Chapter 1

Introduction

The primary motivation for writing this manuscript was to express the Witten index of a certain class of non-Fredholm operators, generated from multi-dimensional, massless Dirac operators, in terms of appropriate underlying spectral shift functions. This goal necessitated a detailed control over continuity properties (more precisely, the existence of Lebesgue points) for the spectral shift functions involved, and hence the bulk of this manuscript is devoted to an exhaustive investigation of the spectral properties of multi-dimensional, massless Dirac operators.

We refer the reader to Appendix E for notational conventions used throughout this manuscript.

To set the stage, let $n \in \mathbb{N}$, $n \geq 2$, $N = 2^{\lfloor (n+1)/2 \rfloor}$, and denote by α_j, $1 \leq j \leq n$, $\alpha_{n+1} := \beta$, $n+1$ anti-commuting Hermitian $N \times N$ matrices with squares equal to I_N, that is,

$$\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} I_N, \quad 1 \leq j, k \leq n+1, \tag{1.1}$$

and introduce in $[L^2(\mathbb{R}^n)]^N$ the free massless Dirac operator

$$H_0 = \alpha \cdot (-i\nabla) = \sum_{j=1}^{n} \alpha_j (-i\partial_j), \quad \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N, \tag{1.2}$$

where $\partial_j = \partial/\partial x_j$, $1 \leq j \leq n$. Introducing the self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1 \leq \ell,\ell' \leq N}$ satisfying for some fixed $\rho \in (1, \infty)$, $C \in (0, \infty)$,

$$\begin{aligned} &V \in [L^\infty(\mathbb{R}^n)]^{N \times N}, \\ &\big|V_{\ell,\ell'}(x)\big| \leq C\langle x \rangle^{-\rho} \quad \text{for a.e. } x \in \mathbb{R}^n, 1 \leq \ell, \ell' \leq N, \end{aligned} \tag{1.3}$$

then permits one to introduce the massless Dirac operator H in $[L^2(\mathbb{R}^n)]^N$ via

$$H = H_0 + V, \quad \operatorname{dom}(H) = \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N. \tag{1.4}$$

In this context we recall our convention

$$[L^2(\mathbb{R}^n)]^N = L^2\big(\mathbb{R}^n; \mathbb{C}^N\big), \quad [W^{1,2}(\mathbb{R}^n)]^N = W^{1,2}\big(\mathbb{R}^n; \mathbb{C}^N\big), \quad \text{etc.},$$

to be used throughout.

Then H_0 and H are self-adjoint in $[L^2(\mathbb{R}^n)]^N$, with essential spectrum covering the entire real line,

$$\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H_0) = \sigma(H_0) = \mathbb{R},$$

In addition,

$$\sigma_{\rm ac}(H_0) = \mathbb{R}, \quad \sigma_{\rm p}(H_0) = \sigma_{\rm sc}(H_0) = \emptyset.$$

Relying on the theory of (local) Kato-smoothness (see, e.g., [140, Section XIII.7], [184, Chapter 4], and [186, Chapters 0–2]) and its variant, strong (local) Kato-smoothness (see, e.g., [184, Chapter 4], [186, Chapters 0–2]), then yields under the stated hypotheses on V that actually,

$$\sigma_{ess}(H) = \sigma_{ac}(H) = \mathbb{R}, \tag{1.5}$$

$$\sigma_{sc}(H) = \emptyset, \tag{1.6}$$

$$\sigma_{s}(H) \cap (\mathbb{R}\backslash\{0\}) = \sigma_{p}(H) \cap (\mathbb{R}\backslash\{0\}), \tag{1.7}$$

with the only possible accumulation points of $\sigma_p(H)$ being 0 and $\pm\infty$. Relations (1.5)–(1.7) describe only the tip of the proverbial iceberg in connection with Chapters 2 and 3. In fact, leading up to (1.7) we establish a limiting absorption principle (LAP) on any compact interval in $\mathbb{R}\backslash\{0\}$ for the free (i.e., non-interacting) massless Dirac operator H_0, prove the absence of singular continuous spectrum of $H = H_0 + V$ for matrix elements $V_{\ell,\ell'}$, $1 \leq \ell, \ell' \leq N$, of V decaying like $O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$, derive Hölder continuity of the boundary values $(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}$ in appropriate weighted L^2-spaces for λ varying in compact subintervals of $\mathbb{R}\backslash\{0\}$, and derive Hölder continuity of the boundary values $(H - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}$ in appropriate weighted L^2-spaces for λ varying in compact subintervals of $\mathbb{R}\backslash\{0\}$ away from the possibly embedded eigenvalues of H. In particular, factoring V into $V = V_1^* V_2$, we derive a $1-1$-correspondence between embedded eigenvalues of H in $\mathbb{R}\backslash\{0\}$ and the eigenvalue -1 of the (normal boundary values of the) Birman–Schwinger-type operator $\overline{V_2(H_0 - (\lambda_0 \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}$.

