Chapter 2

Some background on (locally) smooth operators

In this chapter, we first recall a few basic facts on the notion of (local) Kato-smoothness (see, e.g., [140, Section XIII.7], [184, Chapter 4], and [186, Chapters 0–2]) and then recall a variant, strong (local) Kato-smoothness (see, e.g., [184, Chapter 4], [186, Chapters 0–2]), as these concepts will be useful in subsequent chapters.

Definition 2.1. Let S be self-adjoint in $\mathcal{H}$ and $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ with $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$ and fix $\varepsilon_0 > 0$. Then T is called S-Kato-smooth (in short, S-smooth in the following) if for each $f \in \mathcal{H}$,

$$\begin{aligned} \|T\|_S^2 = \sup_{\varepsilon \in (0,\varepsilon_0),\, \|f\|_{\mathcal{H}}=1} \frac{1}{4\pi^2} \int_{\mathbb{R}} d\lambda \big[\big\| T\big(S - (\lambda + i\varepsilon) I_{\mathcal{H}}\big)^{-1} f \big\|_{\mathcal{H}}^2 \\ + \big\| T\big(S - (\lambda - i\varepsilon) I_{\mathcal{H}}\big)^{-1} f \big\|_{\mathcal{H}}^2 \big] < \infty. \end{aligned} \tag{2.1}$$

It suffices to require (2.1) for a dense set of $f \in \mathcal{H}$ as T is closed. If T is S-smooth, then T is infinitesimally bounded with respect to S.

In terms of unitary groups, T is S-smooth if and only if for all $f \in \mathcal{H}$, $e^{-itS} f \in \operatorname{dom}(T)$ for a.e. $t \in \mathbb{R}$ and

$$\frac{1}{2\pi} \int_{\mathbb{R}} dt \, \big\| T e^{-itS} f \big\|_{\mathcal{H}}^2 \leq C_0 \|f\|_{\mathcal{H}}^2$$

for some constant $C_0 \in (0, \infty)$ (C_0 can be chosen to be $\|T\|_S^2$, but not smaller).

An immediate consequence regarding the absence of singular spectrum derives from the fact that if T is S-smooth then

$$\overline{\operatorname{ran}(T^*)} \subseteq \mathcal{H}_{\mathrm{ac}}(S).$$

In particular,

$$\text{if, in addition, } \ker(T) = \{0\}, \quad \text{then } \mathcal{H}_{\mathrm{ac}}(S) = \mathcal{H},$$

and hence the spectrum of S is purely absolutely continuous,

$$\sigma(S) = \sigma_{\mathrm{ac}}(S), \quad \sigma_{\mathrm{p}}(S) = \sigma_{\mathrm{sc}}(S) = \emptyset.$$

Here $\mathcal{H}_{\mathrm{ac}}(S)$ denotes the absolutely continuous subspace associated with S.

Moreover, as long as $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$ (with $\mathcal{L}$ another complex, separable Hilbert space), BT is S-smooth whenever T is S-smooth.

Finally, if $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and for all $z \in \mathbb{C}\backslash\mathbb{R}$, $T(S - zI_{\mathcal{H}})^{-1}T^*$ has a bounded closure in $\mathcal{H}$ satisfying for some fixed $\varepsilon_0 > 0$,

$$C_1 = \sup_{\lambda\in\mathbb{R},\,\varepsilon\in(0,\varepsilon_0)} \big\|\overline{T(S-(\lambda+i\varepsilon)I_{\mathcal{H}})^{-1}T^*}\big\|_{\mathcal{B}(\mathcal{H})} < \infty, \tag{2.2}$$

then T is S-smooth with $\|T\|_S \le C_1/\pi$.

While Definition 2.1 describes a global condition, a local version can be introduced as follows:

Definition 2.2. Let S be self-adjoint in $\mathcal{H}$ and $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ with $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$. T is called S-Kato-smooth on a Borel set $\Lambda \subseteq \mathbb{R}$ (in short, S-smooth on Λ in the following) if $TE_S(\Lambda)$ is S-smooth.

Again, if $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, then BT is S-smooth on Λ_0 whenever T is.

