Abstract

Applying the theory of strongly smooth operators, we derive a limiting absorption principle (LAP) on any compact interval in $\mathbb{R}\backslash\{0\}$ for the free massless Dirac operator,

$$H_0 = \alpha \cdot (-i\nabla)$$

in $[L^2(\mathbb{R}^n)]^N$, $n \in \mathbb{N}$, $n \geq 2$, $N = 2^{\lfloor (n+1)/2 \rfloor}$. We then use this to demonstrate the absence of singular continuous spectrum of interacting massless Dirac operators $H = H_0 + V$, where the entries of the (essentially bounded) matrix-valued potential V decay like $O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$. This includes the special case of electromagnetic potentials decaying at the same rate. In addition, we derive a one-to-one correspondence between embedded eigenvalues of H in $\mathbb{R}\backslash\{0\}$ and the eigenvalue -1 of the (normal boundary values of the) Birman–Schwinger-type operator

$$\overline{V_2\big(H_0 - (\lambda_0 \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}.$$

Upon expressing $\xi(\,\cdot\,; H, H_0)$ as normal boundary values of regularized Fredholm determinants to the real axis, we then prove that in the concrete case (H, H_0), under appropriate hypotheses on V (implying the decay of V like $O(|x|^{-n-1-\varepsilon})$ as $|x| \to \infty$), the associated spectral shift function satisfies $\xi(\,\cdot\,; H, H_0) \in C((-\infty, 0) \cup (0, \infty))$, and that the left and right limits at zero, $\xi(0_\pm; H, H_0) = \lim_{\varepsilon \downarrow 0} \xi(\pm\varepsilon; H, H_0)$, exist.

This fact is then used to express the resolvent regularized Witten index of the non-Fredholm operator $\boldsymbol{D_A}$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ given by

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A}) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}(\boldsymbol{A_-}),$$

where

$$\boldsymbol{A} = \boldsymbol{A_-} + \boldsymbol{B}, \quad \operatorname{dom}(\boldsymbol{A}) = \operatorname{dom}(\boldsymbol{A_-}).$$

Here $\boldsymbol{A}$, $\boldsymbol{A_-}$, $\boldsymbol{A_+}$, $\boldsymbol{B}$, and $\boldsymbol{B_+}$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ are generated with the help of the Dirac-type operators H, H_0 and potential matrices V, via

$$A(t) = A_- + B(t),\ t \in \mathbb{R}, \quad A_- = H_0, \quad A_+ = A_- + B_+ = H,$$
$$B(t) = b(t)B_+,\ t \in \mathbb{R}, \quad B_+ = V,$$

in $[L^2(\mathbb{R}^n)]^N$, assuming

$$b^{(k)} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dt),\ k \in \mathbb{N}_0, \quad b' \in L^1(\mathbb{R}; dt),$$
$$\lim_{t \to \infty} b(t) = 1, \quad \lim_{t \to -\infty} b(t) = 0.$$

In particular, $A_\pm$ are the asymptotes of the family $A(t)$, $t \in \mathbb{R}$, as $t \to \pm\infty$ in the norm resolvent sense. (Here $L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}}^{\oplus} dt\, \mathcal{H}$ and $\boldsymbol{T} = \int_{\mathbb{R}}^{\oplus} dt\, T(t)$ represent direct integrals of Hilbert spaces and operators.)

Introducing the nonnegative, self-adjoint operators

$$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}}, \quad \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^*$$

in $L^2(\mathbb{R}; [L^2(\mathbb{R}^n)]^N)$, one of the principal results proved in this manuscript expresses the resolvent regularized Witten index $W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}})$ of $\boldsymbol{D}_{\boldsymbol{A}}$ in terms of spectral shift functions via

$$W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}}) = \xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \big[\xi(0_+; H, H_0) + \xi(0_-; H, H_0)\big]/2, \\ k \in \mathbb{N},\ k \geq \lceil n/2 \rceil.$$

Here the notation $\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1)$ indicates that 0 is a right Lebesgue point for $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$, and $W_{k,r}(T)$ represents the kth resolvent regularized Witten index of the densely defined, closed operator T in the complex, separable Hilbert space $\mathcal{K}$, defined by

$$W_{k,r}(T) = \lim_{\lambda \uparrow 0} (-\lambda)^k \operatorname{tr}_{\mathcal{K}} \big((T^*T - \lambda I_{\mathcal{K}})^{-k} - (TT^* - \lambda I_{\mathcal{K}})^{-k}\big),$$

whenever the limit exists for some $k \in \mathbb{N}$.

Keywords. Dirac operators, limiting absorption principle, spectral shift function, Witten index

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