This leaves open the existence of eigenvalues embedded in the essential spectrum, and particularly, the existence of an eigenvalue 0. To deal with these situations one follows [103, Theorem 2.3] and assumes in addition that

$$V : \mathbb{R}^n \to \mathbb{C}^{n\times n} \text{ is Lebesgue measurable and self-adjoint a.e. on } \mathbb{R}^n,$$

and that

$$\text{for some } R > 0,\ V \in [C^1(E_R)]^{N\times N}, \text{ where } E_R = \big\{x \in \mathbb{R}^n \mid |x| > R\big\}, \tag{1.8}$$

and

$$|x|^{1/2}V_{\ell,\ell'}(x) \underset{|x|\to\infty}{=} o(1), \quad (x \cdot \nabla V_{\ell,\ell'})(x) \underset{|x|\to\infty}{=} o(1), \tag{1.9}$$

for $1 \leq \ell, \ell' \leq N$, uniformly with respect to directions. Under all these conditions on V one then obtains

$$\sigma_p(H) \subseteq \{0\}. \tag{1.10}$$

This still leaves open the possibility of an eigenvalue 0. To exclude that as well [103, Theorem 2.1] assume in addition that

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^n} |x| \|V(x)\|_{\mathcal{B}(\mathbb{C}^N)} \leq C \quad \text{for some } C \in \big(0, (n-1)/2\big), \tag{1.11}$$

with $\|\cdot\|_{\mathcal{B}(\mathbb{C}^N)}$ denoting the operator norm of an $N \times N$ matrix in $\mathbb{C}^N$. Then finally,

$$\sigma_{\mathrm{p}}(H) = \emptyset, \tag{1.12}$$

and hence H and H_0 are unitarily equivalent under these conditions on V. The facts (1.10) and (1.12) are discussed in detail in Chapter 4.

We emphasize, however, that in the bulk of this manuscript we will not assume (1.11) as we explicitly intend to include situations with 0 an eigenvalue (and/or a threshold resonance) of H.

Chapter 5 provides a detailed study of the Green's function (matrix) of the free Dirac operator H_0, that is, the integral kernel of the resolvent $(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}$, in terms of the Hankel function of order 1 and half integer index $(n-2)/2$ and $n/2$,

$$\begin{aligned} G_0(z;x,y) &:= (H_0 - zI)^{-1}(x,y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\,[z|x-y|]^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2} H^{(1)}_{n/2}(z|x-y|)\alpha \cdot \frac{(x-y)}{|x-y|}. \end{aligned} \tag{1.13}$$

The Green's function $G_0(z;\cdot,\cdot)$ of H_0 continuously extends to $z \in \overline{\mathbb{C}_+}$, in particular, the limit $z \to 0$ exists,

$$\lim_{\substack{z \to 0, \\ z \in \overline{\mathbb{C}_+}\backslash\{0\}}} G_0(z;x,y) := G_0(0+i\,0;x,y) = i2^{-1}\pi^{-n/2}\Gamma(n/2)\,\alpha \cdot \frac{(x-y)}{|x-y|^n}, \quad x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \tag{1.14}$$

and no blow up occurs for all $n \in \mathbb{N}$, $n \geq 2$.

This chapter ends with various boundedness properties of integral operators $R_{0,\delta}$ and $R_{0,\delta}(z)$ in $[L^2(\mathbb{R}^n)]^N$, $n \geq 2$, associated with integral kernels that are bounded entrywise by

$$\big|R_{0,\delta}(\cdot,\cdot)_{j,k}\big| \leq C\langle\cdot\rangle^{-\delta}\big|G_0(0;\cdot,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta}, \quad \delta \geq 1/2,\ 1 \leq j,k \leq N, \tag{1.15}$$

and

$$\big|R_{0,\delta}(z;\cdot,\cdot)_{j,k}\big| \leq C\langle\cdot\rangle^{-\delta}\big|G_0(z;\cdot,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta}, \quad \delta \geq (n+1)/4,\ z \in \overline{\mathbb{C}_+},\ 1 \leq j,k \leq N, \tag{1.16}$$

for some $C \in (0,\infty)$. In particular, we prove that

$$R_{0,\delta} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad \delta \geq 1/2, \tag{1.17}$$

$$R_{0,\delta}(z) \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad \delta > (n+1)/4,\ z \in \overline{\mathbb{C}_+}. \tag{1.18}$$

Chapter 6 takes the boundedness property of $R_{0,\delta}$ and $R_{0,\delta}(z)$ a step further by proving trace ideal properties. In fact, employing interpolation techniques for trace ideals, we prove, among a variety of related results, that for $n \geq 2$ and $\delta > (n+1)/4$,

$$R_{0,\delta}(z) \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > n, \ z \in \overline{\mathbb{C}_+}. \tag{1.19}$$