For $TE_S(\Lambda)$ to be well defined it suffices that $E_S(\Lambda)\mathcal{H} \cap \operatorname{dom}(S) \subseteq \operatorname{dom}(T)$.

If T is S-smooth on Λ then

$$\overline{\operatorname{ran}\big((TE_S(\Lambda))^*\big)} \subseteq \mathcal{H}_{\mathrm{ac}}(S),$$

in particular,

$$\text{if, in addition, } \ker(T) = \{0\}, \quad \text{then}$$
$$\sigma(S) \cap \Lambda = \sigma_{\mathrm{ac}}(S) \cap \Lambda, \quad \sigma_{\mathrm{p}}(S) \cap \Lambda = \sigma_{\mathrm{sc}}(S) \cap \Lambda = \emptyset.$$

If $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and for all $z \in \mathbb{C}\backslash\mathbb{R}$, $T(S - zI_{\mathcal{H}})^{-1}T^*$ has a bounded closure in $\mathcal{H}$ satisfying for some fixed $\varepsilon_0 > 0$,

$$\sup_{\lambda\in\Lambda,\,\varepsilon\in(0,\varepsilon_0)} \big\|\overline{T(S-(\lambda+i\varepsilon)I_{\mathcal{H}})^{-1}T^*}\big\|_{\mathcal{B}(\mathcal{H})} < \infty, \tag{2.3}$$

or

$$\sup_{\lambda\in\Lambda,\,\varepsilon\in(0,\varepsilon_0)} \varepsilon\big\|\overline{T(S-(\lambda\pm i\varepsilon)I_{\mathcal{H}})^{-1}}\big\|_{\mathcal{B}(\mathcal{H})} < \infty, \tag{2.4}$$

then T is S-smooth on $\overline{\Lambda}$.

Next, following [186, Section 4.4], we turn to the concept of strongly smooth operators on a compact interval $\Lambda_0 = [\lambda_1, \lambda_2]$, $\lambda_j \in \mathbb{R}$, $j = 1, 2$, $\lambda_1 < \lambda_2$ (tailored toward certain applications to differential operators). This requires some preparations: Given a separable complex Hilbert space $\mathcal{H}_0$, one considers the (nonseparable) Banach space of $\mathcal{H}_0$-valued Hölder continuous functions of order $\tau \in (0, 1]$, denoted by $C^\tau(\Lambda_0; \mathcal{H}_0)$, with norm

$$\|f\|_{C^\tau(\Lambda_0;\mathcal{H}_0)} = \sup_{\lambda,\lambda'\in\Lambda_0} \bigg(\|f(\lambda)\|_{\mathcal{H}_0} + \frac{\|f(\lambda)-f(\lambda')\|_{\mathcal{H}_0}}{|\lambda-\lambda'|^\tau}\bigg), \quad f \in C^\tau(\Lambda_0; \mathcal{H}_0).$$

Suppose the self-adjoint operator S in $\mathcal{H}$ has purely absolutely continuous spectrum on Λ_0, that is,

$$\sigma(S) \cap \Lambda_0 = \sigma_{\rm ac}(S) \cap \Lambda_0, \quad \sigma_{\rm p}(S) \cap \Lambda_0 = \sigma_{\rm sc}(S) \cap \Lambda_0 = \emptyset,$$

of constant multiplicity $m_0 \in \mathbb{N} \cup \{\infty\}$ on Λ_0, with $\dim(\mathcal{H}_0) = m_0$. In addition, let

$$\mathcal{F}_0 : \begin{cases} E_S(\Lambda_0)\mathcal{H} \to L^2(\Lambda_0; d\lambda; \mathcal{H}_0), \\ f \mapsto \mathcal{F}_0 f := \tilde{f}, \end{cases} \quad \text{be unitary,}$$

and "diagonalizing" S, that is, turning $SE_S(\Lambda_0)$ into a multiplication operator. More precisely, $\mathcal{F}_0$ generates a spectral representation of S via,

$$(\mathcal{F}_0 E_S(\Omega) f)(\lambda) = \chi_{\Omega \cap \Lambda_0}(\lambda) \tilde{f}(\lambda), \quad f \in E_S(\Lambda_0)\mathcal{H}.$$

