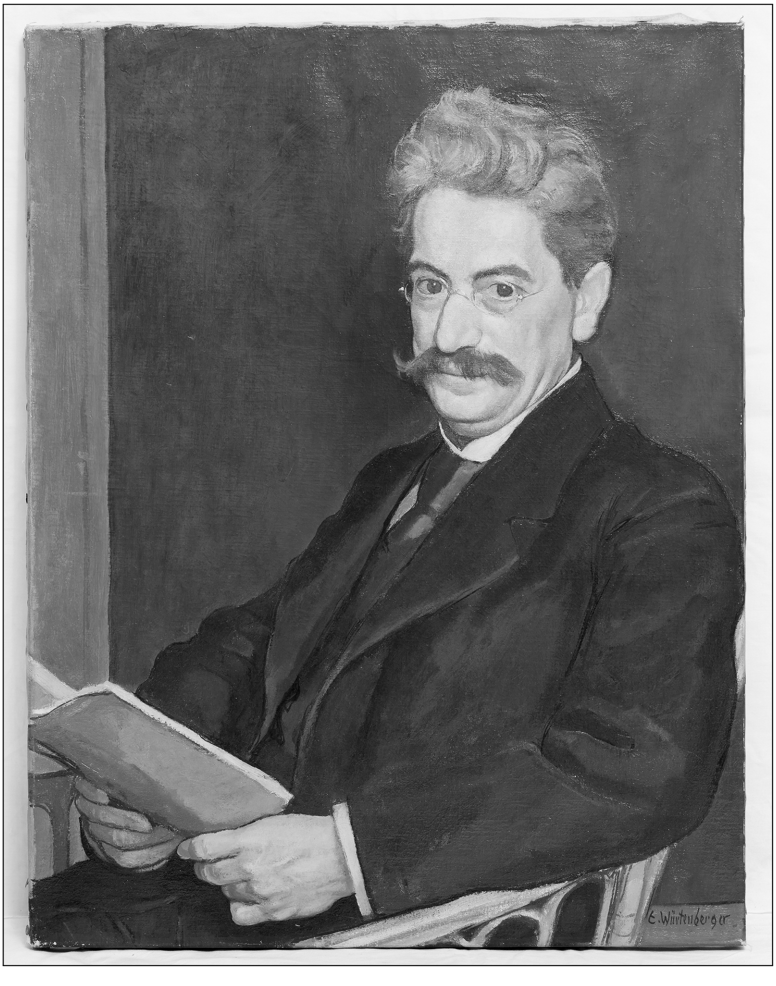
Dedicated to Greta Thunberg



Adolf Hurwitz, 1859–1919. Oil painting by Ernst Württemberger (ca. 1915), stored in the ETH Library, Zürich, Kunstinventar (Inv.-Nr. Ki-00022).

Preface by the Editors

Quaternions are non-commutative generalizations of the complex numbers. They were discovered (or invented) by William Rowan Hamilton about 175 years ago. Their number-theoretical aspects were first investigated by Rudolf Lipschitz in the 1880s. His approach, however, was revised and streamlined by the work of Adolf Hurwitz in 1896. The event of these two studies on quaternions appears to be typical for the state of the art of number theory at the turn of the twentieth century, where new concepts (e.g., ideals) were introduced for a better understanding of a generalized arithmetic of algebraic integers. The quaternion counterpart of integers is interesting not only for the parallels and differences to the ordinary integers and their counterparts in algebraic number fields, but their properties also shed light on the arithmetic of integers. Hurwitz's definition of quaternion integers led him to an elegant proof of the four-square theorem that every positive integer can be represented as a sum of at most four integer squares, e.g.,

$$1919 = 1^2 + 10^2 + 27^2 + 33^2, \quad 2019 = 0^2 + 13^2 + 25^2 + 35^2.$$

This celebrated theorem had been first proved by Joseph-Louis Lagrange in 1770, about 250 years ago, building on Leonhard Euler's four-square identity which not only plays a central role with respect to quaternions but also in the context of orthogonal matrices and magic squares of squares, as we shall explain later. Another feature of Hurwitz's approach is the analogue of the prime factorization of the non-commutative quaternions.

Hurwitz decided to write a textbook on the number theory of quaternions based on his 1896 research paper; it was published in 1919, only a few months before his death 100 years ago. With his experience of courses on number theory in general, and quaternions in particular, given at the Federal Polytechnic School of Zurich (aka *Eidgenössische Polytechnische Hochschule Zürich* since 1911), Hurwitz aimed at making his approach "accessible to a wider circle" of readers; in his preface (see page 19) we can read that he had not only number theorists or professional mathematicians in mind, but everyone with a mathematical education and interest in the topic. Indeed, his booklet, published in autumn 1919 as *Vorlesungen über die Zahlentheorie der Quaternionen*, is easy to read and written with a recognizable emphasis of didactic preparation. In fact, this period saw a number of new textbooks and research monographs pairing pedagogy with science.