Since we are dealing with n-dimensional Dirac operators, $n \geq 2$, the study of resolvents alone is insufficient and certain n-dependent powers of the resolvent of H_0 and H naturally enter the analysis. As a result, in Chapter 7 we prove that for all $k \in \mathbb{N}$, $k \geq n$,

$$\big[(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} - (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\big] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R},$$

as well as for all $\varepsilon > 0$,

$$V\big(H_0^2 + I_{[L^2(\mathbb{R}^n)]^N}\big)^{-(n/2)-\varepsilon} \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big),$$

now assuming additional decay of V of the type, for some $\varepsilon > 0$,

$$V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \ \big|V_{\ell,\ell'}(x)\big| \leq C\,\langle x\rangle^{-n-\varepsilon} \text{ for a.e. } x \in \mathbb{R}^n, \ 1 \leq \ell, \ell' \leq N. \tag{1.20}$$

The next two chapters, Chapters 8 and 9, are devoted to the notion of the spectral shift function for a pair of self-adjoint operators (S, S_0) in $\mathcal{H}$, particularly building on work of Yafaev [185]: We start by introducing the class of functions $\mathfrak{F}_r(\mathbb{R})$, $r \in \mathbb{N}$, by

$$\begin{aligned}\mathfrak{F}_r(\mathbb{R}) := \Big\{ f \in C^2(\mathbb{R}) \,\Big|\, & f^{(\ell)} \in L^\infty(\mathbb{R}); \text{ there exists } \varepsilon > 0 \text{ and } f_0 = f_0(f) \in \mathbb{C} \\ & \text{such that } (d^\ell/d\lambda^\ell)\big[f(\lambda) - f_0\lambda^{-r}\big] \underset{|\lambda|\to\infty}{=} O\big(|\lambda|^{-\ell-r-\varepsilon}\big), \ \ell = 0,1,2\Big\}\end{aligned} \tag{1.21}$$

(it is implied that $f_0 = f_0(f)$ is the same as $\lambda \to \pm\infty$); one observes that $C_0^\infty(\mathbb{R}) \subset \mathfrak{F}_r(\mathbb{R})$, $r \in \mathbb{N}$. Assuming that

$$\begin{aligned} & \operatorname{dom}(S) = \operatorname{dom}(S_0), \quad (S - S_0) \in \mathcal{B}(\mathcal{H}), \\ & \text{for some } 0 < \varepsilon < 1/2, \ (S - S_0)(S_0^2 + I_{\mathcal{H}})^{-(r/2)-\varepsilon} \in \mathcal{B}_1(\mathcal{H}), \end{aligned} \tag{1.22}$$

the following are then the principal results of Chapter 8: Let $r \in \mathbb{N}$, then

$$\big[f(S) - f(S_0)\big] \in \mathcal{B}_1(\mathcal{H}), \quad f \in \mathfrak{F}_r(\mathbb{R}),$$

and there exists a function

$$\xi(\,\cdot\,; S, S_0) \in L^1\big(\mathbb{R}; (1 + |\lambda|)^{-r-1}\,d\lambda\big) \tag{1.23}$$

such that the following trace formula holds,

$$\operatorname{tr}_{\mathcal{H}}\big(f(S)-f(S_0)\big)=\int_{\mathbb{R}}\xi(\lambda;S,S_0)d\lambda\, f'(\lambda), \quad f\in\mathfrak{F}_r(\mathbb{R}). \tag{1.24}$$

In particular, one has

$$\big[(S-zI_{\mathcal{H}})^{-r}-(S_0-zI_{\mathcal{H}})^{-r}\big]\in\mathcal{B}_1(\mathcal{H}), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{1.25}$$

and

$$\operatorname{tr}_{\mathcal{H}}\big((S-zI_{\mathcal{H}})^{-r}-(S_0-zI_{\mathcal{H}})^{-r}\big)=-r\int_{\mathbb{R}}\frac{\xi(\lambda;S,S_0)d\lambda}{(\lambda-z)^{r+1}}, \quad z\in\mathbb{C}\backslash\mathbb{R}. \tag{1.26}$$

The following chapter (Chapter 9) then derives an explicit representation for the spectral shift function $\xi(\,\cdot\,;S,S_0)$ in terms of normal boundary values to the real axis of regularized Fredholm determinants as follows: Slightly extending our set of hypotheses and now assuming that S_0 and S are self-adjoint operators in $\mathcal{H}$ with $(S-S_0)\in\mathcal{B}(\mathcal{H})$, we suppose in addition the following two conditions:

(i) If $r\in\mathbb{N}$ is odd, assume that

$$\big[(S-zI_{\mathcal{H}})^{-r}-(S_0-zI_{\mathcal{H}})^{-r}\big]\in\mathcal{B}_1(\mathcal{H}), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{1.27}$$

and

$$(S-S_0)(S_0-zI_{\mathcal{H}})^{-j}\in\mathcal{B}_{(r+1)/j}(\mathcal{H}), \quad j\in\mathbb{N},\ 1\le j\le r+1. \tag{1.28}$$

(ii) If $r\in\mathbb{N}$ is even, assume that for some $0<\varepsilon<1/2$,

$$(S-S_0)(S_0^2+I_{\mathcal{H}})^{-(r/2)-\varepsilon}\in\mathcal{B}_1(\mathcal{H}). \tag{1.29}$$

(In this case one can show that (1.28) holds as well).