With these preparations in place, we are now in position to define the notion of strongly smooth operators (cf. [184, Section 4.4], where a more general concept is introduced):

Definition 2.3. Let S be self-adjoint in $\mathcal{H}$ with purely absolutely continuous spectrum of constant (possibly, infinite) multiplicity on Λ_0 and suppose that $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ with $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$. Then T is called strongly S-Kato-smooth on Λ_0 (in short, strongly S-smooth on Λ_0 in the following), with exponent $\tau \in (0,1]$, if $\mathcal{F}_0(TE_S(\Lambda))^* : \mathcal{K} \to C^\tau(\Lambda_0; \mathcal{H}_0)$ is continuous, that is, for $f = (TE_S(\Lambda_0))^*\xi$, $\xi \in \mathcal{K}$,

$$\begin{aligned} &\|\tilde{f}(\lambda)\|_{\mathcal{H}_0} = \big\|\big(\mathcal{F}_0\big(TE_S(\Lambda_0)\big)^*\xi\big)(\lambda)\big\|_{\mathcal{H}_0} \le C\|\xi\|_{\mathcal{K}}, \\ &\|\tilde{f}(\lambda) - \tilde{f}(\lambda')\|_{\mathcal{H}_0} \le C|\lambda - \lambda'|^\tau \|\xi\|_{\mathcal{K}}, \end{aligned}$$

with $C \in (0,\infty)$ independent of $\lambda, \lambda' \in \Lambda_0$ and $\xi \in \mathcal{K}$.

Not surprisingly, the terminology chosen is consistent with the fact that

if T is strongly S-smooth on Λ_0, then it is S-smooth on Λ_0.

Moreover, as long as $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$ (with $\mathcal{L}$ another complex, separable Hilbert space) and T is strongly S-smooth with exponent $\tau \in (0,1]$ on Λ_0, then BT is strongly S-smooth on Λ_0 with the same exponent $\tau \in (0,1]$.

Next, we recall a perturbation approach in which S corresponds to the "sum" of an unperturbed self-adjoint operator S_0 in $\mathcal{H}$ and a perturbation V in $\mathcal{H}$ that can be factorized into a product $V_1^* V_2$ as follows: Suppose $V_j \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, $j = 1,2$, with

$$V_j\big(|S_0| + I_{\mathcal{H}}\big)^{-1/2} \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad j = 1,2, \tag{2.5}$$

and the symmetry condition,

$$(V_1 f, V_2 g)_{\mathcal{K}} = (V_2 f, V_1 g)_{\mathcal{K}}, \quad f, g \in \operatorname{dom}\big(|S_0|^{1/2}\big).$$

In addition, suppose that for some (and hence for all) $z \in \rho(S_0)$, $V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*$ has a bounded extension in $\mathcal{K}$, which is then given by its closure

$$\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*} = V_2(S_0 - zI_{\mathcal{H}})^{-1/2}\big[V_1(S_0 - \overline{z}I_{\mathcal{H}})^{-1/2}\big]^*. \tag{2.6}$$

Here the operator $\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}$ represents an abstract Birman–Schwinger-type operator.

Finally, we assume that

$$\big[I_{\mathcal{K}} + \overline{V_2(S_0 - z_0 I_{\mathcal{H}})^{-1}V_1^*}\big]^{-1} \in \mathcal{B}(\mathcal{K}) \quad \text{for some } z_0 \in \rho(S_0).$$

Then the equation

$$\begin{aligned} R(z) &= (S_0 - zI_{\mathcal{H}})^{-1} \\ &\quad - \big[V_1(S_0 - \overline{z}I_{\mathcal{H}})^{-1}\big]^*\big[I_{\mathcal{H}} + \overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big]^{-1}V_2(S_0 - zI_{\mathcal{H}})^{-1}, \\ &\qquad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{2.7}$$

defines the resolvent of a self-adjoint operator S in $\mathcal{H}$, that is,

$$R(z) = (S - zI_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.8}$$

with $S \supseteq S_0 + V_1^*V_2$ (the latter defined on $\operatorname{dom}(S_0) \cap \operatorname{dom}(V_1^*V_2)$, which may consist of $\{0\}$ only); for details we refer to [107] (see also [79], [184, Section 1.9]).

