Appendix D

Glossary of notation

Basic notation

- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ have their usual meaning.
- For $a \in \mathbb{R}$, $\mathbb{Z}_{\geq a} := \mathbb{Z} \cap [a, \infty)$, $\mathbb{R}_{\geq a} := \mathbb{R} \cap [a, \infty)$, and $\mathbb{R}_{>a} := \mathbb{R} \cap (a, \infty)$.
- z^* denotes the complex conjugate of $z \in \mathbb{C}$.
- $\mathcal{I} \subseteq \mathbb{R}$ usually denotes an interval, that is, a connected subset of $\mathbb{R}$.
- $\mathcal{I}_\ell = [0, \ell]$, $\ell \in \mathbb{R}_{>0}$.
- For a given set S, $\#S$ denotes its cardinality if S is finite; otherwise set $\#S = \infty$.
- We shall denote by (x_n) or sometimes $(x_n)_{n\geq 0}$ a sequence $(x_n)_{n=0}^{\infty}$.

Graphs

- $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a graph with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$.
- $\mathcal{E}_v$ is the set of edges at $v \in \mathcal{V}$.
- $\vec{\mathcal{G}}_d = (\mathcal{V}, \vec{\mathcal{E}})$ is an oriented graph and $\vec{\mathcal{E}}$ the set of oriented edges.
- $\vec{\mathcal{E}}_v$ is the set of oriented (both incoming and outgoing) edges at v.
- e_ι and e_τ are the initial and terminal vertices of an oriented edge $\vec{e}$.
- deg is the vertex degree function.
- Deg is the weighted vertex degree.
- b or $(\mathcal{V}, m; b)$ is a weighted graph on $\mathcal{V}$.
- (b, c) or $(\mathcal{V}, m; b, c)$ is a weighted graph with killing term c on $\mathcal{V}$.
- $\mathcal{G}_b = (\mathcal{V}, \mathcal{E}_b)$ is the underlying simple graph of b.
- $\mathcal{G} = (\mathcal{G}_d, |\cdot|)$ is a metric graph or its model.
- $(\mathcal{G}, \mu, \nu) = (\mathcal{G}_d, |\cdot|, \mu, \nu)$ is a weighted metric graph or its model.
- ϱ_0 is the length metric on $\mathcal{G}$, i.e., the natural path metric on $\mathcal{G}$.
- ϱ_η is the intrinsic metric on $(\mathcal{G}, \mu, \nu)$ and $\eta = \sqrt{\frac{\mu}{\nu}}$ is the intrinsic weight.
- ϱ_m is the star path metric on $\mathcal{V}$ corresponding to the star weight m.
- S_n is the n-th combinatorial sphere of a rooted graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$.
- $\mathfrak{C}(\mathcal{G})$ is the space of topological ends of a metric graph $\mathcal{G}$.
- $\mathfrak{C}_0(\mathcal{G}; \mu)$ is the set of finite volume (with respect to μ) ends of $\mathcal{G}$.

Function spaces

- X is a locally compact Hausdorff space X, and μ is a Borel measure on X.
- $C(X)$ is the space of continuous functions on X.
- $C(X)$ is the set of complex-valued functions on X if X is countable.
- $C_b(X)$, $C_0(X)$, and $C_c(X)$ are, respectively, the spaces bounded, vanishing at infinity, and compactly supported continuous functions on X.
- $C^+(X)$ is the cone of positive functions on X.
- $\mathcal{F}_b(\mathcal{V})$ denotes the domain of definition of the formal graph Laplacian $L_{c,b,m}$.
- $\mathrm{CA}(\mathcal{G} \setminus \mathcal{V})$ is the set of continuous, edgewise affine functions on a metric graph $\mathcal{G}$.
- $L^p(X;\mu)$ is the complex Banach space of measurable functions, $p \in [1,\infty]$.
- $L^p_c(X;\mu)$ is the subspace of compactly supported functions in $L^p(X;\mu)$.
- $\ell^p(X;m) := L^p(X;m)$, $\ell^p_c(X;m) := L^p_c(X;m)$ if X is countable.
- $H^1_{\mathrm{loc}}(\mathcal{G} \setminus \mathcal{V})$ is the space of all edgewise H^1 functions.
- $H^1_{\mathrm{loc}}(\mathcal{G}) = H^1_{\mathrm{loc}}(\mathcal{G} \setminus \mathcal{V}) \cap C(\mathcal{G})$.
- $H^1_c(\mathcal{G}) = H^1_{\mathrm{loc}}(\mathcal{G}) \cap C_c(\mathcal{G})$.
- $H^1(\mathcal{G}) = H^1(\mathcal{G};\mu,\nu)$ is the first (weighted) Sobolev space on $\mathcal{G}$.
- $H^1_0(\mathcal{G}) = H^1_0(\mathcal{G};\mu,\nu) = \overline{H^1_c(\mathcal{G})}^{\|\cdot\|_{H^1(\mathcal{G};\mu,\nu)}}$.
- $H^1_0(\mathcal{G} \setminus \mathcal{V})$ is the subspace of $H^1(\mathcal{G})$-functions vanishing at all vertices.
- $\dot{H}^1(\mathcal{G}) = \dot{H}^1(\mathcal{G},\nu)$ is the space of functions of finite energy on $\mathcal{G}$.

