

Contents

Preface by the Editors	ix
Acknowledgments	xiii
I Introduction	1
II Lectures on the number theory of quaternions	17
Preface	19
1 The quaternions and how to compute with them	21
Box A: Cayley's definition of quaternions using matrices	27
2 The field of quaternions and their permutations and inversions	33
Box B: Structure! About fields, skew fields, rings, algebras, and their automorphisms	37
3 The field R and its permutations	45
Box C: Algebraic numbers and quadratic fields	48
Box D: How to introduce algebraic integers? Some thoughts about number theory	53
4 The integer quaternions	59
Box E: Integral closure and orders in number fields	67
5 The permutations of integer quaternions	73
Box F: Commutative divisibility	76
6 Greatest common divisor and quaternion ideals	77
Box G: Euclidean rings: From geometry to arithmetic	81
7 Even and odd quaternions. Associated and primary quaternions	91
Box H: Modular arithmetic and the Chinese remainder theorem	95

8	The integer quaternions modulo an odd number	97
	Box I: Arithmetical functions, Dirichlet series and Euler products	105
9	The prime quaternions	109
	Box J: Prime quaternions and prime numbers	113
10	The factorization theorem	119
	Box K: Almost unique factorization of quaternion integers and consequences	124
11	The representations of a positive integer as a sum of four squares	129
	Box L: The number of representations as a sum of squares	131
12	A problem due to Euler	139
	Box M: Magic squares of squares	150
	Notes and addendum	157
III	Two Arithmetics	161
	III.1 Rudolf Lipschitz and Adolf Hurwitz	161
	III.2 Differences and references	168
	Box N: Another proof of the four-square theorem following Lipschitz and Dickson	176
	III.3 The notion of integers in the theory of algebraic numbers	178
	III.4 Research in <i>Vorlesungen</i>	184
IV	A view into Hurwitz's <i>Mathematische Tagebücher</i>	191
	IV.1 The legacy	191
	IV.2 Diaries No. 14, 15 and 25	195
	IV.3 Quaternions in the later diaries	200
	Box O: Dickson's generalized maximal orders	207
V	On the composition of quadratic forms	211
	I.	214
	II.	219
	III.	221
	IV.	228
	Box P: Octonions and the eight-square identity	237

VI Abstraction and generalization	241
VI.1 Octonions and algebras	241
Box Q: A letter from Hades	243
VI.2 Sums of squares, quadratic forms, and more	253
VII Epilogue	257
Box R: Fermat’s method of infinite descent and Euler’s proof of the four-square theorem	259
Appendices	267
Appendix A: Theses	267
Appendix B: Elementary number theory in a nutshell	268
References	273
List of protagonists	287
Index	289