Introducing

$$F_{S,S_0}(z):=\ln\big(\det_{\mathcal{H},r+1}\big((S-zI_{\mathcal{H}})(S_0-zI_{\mathcal{H}})^{-1}\big)\big), \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{1.30}$$

where $\det_{\mathcal{H},r+1}(\,\cdot\,)$ denotes the $(r+1)$st regularized Fredholm determinant, and introducing the analytic function $G_{S,S_0}(\,\cdot\,)$ in $\mathbb{C}\backslash\mathbb{R}$ such that

$$\begin{aligned}&\frac{d^r}{dz^r}G_{S,S_0}(z)\\ &=\operatorname{tr}_{\mathcal{H}}\Bigg(\frac{d^{r-1}}{dz^{r-1}}\sum_{j=0}^{r-1}(-1)^{r-j}(S_0-zI_{\mathcal{H}})^{-1}\big[(S-S_0)(S_0-zI_{\mathcal{H}})^{-1}\big]^{r-j}\Bigg),\\ &\qquad z\in\mathbb{C}\backslash\mathbb{R},\end{aligned} \tag{1.31}$$

the main result of Chapter 9 then reads as follows: If F_{S,S_0} and G_{S,S_0} have normal (or nontangential) boundary values on $\mathbb{R}$, then for a.e. $\lambda \in \mathbb{R}$,

$$\begin{aligned}\xi(\lambda; S, S_0) = \pi^{-1} \operatorname{Im}\big(F_{S,S_0}(\lambda + i0)\big) - \pi^{-1} \operatorname{Im}\big(G_{S,S_0}(\lambda + i0)\big) \\ + P_{r-1}(\lambda) \text{ for a.e. } \lambda \in \mathbb{R},\end{aligned} \tag{1.32}$$

where P_{r-1} is a polynomial of degree less than or equal to $r-1$.

The subsequent two chapters then analyze (1.30) and (1.31) and their normal boundary values to the real axis in the concrete case where $S = H$ and $S_0 = H_0$.

More precisely, Chapter 10 establishes continuity properties of

$$\operatorname{Im}\big(F_{H,H_0}(\lambda + i0)\big), \quad \lambda \in \mathbb{R},$$

by invoking a lengthy study of threshold spectral properties of H, following an approach by Jensen and Nenciu [99], and, especially, by Erdoğan, Goldberg, and Green [59, 60, 64, 65]. In particular, we recall an exhaustive study of eigenvalues 0 and/or resonances at 0 and finally prove that under assumptions (1.8), (1.9), and (1.20), $F_{H,H_0}(\,\cdot\,)$, has normal boundary values on $\mathbb{R}\backslash\{0\}$. In addition, the boundary values to $\mathbb{R}$ of the function $\operatorname{Im}(F_{H,H_0}(z))$, $z \in \mathbb{C}_+$, are continuous on $(-\infty, 0) \cup (0, \infty)$,

$$\operatorname{Im}\big(F_{H,H_0}(\lambda + i0)\big) \in C\big((-\infty, 0) \cup (0, \infty)\big), \tag{1.33}$$

and the left and right limits at zero,

$$\operatorname{Im}\big(F_{H,H_0}(0_\pm + i0)\big) = \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(F_{H,H_0}(\pm\varepsilon + i0)\big), \tag{1.34}$$

exist. In particular, if 0 is a regular point for H (i.e., in the absence of any zero energy eigenvalue and resonance of H), then

$$\operatorname{Im}\big(F_{H,H_0}(\lambda + i0)\big) \in C(\mathbb{R}). \tag{1.35}$$

Under the following strengthened decay assumption on V, for some $\varepsilon > 0$,

$$\begin{aligned}&V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \\ &\big|V_{\ell,\ell'}(x)\big| \leq C\langle x\rangle^{-n-1-\varepsilon} \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell, \ell' \leq N,\end{aligned} \tag{1.36}$$

an unrelenting barrage of estimates finally proves in Chapter 11 that if $n \in \mathbb{N}$ is odd, $n \geq 3$, then $\frac{d^n}{dz^n} G_{H,H_0}(\,\cdot\,)$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$. If $n \in \mathbb{N}$ is even, then $\frac{d^n}{dz^n} G_{H,H_0}(\,\cdot\,)$ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}\backslash\{0\}$. Moreover, if $n \geq 4$, then

$$\left\| \frac{d^n}{dz^n} G_{H,H_0}(\,\cdot\,) \right\|_{\mathcal{B}(\mathbb{C}^N)} \underset{\substack{z\to 0,\\ z \in \overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{-[n-(n/(n-1))]}\big), \tag{1.37}$$

and if $n = 2$, then for any $\delta \in (0, 1)$,

$$\left\| \frac{d^2}{dz^2} G_{H,H_0}(\cdot) \right\|_{\mathcal{B}(\mathbb{C}^2)} \underset{\substack{z \to 0, \\ z \in \overline{\mathbb{C}_+} \setminus \{0\}}}{=} O\big(|z|^{-(1+\delta)}\big). \tag{1.38}$$