One also has

$$\begin{aligned} (S - zI_{\mathcal{H}})^{-1} - (S_0 - zI_{\mathcal{H}})^{-1} &= -\big[V_1(S - \overline{z}I_{\mathcal{H}})^{-1}\big]^*V_2(S_0 - zI_{\mathcal{H}})^{-1} \\ &= -\big[V_1(S_0 - \overline{z}I_{\mathcal{H}})^{-1}\big]^*V_2(S - zI_{\mathcal{H}})^{-1}, \\ &\qquad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} (S_0 - zI_{\mathcal{H}})^{-1} &= (S - zI_{\mathcal{H}})^{-1} \\ &\quad - \big[V_1(S - \overline{z}I_{\mathcal{H}})^{-1}\big]^*\big[I_{\mathcal{H}} - \overline{V_2(S - zI_{\mathcal{H}})^{-1}V_1^*}\big]^{-1}V_2(S - zI_{\mathcal{H}})^{-1}, \\ &\qquad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

as well as,

$$\begin{aligned} \overline{V_1(S - zI_{\mathcal{H}})^{-1}V_1^*} &= \overline{V_1(S_0 - zI_{\mathcal{H}})^{-1}V_1^*} \\ &\quad - \big[\overline{V_1(S - zI_{\mathcal{H}})^{-1}V_1^*}\big]\big[\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big], \quad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

implying

$$\begin{aligned} &\overline{V_1(S - zI_{\mathcal{H}})^{-1}V_1^*} \\ &\quad = \overline{V_1(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big[I_{\mathcal{K}} + \overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big]^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{2.9}$$

Similarly,

$$\overline{V_2(S - zI_{\mathcal{H}})^{-1}V_1^*} = I_{\mathcal{H}} - \big[I_{\mathcal{K}} + \overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big]^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \quad (2.10)$$

The remaining results in this chapter are all taken from [186, Sections 4.4–4.7].

Theorem 2.4. *Assuming the hypotheses on S_0, V_j, $j = 1, 2$, employed in* (2.5)–(2.9), *suppose the following additional conditions* (i)–(iii) *hold:*

(i) *V_2 is S_0-smooth on $\Lambda_0 = [\lambda_1, \lambda_2]$, $\lambda_j \in \mathbb{R}$, $j = 1, 2$, $\lambda_1 < \lambda_2$.*

(ii) *The analytic, operator-valued functions*

$$\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}, \quad \overline{V_1(S_0 - zI_{\mathcal{H}})^{-1}V_1^*} \quad \text{on } \operatorname{Re}(z) \in (\lambda_1, \lambda_2),\ \operatorname{Im}(z) \neq 0,$$

are continuous in $\mathcal{B}(\mathcal{K})$-norm up to and including the two rims of the "cut" along (λ_1, λ_2).

(iii) *For some $k \in \mathbb{N}$, $\big[\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}\big]^k \in \mathcal{B}_\infty(\mathcal{K})$, $\operatorname{Im}(z) \neq 0$.*

Define

$$\begin{aligned}&\mathcal{N}_\pm = \big\{\lambda \in \Lambda_0 \mid \text{there exists } 0 \neq f \in \mathcal{K} \text{ s.t. } -f = \overline{V_2(S_0 - (\lambda \pm i0)I_{\mathcal{H}})^{-1}V_1^*}\, f\big\},\\ &\mathcal{N} = \mathcal{N}_- \cup \mathcal{N}_+. \end{aligned} \quad (2.11)$$

