Chapter 3

A limiting absorption principle for interacting, massless Dirac operators

In this chapter, following [186, Sections 1.11, 2.1, and 2.2], we apply the abstract framework of strongly smooth operators of the preceding chapter to the concrete case of massless Dirac operators with electromagnetic potentials.

To rigorously define the free massless n-dimensional Dirac operators to be studied in the sequel, we now introduce the following set of basic hypotheses assumed for the remainder of this manuscript (these hypotheses will have to be strengthened later on).

Hypothesis 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(i) *Set $N = 2^{\lfloor (n+1)/2 \rfloor}$ and let α_j, $1 \leq j \leq n$, $\alpha_{n+1} := \beta$, denote $n+1$ anti-commuting Hermitian $N \times N$ matrices with squares equal to I_N, that is,*

$$\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} I_N, \quad 1 \leq j,k \leq n+1. \tag{3.1}$$

(ii) *Introduce in $[L^2(\mathbb{R}^n)]^N$ the free massless Dirac operator*

$$H_0 = \alpha \cdot (-i\nabla) = \sum_{j=1}^{n} \alpha_j (-i\partial_j), \quad \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N, \tag{3.2}$$

where $\partial_j = \partial/\partial x_j$, $1 \leq j \leq n$.

(iii) *Next, consider the self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1 \leq \ell,\ell' \leq N}$ satisfying for some fixed $\rho \in (1,\infty)$, $C \in (0,\infty)$,*

$$\begin{aligned} &V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \\ &|V_{\ell,\ell'}(x)| \leq C \langle x \rangle^{-\rho} \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N. \end{aligned} \tag{3.3}$$

Under these assumptions on V, the massless Dirac operator H in $[L^2(\mathbb{R}^n)]^N$ is defined via

$$H = H_0 + V, \quad \operatorname{dom}(H) = \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N. \tag{3.4}$$

Then H_0 and H are self-adjoint in $[L^2(\mathbb{R}^n)]^N$, with essential spectrum covering the entire real line,

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = \mathbb{R},$$

a consequence of relative compactness of V with respect to H_0. In addition,

$$\sigma_{\text{ac}}(H_0) = \mathbb{R}, \quad \sigma_{\text{p}}(H_0) = \sigma_{\text{sc}}(H_0) = \emptyset.$$

On occasion (cf. Chapter 7) we will drop the self-adjointness hypothesis on the $N \times N$ matrix V and still define a closed operator H in $[L^2(\mathbb{R}^n)]^N$ as in (3.4).

For completeness we also recall that the massive free Dirac operator in $[L^2(\mathbb{R}^n)]^N$ associated with the mass parameter $m > 0$ then would be of the form

$$H_0(m) = H_0 + m\beta, \quad \operatorname{dom}\big(H_0(m)\big) = [W^{1,2}(\mathbb{R}^n)]^N, \ m > 0, \ \beta = \alpha_{n+1},$$

but we will primarily study the massless case $m = 0$ in this manuscript.

In the special one-dimensional case $n = 1$, one can choose for α_1 either a real constant or one of the three Pauli matrices. Similarly, in the massive case, β would typically be a second Pauli matrix (different from α_1). For simplicity we confine ourselves to $n \in \mathbb{N}$, $n \geq 2$, in the following.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. In addition, for any $n \in \mathbb{N}$, we also introduce the scale of weighted L^2-spaces,

$$L^2_s(\mathbb{R}^n) = \big\{f \in \mathcal{S}'(\mathbb{R}^n) \,|\, \|f\|_{L^2_s(\mathbb{R}^n)} := \big\|\langle x\rangle^s f\big\|_{L^2(\mathbb{R}^n)} < \infty\big\}, \quad s \in \mathbb{R}.$$

Defining Q_j as the operator of multiplication by x_j, $1 \leq j \leq n$, in $L^2(\mathbb{R}^n)$, and introducing $Q = (Q_1, \dots, Q_n)$, one notes that

$$\operatorname{dom}\big(\langle Q\rangle^s\big) = L^2_s(\mathbb{R}^n), \quad s \in \mathbb{R}.$$

Employing the relations (3.1), one observes that

$$H_0(m)^2 = I_N[-\Delta + m^2 I_{L^2(\mathbb{R}^n)}], \ \operatorname{dom}\big(H_0(m)^2\big) = [W^{2,2}(\mathbb{R}^n)]^N, \ m \geq 0. \quad (3.5)$$

Remark 3.2. Since we permit a (sufficiently decaying) matrix-valued potential V in H, this includes, in particular, the case of electromagnetic interactions introduced via minimal coupling, that is, V describes also special cases of the form,

$$\begin{aligned} &H(q, A) := \alpha \cdot (-i\nabla - A) + qI_N = H_0 + [qI_N - \alpha \cdot A], \\ &\operatorname{dom}\big(H(q, A)\big) = [W^{1,2}(\mathbb{R}^n)]^N, \end{aligned}$$

where (q, A) represent the electromagnetic potentials on $\mathbb{R}^n$, with $q : \mathbb{R}^n \to \mathbb{R}$, $q \in L^\infty(\mathbb{R}^n)$, $A = (A_1, \dots, A_n)$, $A_j : \mathbb{R}^n \to \mathbb{R}$, $A_j \in L^\infty(\mathbb{R}^n)$, $1 \leq j \leq n$, and for some fixed $\rho > 1$, $C \in (0, \infty)$,

$$\big|q(x)\big| + \big|A_j(x)\big| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^n, \ 1 \leq j \leq n. \quad (3.6)$$

◇

To analyze the spectral properties of H we first turn to the spectral representation of $H_0 = \alpha \cdot (-i\nabla)$ (see also Thaller [165, Section 5.6] and Yafaev [186, Section 2.4]).