Thus, combining (1.32)–(1.35), (1.37), (1.38) finally yields the first principal result of Chapter 12 in the following form: Under the hypotheses (1.8), (1.9), and (1.36),

$$\xi(\,\cdot\,; H, H_0) \in C\big((-\infty, 0) \cup (0, \infty)\big), \tag{1.39}$$

and the left and right limits at zero,

$$\xi(0_\pm; H, H_0) = \lim_{\varepsilon \downarrow 0} \xi(\pm\varepsilon; H, H_0), \tag{1.40}$$

exist. In particular, if 0 is a regular point for H, then

$$\xi(\,\cdot\,; H, H_0) \in C(\mathbb{R}). \tag{1.41}$$

This represents the main spectral theoretic result derived in this manuscript. The remainder of Chapter 12 then describes our application to the (resolvent regularized) Witten index of a particular class of non-Fredholm operators acting in the Hilbert space $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ in connection with multi-dimensional, massless Dirac operators.

This requires some preparations to which we turn next. We recall a bit of notation: Linear operators in the Hilbert space $L^2(\mathbb{R}; dt; \mathcal{H})$, in short, $L^2(\mathbb{R}; \mathcal{H})$, will be denoted by boldface symbols of the type $\boldsymbol{T}$, to distinguish them from operators T in $\mathcal{H}$. In particular, operators denoted by $\boldsymbol{T}$ in the Hilbert space $L^2(\mathbb{R}; \mathcal{H})$ represent operators associated with a family of operators $\{T(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$, defined by

$$\begin{aligned}
&(\boldsymbol{T} f)(t) = T(t) f(t) \text{ for a.e. } t \in \mathbb{R}, \\
&f \in \operatorname{dom}(\boldsymbol{T}) = \bigg\{ g \in L^2(\mathbb{R}; \mathcal{H}) \,\bigg|\, g(t) \in \operatorname{dom}\big(T(t)\big) \text{ for a.e. } t \in \mathbb{R}; \\
&\quad t \mapsto T(t) g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}} dt \, \big\|T(t) g(t)\big\|_{\mathcal{H}}^2 < \infty \bigg\}.
\end{aligned} \tag{1.42}$$

In the special case, where $\{T(t)\}$ is a family of bounded operators on $\mathcal{H}$ with

$$\sup_{t \in \mathbb{R}} \big\|T(t)\big\|_{\mathcal{B}(\mathcal{H})} < \infty,$$

the associated operator $\boldsymbol{T}$ is a bounded operator on $L^2(\mathbb{R}; \mathcal{H})$ with

$$\|\boldsymbol{T}\|_{\mathcal{B}(L^2(\mathbb{R};\mathcal{H}))} = \sup_{t \in \mathbb{R}} \big\|T(t)\big\|_{\mathcal{B}(\mathcal{H})}.$$

For brevity we will abbreviate $\boldsymbol{I} := I_{L^2(\mathbb{R};\mathcal{H})}$ in the following and note that in the concrete situation of n-dimensional, massless Dirac operators at hand, $\mathcal{H} = [L^2(\mathbb{R}^n)]^N$.

Denoting

$$A_- = H_0, \quad B_+ = V, \quad A_+ = A_- + B_+ = H,$$

we introduce two families of operators in $[L^2(\mathbb{R}^n)]^N$ by

$$\begin{aligned} &B(t) = b(t)B_+, \quad t \in \mathbb{R}, \\ &b^{(k)} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dt), \ k \in \mathbb{N}_0, \quad b' \in L^1(\mathbb{R}; dt), \\ &\lim_{t\to\infty} b(t) = 1, \quad \lim_{t\to-\infty} b(t) = 0, \\ &A(t) = A_- + B(t), \quad t \in \mathbb{R}. \end{aligned} \tag{1.43}$$

Next, following the general setups described in [38, 41–44, 78, 137], the operators $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}' = \boldsymbol{B}'$ are now given in terms of the families $A(t)$, $B(t)$, and $B'(t)$, $t \in \mathbb{R}$, as in (1.42). In addition, $\boldsymbol{A}_-$ in $L^2(\mathbb{R}; [L^2(\mathbb{R}^n)]^N)$ represents the self-adjoint (constant fiber) operator defined by

$$\begin{aligned} &(\boldsymbol{A}_- f)(t) = A_- f(t) \text{ for a.e. } t \in \mathbb{R}, \\ f \in \operatorname{dom}(\boldsymbol{A}_-) = \Big\{ &g \in L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \,\Big|\, g(t) \in \operatorname{dom}(A_-) \text{ for a.e. } t \in \mathbb{R}, \\ &t \mapsto A_- g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} dt \, \|A_- g(t)\|^2_{[L^2(\mathbb{R}^n)]^N} < \infty \Big\}. \end{aligned} \tag{1.44}$$