Laplacians and their quadratic/energy forms

- $L = L_{c,b,m}$ is the formal graph Laplacian on $(\mathcal{V}, m; b, c)$.
- $\mathbf{h}$, $\mathbf{h}'$, $\mathbf{h}^0$ are the maximal, pre-minimal, minimal graph Laplacians in $\ell^2(\mathcal{V};m)$.
- $\mathbf{h}_D$ and $\mathbf{h}_N$ are the Dirichlet and Neumann Laplacians in $\ell^2(\mathcal{V};m)$.
- $\mathfrak{q} = \mathfrak{q}_{c,b}$ is the energy form on (b,c).
- $\mathfrak{q}_D$ and $\mathfrak{q}_N$ are the maximal and the minimal forms in $\ell^2(\mathcal{V};m)$.
- Δ is the weighted Laplacian on $(\mathcal{G},\mu,\nu)$.
- $\mathbf{H}$, $\mathbf{H}'$ and $\mathbf{H}^0$ are the maximal, pre-minimal and minimal Kirchhoff Laplacians in $L^2(\mathcal{G};\mu)$.
- $\mathbf{H}_D$ and $\mathbf{H}_N$ are the Dirichlet and Neumann Laplacians in $L^2(\mathcal{G};\mu)$.
- $\mathbf{H}_G$ and $\mathbf{H}_{G,\min}$ are the maximal and minimal Gaffney Laplacians in $L^2(\mathcal{G};\mu)$.
- $\mathbf{H}_\alpha$, $\mathbf{H}'_\alpha$ and $\mathbf{H}^0_\alpha$ are the maximal, pre-minimal and minimal Laplacians with δ-couplings.

- $\mathfrak{Q}$ is the energy form on $(\mathcal{G}, \mu, \nu)$.
- $\mathfrak{Q}_D$ and $\mathfrak{Q}_N$ are the maximal and the minimal forms in $L^2(\mathcal{G}; \mu)$.

Operator theory

- $\mathcal{H}$ and $\mathfrak{H}$ are separable complex Hilbert spaces.
- $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$.
- $\mathfrak{S}_p(\mathcal{H})$, $p \in (0, \infty]$ are the Schatten–von Neumann ideals in $\mathcal{B}(\mathcal{H})$.
- $\mathrm{I}_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$, and $\mathrm{I}_n := \mathrm{I}_{\mathbb{C}^n}$.
- $\mathbb{O}_{\mathcal{H}}$ is the zero operator in $\mathcal{H}$, and $\mathbb{O}_n := \mathbb{O}_{\mathbb{C}^n}$.
- For a self-adjoint operator A in $\mathcal{H}$, $\lambda_0(A)$ and $\lambda_0^{\mathrm{ess}}(A)$ denote the bottoms of the spectrum, respectively, of the essential spectrum

  $$\lambda_0(A) = \inf \sigma(A), \quad \lambda_0^{\mathrm{ess}}(A) = \inf \sigma_{\mathrm{ess}}(A),$$

  and $A^- := A\mathbb{1}_{(-\infty,0)}(A)$, where $\mathbb{1}_{(-\infty,0)}(A)$ is the spectral projection on the negative subspace of A.
- For a closed symmetric operator A,
  - $\mathrm{Ext}(A)$ is the set of its proper extensions,
  - $\mathrm{Ext}_S(A)$ is the set of its self-adjoint extensions.
- For a non-negative symmetric operator A,
  - $\mathrm{Ext}_S^+(A)$ is the set of its non-negative self-adjoint extensions,
  - $\mathrm{Ext}_S^\kappa(A)$, $\kappa \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, are self-adjoint extensions of A with the total multiplicity of the negative spectrum equal to κ,
  - $\mathrm{Ext}_M(A)$ is the set of Markovian extensions of A.