Then $\mathcal{N}_\pm$, $\mathcal{N}$ are closed and of Lebesgue measure zero. Moreover, the analytic, operator-valued function $\overline{V_1(S - zI_{\mathcal{H}})^{-1}V_1^*}$ on $\operatorname{Re}(z) \in (\lambda_1, \lambda_2)$, $\operatorname{Im}(z) \neq 0$, is continuous in $\mathcal{B}(\mathcal{K})$-norm up to and including the two rims of the "cut" along $(\lambda_1, \lambda_2)\backslash\mathcal{N}$. If, in addition, $\ker(V_1) = \{0\}$, then S has purely absolutely continuous spectrum on $\Lambda_0\backslash\mathcal{N}$, that is,

$$\sigma(S) \cap (\Lambda_0\backslash\mathcal{N}) = \sigma_{\rm ac}(S) \cap (\Lambda_0\backslash\mathcal{N}), \quad \sigma_{\rm p}(S) \cap (\Lambda_0\backslash\mathcal{N}) = \sigma_{\rm sc}(S) \cap (\Lambda_0\backslash\mathcal{N}) = \emptyset.$$

As detailed in [184, Remark 4.6.3], condition (iii) in Theorem 2.4 can be replaced by the following one:

(iii′) Suppose there exist $z_\pm \in \rho(S_0)$, $\pm\operatorname{Im}(z_\pm) > 0$, such that

$$\big[I_{\mathcal{K}} + \overline{V_2(S_0 - z_\pm I_{\mathcal{H}})^{-1}V_1^*}\big]^{-1} \in \mathcal{B}(\mathcal{K}),$$

and

$$\begin{aligned}&\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*} - \overline{V_2(S_0 - z'I_{\mathcal{H}})^{-1}V_1^*}\\ &\quad = (z - z')\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}(S_0 - z'I_{\mathcal{H}})^{-1}V_1^*} \in \mathcal{B}_\infty(\mathcal{K}), \quad z, z' \in \rho(S_0).\end{aligned}$$

Next we strengthen the hypotheses in Theorem 2.4 by invoking the notion of strong S_0-smoothness:

Theorem 2.5. *Assuming the hypotheses on S_0, V_j, $j = 1, 2$, employed in* (2.5)–(2.9), *suppose in addition the following conditions* (i)–(iii) *hold:*

(i) S_0 *has purely absolutely continuous spectrum of constant multiplicity* $m_0 \in \mathbb{N} \cup \{\infty\}$ *on* $\Lambda_0 = [\lambda_1, \lambda_2]$, $\lambda_j \in \mathbb{R}$, $j = 1, 2$, $\lambda_1 < \lambda_2$.

(ii) V_j *are strongly* S_0*-smooth on any compact subinterval of* Λ_0 *with exponents* $\tau_j > 0$, $j = 1, 2$.

(iii) *For some* $k \in \mathbb{N}$, $[\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}]^k \in \mathcal{B}_\infty(\mathcal{K})$, $\operatorname{Im}(z) \neq 0$.
Then the analytic, operator-valued functions

$$\overline{V_1(S_0 - zI_{\mathcal{H}})^{-1}V_1^*}, \quad \overline{V_2(S - zI_{\mathcal{H}})^{-1}V_1^*}, \quad (\textit{resp.}, \ \overline{V_1(S - zI_{\mathcal{H}})^{-1}V_1^*})$$

on $\operatorname{Re}(z) \in (\lambda_1, \lambda_2)$, $\operatorname{Im}(z) \neq 0$, *are Hölder continuous in* $\mathcal{B}(\mathcal{K})$*-norm with exponent* $\min\{\tau_1, \tau_2\}$ *up to and including the two rims of the "cut" along* (λ_1, λ_2) *(resp.,* $(\lambda_1, \lambda_2)\backslash\mathcal{N}$*).*

Moreover, the local wave operators

$$W_\pm(S, S_0; \Lambda_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itS} e^{-itS_0} P_{S_0,\mathrm{ac}}(\Lambda_0),$$

with $P_{S_0,\mathrm{ac}}(\Lambda_0) = E_{S_0}(\Lambda_0)E_{S_0,\mathrm{ac}}$, *and* $E_{S_0,\mathrm{ac}}$ *the projection onto the absolutely continuous subspace of* S_0, *exist and are complete, that is,*

$$\ker\big(W_\pm(S, S_0; \Lambda_0)\big) = \mathcal{H} \ominus E_{S_0,\mathrm{ac}}(\Lambda_0)\mathcal{H}, \quad \operatorname{ran}\big(W_\pm(S, S_0; \Lambda_0)\big) = P_{S,\mathrm{ac}}(\Lambda_0)\mathcal{H},$$

with $P_{S,\mathrm{ac}}(\Lambda_0) = E_S(\Lambda_0)E_{S,\mathrm{ac}}$, *and* $E_{S,\mathrm{ac}}$ *the projection onto the absolutely continuous subspace of* S.