At this point one can introduce the fundamental operator $\boldsymbol{D}_{\boldsymbol{A}}$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ by

$$\boldsymbol{D}_{\boldsymbol{A}} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D}_{\boldsymbol{A}}) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}(\boldsymbol{A}_-), \tag{1.45}$$

where

$$\boldsymbol{A} = \boldsymbol{A}_- + \boldsymbol{B}, \quad \operatorname{dom}(\boldsymbol{A}) = \operatorname{dom}(\boldsymbol{A}_-), \quad \boldsymbol{B} \in \mathcal{B}\big(L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big).$$

Here the operator d/dt in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ is defined by

$$\begin{aligned} &\left(\frac{d}{dt} f\right)(t) = f'(t) \text{ for a.e. } t \in \mathbb{R}, \\ &f \in \operatorname{dom}(d/dt) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big). \end{aligned} \tag{1.46}$$

Since $\boldsymbol{D}_{\boldsymbol{A}}$ is densely defined and closed in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$, one can introduce the nonnegative, self-adjoint operators $\boldsymbol{H}_j$, $j = 1, 2$, in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ by

$$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}}, \quad \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^*.$$

Introducing the operator $\boldsymbol{H}_0$ in $L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N)$ by

$$\boldsymbol{H}_0 = -\frac{d^2}{dt^2} + \boldsymbol{A}_-^2, \quad \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}\big(\mathbb{R};[L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}\big(\boldsymbol{A}_-^2\big), \tag{1.47}$$

then $\boldsymbol{H}_0$ is self-adjoint and one obtains the following decomposition of the operators $\boldsymbol{H}_j$, $j = 1,2$,

$$\begin{aligned} \boldsymbol{H}_j &= -\frac{d^2}{dt^2} + \boldsymbol{A}^2 + (-1)^j \boldsymbol{A}' \\ &= \boldsymbol{H}_0 + \boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 + (-1)^j \boldsymbol{B}', \\ \operatorname{dom}(\boldsymbol{H}_j) &= \operatorname{dom}(\boldsymbol{H}_0), \quad j = 1,2. \end{aligned} \tag{1.48}$$

Next, we turn to a canonical approximation procedure: Consider the characteristic function for the interval $[-\ell,\ell] \subset \mathbb{R}$,

$$\chi_\ell(\nu) = \chi_{[-\ell,\ell]}(\nu), \quad \nu \in \mathbb{R}, \ \ell \in \mathbb{N}, \tag{1.49}$$

and hence $\text{s-lim}_{\ell\to\infty} \chi_\ell(A_-) = I_{[L^2(\mathbb{R}^n)]^N}$. Introducing

$$\begin{aligned} &A_\ell(t) = A_- + \chi_\ell(A_-)B(t)\chi_\ell(A_-) = A_- + B_\ell(t), \\ &\operatorname{dom}\big(A_\ell(t)\big) = \operatorname{dom}(A_-), \quad \ell \in \mathbb{N}, \ t \in \mathbb{R}, \\ &B_\ell(t) = \chi_\ell(A_-)B(t)\chi_\ell(A_-), \quad \operatorname{dom}\big(B_\ell(t)\big) = [L^2(\mathbb{R}^n)]^N, \quad \ell \in \mathbb{N}, \ t \in \mathbb{R}, \\ &A_{+,\ell} = A_- + \chi_\ell(A_-)B_+\chi_\ell(A_-), \quad \operatorname{dom}(A_{+,\ell}) = \operatorname{dom}(A_-), \quad \ell \in \mathbb{N}, \end{aligned} \tag{1.50}$$

one concludes that

$$A_{+,\ell} - A_- = \chi_\ell(A_-)B_+\chi_\ell(A_-) \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \qquad \ell \in \mathbb{N}, \tag{1.51}$$

$$A_\ell'(t) = B_\ell'(t) = \chi_\ell(A_-)B'(t)\chi_\ell(A_-) \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad \ell \in \mathbb{N}, \ t \in \mathbb{R}. \tag{1.52}$$

As a consequence of (1.51), the spectral shift functions $\xi(\,\cdot\,; A_{+,\ell}, A_-)$, $\ell \in \mathbb{N}$, exist and are uniquely determined by

$$\xi(\,\cdot\,; A_{+,\ell}, A_-) \in L^1(\mathbb{R}; d\nu), \quad \ell \in \mathbb{N}. \tag{1.53}$$

