$

The above notions lead to another convenient characterization of recurrence (see [78, Theorem 1.6.3]):

Lemma B.7. *Let* $\mathfrak{t}$ *be a Dirichlet form on* $\mathcal{H}$*. Then* $\mathfrak{t}$ *is recurrent if and only if* $\mathbb{1}$ *belongs to* $\operatorname{dom}(\mathfrak{t}_e)$ *and* $\mathfrak{t}_e[\mathbb{1}] = 0$.