For the remainder of this theorem suppose in addition that $\tau_1 > 1/2$. *Then*

$$\mathcal{N}_\pm = \mathcal{N} = \sigma_{\mathrm{p}}(S) \cap \Lambda_0$$

and the (geometric) multiplicities of the eigenvalue $\lambda_0 \in \Lambda_0$ *of* S *and the eigenvalue* -1 *of* $\overline{V_2(S_0 - (\lambda_0 \pm i0)I_{\mathcal{H}})^{-1}V_1^*}$ *coincide. If in addition,* $\ker(V_1) = \{0\}$, *then* S *has no singularly continuous spectrum on* Λ_0, *that is,*

$$\sigma(S) \cap \Lambda_0 = \sigma_{\mathrm{ac}}(S) \cap \Lambda_0, \quad \sigma_{\mathrm{sc}}(S) \cap \Lambda_0 = \emptyset,$$

and the singular spectrum of S *on the interior,* (λ_1, λ_2), *of* Λ_0 *consists only of eigenvalues of finite multiplicity with no accumulation point in* (λ_1, λ_2), *in particular,*

$$\sigma_{\mathrm{s}}(S) \cap (\lambda_1, \lambda_2) = \sigma_{\mathrm{p}}(S) \cap (\lambda_1, \lambda_2).$$

Again, condition (iii) in Theorem 2.5 can be replaced by condition (iii′) above.

To make the transition from local to global wave operators we also recall the following result.

Theorem 2.6. *Assuming the hypotheses on* S_0, V_j, $j = 1, 2$, *employed in* (2.5)–(2.9), *suppose in addition the following conditions* (i)–(iii) *hold:*

(i) S_0 *has purely absolutely continuous spectrum of constant multiplicity* $m_{0,\ell} \in \mathbb{N} \cup \{\infty\}$ *on a system of intervals* $\Lambda_{0,\ell} = [\lambda_{1,\ell}, \lambda_{2,\ell}]$, $\lambda_{j,\ell} \in \mathbb{R}$, $j = 1, 2$, $\lambda_{1,\ell} < \lambda_{2,\ell}$, $\ell \in \mathcal{I}$, $\mathcal{I} \subseteq \mathbb{N}$ *an appropriate index set, such that*

$$\sigma(S_0) \backslash \bigcup_{\ell \in \mathcal{I}} \Lambda_{0,\ell} \text{ has Lebesgue measure zero.}$$

(ii) V_j *are strongly* S_0*-smooth on any compact subinterval of* $\Lambda_{0,\ell}$ *with exponents* $\tau_{j,\ell} > 0$, $j = 1, 2$, $\ell \in \mathcal{I}$.

(iii) *For some* $k \in \mathbb{N}$, $[\overline{V_2(S_0 - zI_{\mathcal{H}})^{-1} V_1^*}]^k \in \mathcal{B}_\infty(\mathcal{K})$, $\operatorname{Im}(z) \neq 0$.

Then the (global) wave operators

$$W_\pm(S, S_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itS} e^{-itS_0},$$

exist and are complete, that is,

$$\ker\big(W_\pm(S, S_0)\big) = \{0\}, \quad \operatorname{ran}\big(W_\pm(S, S_0)\big) = E_{S,\mathrm{ac}} \mathcal{H},$$

with $E_{S,\mathrm{ac}}$ *the projection onto the absolutely continuous subspace of* S.

For additional references in the context of smooth operator theory, limiting absorption principles, and completeness of wave operators, see, for instance, [6, 11, 34], [19, Chapter 17], [21, 75, 87, 107, 112, 118], [140, Section XIII.7], [142], [184, Chapter 4], [186, Chapters 0–2].