Hurwitz's mentor, the eminent Felix Klein, wrote:

> *Hurwitz had been called an aphorist. In full mastery of the disciplines in question, he selects an important problem here and there, which he promotes by a significant amount. Each of his works stands on its own and is a complete work.*[1]

In particular, his booklet on quaternions received very positive reviews. For example, Philipp Furtwängler remarked that "the book is, as all of Hurwitz's treatise, written very clearly and captivatingly and inspires some new questions."[2] However, the developments of his time have rendered the work almost forgotten.

With our translation (see Chapter II) we wish to make Hurwitz's beautiful *Lectures on the number theory of quaternions* accessible to a broad international readership. However, we pursue further aims with this book.

First of all, we wish to explain in detail the advantage of Hurwitz's approach to developing an arithmetic of quaternions, as compared to Lipschitz's straightforward attempt. In fact, in number-theoretical questions around quaternions it is the algebraic structure which leads to deeper results; more precisely, the correct notion of "quaternion integers" within the set of all quaternions. Hurwitz himself stressed in his preface (still page 19) the proximity to algebraic number theory.

As a matter of fact, the universal mathematician Hurwitz was well aware of the latest methods in algebraic number theory. Around the time that Hurwitz began his studies on quaternions, his former pupil David Hilbert wrote his influential *Zahlbericht* on the recent progress in this quickly developing discipline. Jeremy Gray described their relationship as follows:

> *From 1886 to 1892, when Minkowski was in Bonn, Hilbert went almost daily on mathematical walks with Hurwitz. Learning to walk and talk mathematics right across the syllabus was very important for Hilbert, and he later took the tradition with him to Göttingen. With Hurwitz, he said, he rummaged through every corner of mathematics, with Hurwitz always in the lead.*[3]

Indeed, Hurwitz had a great expertise in algebraic questions and he was proof-reading parts of Hilbert's *Zahlbericht*. So Hurwitz was definitely well prepared to build up a number theory of quaternions.

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[1] „Man hat Hurwitz einen Aphoristiker genannt. In voller Beherrschung der in Betracht kommenden Disziplinen sucht er sich hier und dort ein wichtiges Problem heraus, das er jeweils um ein bedeutendes Stück fördert. Jede seiner Arbeiten steht für sich und ist ein abgeschlossenes Werk." [156, p. 328]

[2] „Das Buch ist, wie alle Abhandlungen von Hurwitz, sehr klar und fesselnd geschrieben und regt zu manchen neuen Fragestellungen an." [97]

[3] see [107, p. 18]

Another purpose of this book is to serve as a number theory *textbook*, giving an introduction to the arithmetic of quaternions and disclosing analogies with some basic algebraic number theory. Jeremy Gray's recent *History of Abstract Algebra* has been a great inspiration with respect to this aim, although our book takes a rather different road. The quaternion part of the present book is in Hurwitz's spirit, whereas the additions with respect to algebraic number theory match the contemporary perspective on number theory. For this goal *we comment in gray boxes on Hurwitz's lectures and derive the corresponding arithmetic for the ring of Gaussian integers and other rings of integers besides their quaternion counterpart!* Here our notation differs slightly from Hurwitz's notation for didactical reasons in order to stress the differences between both parts.

Hurwitz's booklet is the core of this publication. Moreover, we provide a translation of his article *Über die Komposition der quadratischen Formen*, posthumously published in the renowned *Mathematische Annalen* in 1922 (see Chapter V). This paper includes a refinement of the older article *Ueber die Composition der quadratischen Formen von beliebig vielen Variabeln* of Hurwitz, published in the *Göttinger Nachrichten* in 1898; both contain a proof of the so-called 1-2-4-8 theorem that there are no bilinear n-square identities except for $n = 1$ (the trivial case), $n = 2$ (complex numbers), $n = 4$ (quaternions), and $n = 8$ (related to so-called octonions discovered by John Graves).

The translation follows Hurwitz's language closely. In a few places, however, we have chosen to deviate from literal translation by using better fitting expressions; there should be no change in content at any point. A glossary helps the reader to understand differences in the old-fashioned naming beyond linguistic issues. Further chapters highlight the historical context, Hurwitz's mathematical diaries and the impact of his quaternion arithmetic. We hope that these additions make the reading a convenient experience.

We share Hurwitz's opinion that no deeper knowledge of number theory or algebra is needed for this introduction to quaternion arithmetic beyond basic knowledge of linear algebra, simple divisibility properties of the integers, and modular arithmetic. In the recent past, we have tested part of the present material by giving lectures on Hurwitz's number theory of quaternions at workshops and in university courses in Denmark (Aarhus), Germany (Augsburg, Berlin, Paderborn and Würzburg), Austria (Linz) and Thailand (Walailak University in Nakhon Si Thammarat). We completely agree with Hurwitz's final words of his preface in the hope that (t)his book encourages the promotion of this topic further.

Oberwolfach, September $2019 = 1^2 + 13^2 + 43^2$

Nicola Oswald
Jörn Steuding

Post scriptum. In the meantime, John Voight has published his monograph [258], an excellent and comprehensive source to quaternions and their generalizations.