Appendix C

Heat kernel bounds

In this appendix, we collect some useful results relating heat kernel decay with Sobolev- and Nash-type inequalities. Throughout this section we shall assume that $A = A^* \ge 0$ is a generator of a Markovian semigroup in $L^2(X;\mu)$ (see Appendix B for details). The corresponding quadratic form, which is a Dirichlet form on $L^2(X;\mu)$, is denoted by $\mathfrak{Q}_A$, that is,

$$\mathfrak{Q}_A[f] = \|A^{1/2} f\|_2^2, \quad \mathrm{dom}(\mathfrak{Q}_A) = \mathrm{dom}(A^{1/2}),$$

where A^γ, $\gamma > 0$, is a non-negative self-adjoint operator. Recall that (see, e.g., [51, Section 2.1]), the semigroup $T_t = \mathrm{e}^{-tA}$ is called *ultracontractive* if e^{-tA} is bounded as an operator from $L^2(X;\mu)$ to $L^\infty(X;\mu)$ for all $t > 0$. By duality, the latter is equivalent to e^{-tA} being bounded from $L^1(X;\mu)$ to $L^\infty(X;\mu)$ for all $t > 0$.

We begin with the following simple result (see [51, Theorem 2.4.1]).

Proposition C.1. *Let $\gamma > 0$ be fixed. If*[1]

$$\|f\|_\infty \le C_1 \|(A + \mathrm{I})^{\gamma/2} f\|_2$$

for all $f \in \mathrm{dom}(A + \mathrm{I})^{\gamma/2}$, then e^{-tA} is ultracontractive and there is a positive constant $C_2 > 0$ such that

$$\|\mathrm{e}^{-tA}\|_{1\to\infty} \le C_2 t^{-\gamma} \tag{C.1}$$

for all $t \in (0,1)$. Conversely, if (C.1) holds on $(0,1)$ for some $\gamma > 0$, then

$$\|f\|_\infty \le C(\varepsilon) \|(A + \mathrm{I})^{\gamma/2+\varepsilon} f\|_2, \quad f \in \mathrm{dom}(A + \mathrm{I})^{\gamma/2+\varepsilon},$$

is valid for any $\varepsilon > 0$.

The next result is a famous theorem of N. Th. Varopoulos (see [204], [206, Theorem II.5.2], [51, Theorem 2.4.2]).

Theorem C.2 ([204]). *Let $D > 2$ be fixed. Then a bound of the form*

$$\|\mathrm{e}^{-tA}\|_{1\to\infty} \le C_1 t^{-D/2} \tag{C.2}$$

for all $t > 0$ is equivalent to the validity of the Sobolev-type inequality

$$\|f\|^2_{\frac{2D}{D-2}} \le C_2 \mathfrak{Q}_A[f] \quad \textit{for all } f \in \mathrm{dom}(\mathfrak{Q}_A).$$

[1] Here we use the standard notation $\|f\|_p := \|f\|_{L^p(X;\mu)}$ for $f \in L^p(X;\mu)$ and $\|T\|_{p\to q}$ denotes the norm of a linear operator T acting from $L^p(X;\mu)$ to $L^q(X;\mu)$.

As an immediate corollary we get the following claim relating the behavior of the heat kernel as $t \to 0$ with the Sobolev inequality (see [51, Corollary 2.4.3]).

Corollary C.3. *Let $D > 2$ be fixed. Then (C.2) holds for all $t \in (0,1)$ if and only if*

$$\|f\|^2_{\frac{2D}{D-2}} \le C(\mathfrak{Q}_A[f] + \|f\|_2^2)$$

for all $f \in \mathrm{dom}(\mathfrak{Q}_A)$.

Notice that $\|\cdot\|_{\mathfrak{Q}} = \mathfrak{Q}_A[\cdot] + \|\cdot\|_2^2$ is the graph norm and it is equivalent to the energy (semi-)norm $\mathfrak{Q}_A[\cdot]$ if and only if A has a positive spectral gap, $\lambda_0(A) > 0$.

Let us also recall the following result relating on-diagonal heat kernel estimates with Nash-type inequalities ([39, Theorem 2.1], [51, Theorem 2.4.6]).

Theorem C.4 ([39]). *Estimate (C.2) holds true for all $t > 0$ with some fixed $D > 0$ if and only if the inequality*

$$\|f\|_2^{2+4/D} \le C\, \mathfrak{Q}_A[f]\, \|f\|_1^{4/D} \tag{C.3}$$

holds true for all $f \in \mathrm{dom}(\mathfrak{Q}_A) \cap L^1(X;\mu)$. Moreover, the inequality

$$\|f\|_2^{2+4/D} \le C(\mathfrak{Q}_A[f] + \|f\|_2^2)\|f\|_1^{4/D} \tag{C.4}$$

holds for all $f \in \mathrm{dom}(\mathfrak{Q}_A) \cap L^1(X;\mu)$ if and only if (C.2) holds for all $t \in (0,1)$.

Remark C.5. Taking into account that both (C.3) and (C.4) are homogeneous (with respect to $f \to cf$, $c \in \mathbb{C}$), one can restrict in (C.3) to functions with $\|f\|_1 = 1$ or $\|f\|_1 = c$ for any fixed $c > 0$. Moreover, $\mathfrak{Q}_A[|f|] \le \mathfrak{Q}_A[f]$ for all $f \in \mathrm{dom}(\mathfrak{Q}_A)$ since $\mathfrak{Q}_A$ is a Dirichlet form. Therefore, in all the above theorems one can further restrict to non-negative functions.

The following extension of Theorem C.2 and Theorem C.4 to sub-exponential scales is due to T. Coulhon (see [46, Theorem II.5]).

Theorem C.6. *Let $m\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a decreasing bijection such that its logarithmic derivative has polynomial growth, i.e., $M := -\log m$ satisfies for some $\alpha > 0$*

$$M'(x) \ge \alpha M'(s), \quad \textit{for all } s > 0 \quad \textit{and} \quad x \in [s, 2s]. \tag{C.5}$$

Then the following conditions are equivalent:

(i) e^{-tA} *is ultracontractive and there is $C_1 > 0$ such that*

$$\|\mathrm{e}^{-tA}\|_{1\to\infty} \le m(C_1 t) \quad \textit{for all } t > 0.$$

(ii) *there is $C_2 > 0$ such that for all $f \in \mathrm{dom}(\mathfrak{Q})$ with $\|f\|_{L^1} = 1$,*

$$\theta_m(\|f\|_2^2) \le C_2 \mathfrak{Q}_A[f],$$

where $\theta_m := -m' \circ m^{-1}$.