We also note the analogous decompositions,

$$\begin{aligned} &\boldsymbol{H}_{j,\ell} = -\frac{d^2}{dt^2} + \boldsymbol{A}_\ell^2 + (-1)^j \boldsymbol{A}_\ell' = \boldsymbol{H}_0 + \boldsymbol{B}_\ell\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B}_\ell + \boldsymbol{B}_\ell^2 + (-1)^j \boldsymbol{B}_\ell', \\ &\operatorname{dom}(\boldsymbol{H}_{j,\ell}) = \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}\big(\mathbb{R};[L^2(\mathbb{R}^n)]^N\big), \quad \ell \in \mathbb{N}, \ j = 1,2, \end{aligned}$$

with

$$\boldsymbol{B}_\ell = \chi_\ell(\boldsymbol{A}_-)\boldsymbol{B}\chi_\ell(\boldsymbol{A}_-), \quad \boldsymbol{B}_\ell' = \chi_\ell(\boldsymbol{A}_-)\boldsymbol{B}'\chi_\ell(\boldsymbol{A}_-), \quad \ell \in \mathbb{N}.$$

Under hypotheses (1.8), (1.9), (1.36), and

$$V_{\ell,\ell'} \in W^{4n,\infty}(\mathbb{R}^n), \quad 1 \leq \ell, \ell' \leq N, \tag{1.54}$$

it is proven in [44] that

$$\begin{aligned} &\big[(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-r} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-r}\big], \big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-r} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-r}\big] \\ &\quad \in \mathcal{B}_1\big(L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big), \quad \ell \in \mathbb{N},\ r \in \mathbb{N},\ r \geq \lceil n/2 \rceil, \end{aligned} \tag{1.55}$$

and

$$\begin{aligned} \lim_{\ell \to \infty} &\big\| \big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-r} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-r}\big] \\ &\quad - \big[(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-r} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-r}\big]\big\|_{\mathcal{B}_1(L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N))} = 0, \\ &\qquad z \in \mathbb{C}\backslash[0,\infty). \end{aligned} \tag{1.56}$$

Relations (1.55) together with the fact that $\boldsymbol{H}_j \geq 0$, $\boldsymbol{H}_{j,\ell} \geq 0$, $\ell \in \mathbb{N}$, $j = 1, 2$, imply the existence and uniqueness of spectral shift functions $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ and $\xi(\,\cdot\,; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})$ for the pair of operators $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ and $(\boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})$, $\ell \in \mathbb{N}$, respectively, employing the normalization

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = 0, \quad \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) = 0, \quad \lambda < 0,\ \ell \in \mathbb{N} \tag{1.57}$$

(cf. [184, Section 8.9]). Moreover,

$$\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1) \in L^1\big(\mathbb{R}; (1 + |\lambda|)^{-r-1} d\lambda\big). \tag{1.58}$$

Since

$$\int_{\mathbb{R}} dt \,\big\| A_\ell'(t) \big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} < \infty, \quad \ell \in \mathbb{N}, \tag{1.59}$$

employing $b'(\,\cdot\,) \in L^1(\mathbb{R}; dt)$, one obtains (cf. [78, 137])

$$\big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-1} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-1}\big] \in \mathcal{B}_1\big(L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big), \quad \ell \in \mathbb{N},$$

and hence

$$\xi(\,\cdot\,; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) \in L^1(\mathbb{R}; d\lambda), \quad \ell \in \mathbb{N}.$$

In addition, (12.20), (1.57), and (1.59) imply the approximate trace formula,

$$\int_{[0,\infty)} \frac{\xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})\, d\lambda}{(\lambda - z)^2} = \frac{1}{2} \int_{\mathbb{R}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\nu^2 - z)^{3/2}}, \quad \ell \in \mathbb{N},\ z \in \mathbb{C}\backslash[0,\infty), \tag{1.60}$$

which in turn implies

$$\xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) = \begin{cases} \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}}, & \text{for a.e. } \lambda > 0, \\ 0, & \lambda < 0, \end{cases} \quad \ell \in \mathbb{N}, \tag{1.61}$$

via a Stieltjes inversion argument.

Given hypothesis (1.20), we will prove in Theorem 7.4 that

$$\big[(A_+ - zI_{[L^2(\mathbb{R}^n)]^N})^{-r_0} - (A_- - zI_{[L^2(\mathbb{R}^n)]^N})^{-r_0}\big] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big),$$
$$r_0 \in \mathbb{N},\ r_0 \geq 2\lfloor n/2 \rfloor + 1,\ z \in \mathbb{C}\backslash\mathbb{R}. \quad (1.62)$$

Since $2\lfloor n/2 \rfloor + 1$ is always odd, [185, Theorem 2.2] yields the existence of a spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ for the pair (A_+, A_-) satisfying

$$\xi(\,\cdot\,; A_+, A_-) \in L^1\big(\mathbb{R}; (1 + |\nu|)^{-r_0-1} d\nu\big). \quad (1.63)$$

While (1.63) does not determine $\xi(\,\cdot\,; A_+, A_-)$ uniquely, one can show (following [40, Theorem 4.7]) that there exists a unique spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ given by the limiting relation

$$\lim_{\ell\to\infty} \xi(\,\cdot\,; A_{+,\ell}, A_-) = \xi(\,\cdot\,; A_+, A_-) \quad \text{in } L^1\big(\mathbb{R}; (1 + |\nu|)^{-r_0-1} d\nu\big), \quad (1.64)$$

and hence we will always choose this particular spectral shift function in (1.64) for the pair (A_+, A_-) in the following.

At this point one can entertain the limit $\ell \to \infty$ in (1.61): Indeed, (1.61) yields

$$\begin{aligned}
&\int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda) \\
&\quad = \frac{1}{\pi} \int_{[0,\infty)} d\lambda\, f'(\lambda) \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \\
&\quad = \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-) d\nu\, F'(\nu), \quad \ell \in \mathbb{N},
\end{aligned} \quad (1.65)$$

where F' is defined by

$$F'(\nu) = \int_{\nu^2}^{\infty} d\lambda\, f'(\lambda)(\lambda - \nu^2)^{-1/2}, \quad \nu \in \mathbb{R}. \quad (1.66)$$

The limit $\ell \to \infty$ on left-hand side of (1.65) is controlled via (1.56), and, since $F' \in C_0^\infty(\mathbb{R})$, the right-hand side of (1.65) is controlled via (1.64), implying

$$\begin{aligned}
&\int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) d\lambda\, f'(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_+, A_-) d\nu\, F'(\nu) \\
&= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda\, f'(\lambda) \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \chi_{[0,\infty)}(\lambda), \quad f \in C_0^\infty(\mathbb{R}),
\end{aligned}$$

and hence

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \quad \text{for a.e. } \lambda > 0, \quad (1.67)$$

due to our normalization in (1.57). This establishes the limiting relation $\ell \to \infty$ of (1.61).

Having established (1.67), we turn to the resolvent regularized Witten index of the operator $\boldsymbol{D}_{\boldsymbol{A}}$. Since $\sigma(A_\pm) = \mathbb{R}$, in particular, $0 \notin \rho(A_+) \cap \rho(A_-)$, $\boldsymbol{D}_{\boldsymbol{A}}$ is a non-Fredholm operator. Even though $\boldsymbol{D}_{\boldsymbol{A}}$ is a non-Fredholm operator, its Witten index is well defined and expressible in terms of the spectral shift functions for the pair of operators $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ and (A_+, A_-) as will be shown below.

To introduce an appropriately (resolvent regularized) Witten index of $\boldsymbol{D}_{\boldsymbol{A}}$, we consider a densely defined, closed operator T in the complex, separable Hilbert space $\mathcal{K}$ and assume that for some $k \in \mathbb{N}$, and all $\lambda < 0$

$$\big[(T^*T - \lambda I_{\mathcal{K}})^{-k} - (TT^* - \lambda I_{\mathcal{K}})^{-k}\big] \in \mathcal{B}_1(\mathcal{K}).$$

Then the kth resolvent regularized Witten index of T is defined by

$$W_{k,r}(T) = \lim_{\lambda \uparrow 0} (-\lambda)^k \operatorname{tr}_{\mathcal{K}} \big((T^*T - \lambda I_{\mathcal{K}})^{-k} - (TT^* - \lambda I_{\mathcal{K}})^{-k}\big), \qquad (1.68)$$

whenever the limit exists. The analogous semigroup regularized definition reads,

$$W_s(T) = \lim_{t \uparrow \infty} \operatorname{tr}_{\mathcal{K}} (e^{-tT^*T} - e^{-tTT^*}),$$

but in this manuscript it suffices to employ (1.68).

The second main result of Chapter 12, and at the same time the main result of this manuscript, the characterization of the Witten index of $\boldsymbol{D}_{\boldsymbol{A}}$ in terms of spectral shift functions, can thus be summarized as follows:

Theorem 1.1. *Assume hypotheses* (1.8), (1.9), (1.36), *and* (1.54). *Then* 0 *is a right Lebesgue point of* $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$, *denoted by* $\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1)$, *and*

$$\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \big[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)\big]/2.$$

In addition, the resolvent regularized Witten index $W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}})$ *of* $\boldsymbol{D}_{\boldsymbol{A}}$ *exists for all* $k \in \mathbb{N}$, $k \geq \lceil n/2 \rceil$, *and equals*

$$\begin{aligned} W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}}) = \xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) &= \big[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)\big]/2 \\ &= \big[\xi(0_+; H, H_0) + \xi(0_-; H, H_0)\big]/2. \qquad (1.69) \end{aligned}$$

This is the first result of this kind applicable to non-Fredholm operators in a partial differential operator setting involving multi-dimensional massless Dirac operators. In a sense, a project that started with Pushnitski in 2008, was considerably extended in scope in [78], and further developed with the help of [38,41–44,82,83], finally comes full circle.

Appendix A collects some useful results on block matrix operators, Appendix B is devoted to asymptotic results for Hankel functions, Appendix C presents low-energy expansions and estimates for the free Dirac Green's function in the massless case, Appendix D recalls a product formula for modified (regularized) Fredholm determinants, and finally, Appendix E collects some of the notational conventions used throughout this manuscript.