Introducing the unitary Fourier transform in $L^2(\mathbb{R}^n)$ via

$$\mathcal{F}: \begin{cases} L^2(\mathbb{R}^n; d^n x) \to L^2(\mathbb{R}^n; d^n p), \\ f \mapsto (\mathcal{F} f)(p) := f^\wedge(p) = \operatorname*{s-lim}_{R\to\infty} (2\pi)^{-n/2} \int_{|x|\leq R} d^n x \, e^{-ip\cdot x} f(x), \end{cases} \tag{3.7}$$

with s-lim abbreviating the limit in the topology of $L^2(\mathbb{R}^n)$, one obtains

$$H_0 = \mathcal{F}^{-1} |p| (\alpha \cdot \omega) \mathcal{F}, \tag{3.8}$$

employing polar coordinates in Fourier space, $p = |p|\,\omega$, $\omega \in S^{n-1}$. Since by (3.1) (see also (3.5))

$$(\alpha \cdot \omega)^2 = I_N, \quad \omega \in S^{n-1},$$

the self-adjoint matrix $\alpha \cdot \omega$ has eigenvalues ± 1 of multiplicity $N/2$ with associated spectral projection matrices of rank $N/2$ denoted by $\Pi_\pm(\omega)$,

$$\alpha \cdot \omega = \Pi_+(\omega) - \Pi_-(\omega), \quad \omega \in S^{n-1}.$$

Introducing

$$T(\omega) = 2^{-1/2}(\alpha_{n+1} + \alpha \cdot \omega), \quad \omega \in S^{n-1}, \tag{3.9}$$

one infers that $T(\omega) \in \mathbb{C}^{N\times N}$ is Hermitian symmetric for each $\omega \in S^{n-1}$. In addition, the anti-commutation property in (3.1) implies

$$\begin{aligned} T(\omega)T(\omega)^* &= 2^{-1}\big[\alpha_{n+1}^2 + (\alpha\cdot\omega)\alpha_{n+1} + \alpha_{n+1}(\alpha\cdot\omega) + (\alpha\cdot\omega)^2\big] \\ &= I_N, \quad \omega \in S^{n-1}, \end{aligned}$$

so that $T(\omega)$ is actually unitary for each $\omega \in S^{n-1}$. The reason for introducing the unitary matrix $T(\omega)$, $\omega \in S^{n-1}$, is that it can be used to diagonalize the matrix $\alpha \cdot p$. Indeed, writing $p \in \mathbb{R}^n$ in polar coordinates as $p = |p|\omega$ with $\omega \in S^{n-1}$, one obtains

$$\begin{aligned} T(\omega)|p|\alpha_{n+1}T(\omega)^* &= 2^{-1}|p|\big[\alpha_{n+1} + 2\alpha\cdot\omega - (\alpha\cdot\omega)^2\alpha_{n+1}\big] \\ &= |p|(\alpha\cdot\omega) \\ &= \alpha \cdot p, \end{aligned} \tag{3.10}$$

so that $\alpha \cdot p$ is unitarily equivalent to $|p|\alpha_{n+1}$ in $\mathbb{C}^N$. Of course, α_{n+1} is Hermitian symmetric, so it may be diagonalized by conjugating with a fixed (i.e., p-independent) unitary matrix $U \in \mathbb{C}^{N\times N}$. We may assume without loss that the columns of U are arranged so that

$$\alpha_{n+1} = U \begin{pmatrix} -I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{pmatrix} U^*, \tag{3.11}$$

where $0_{N/2}$ denotes the zero matrix in $\mathbb{C}^{(N/2)\times(N/2)}$. The facts (3.10) and (3.11) combine to yield

$$\alpha \cdot p = \widetilde{T}(\omega)|p| \begin{pmatrix} -I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{pmatrix} \widetilde{T}(\omega)^*, \quad p = |p|\omega \in \mathbb{R}^n, \tag{3.12}$$

where

$$\widetilde{T}(\omega) := T(\omega)U, \quad \omega \in S^{n-1},$$

and then (3.8) implies

$$H_0 = \mathcal{F}^{-1}\widetilde{T}(\omega)|p| \begin{pmatrix} -I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{pmatrix} \widetilde{T}(\omega)^* \mathcal{F}. \tag{3.13}$$

A simple manipulation in (3.13) yields

$$\widetilde{T}(\omega)^* \mathcal{F} H_0 = |p| \begin{pmatrix} -I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{pmatrix} \widetilde{T}(\omega)^* \mathcal{F}. \tag{3.14}$$

To "diagonalize" H_0, we introduce the notation

$$P_- := \begin{pmatrix} I_{N/2} & 0_{N/2} \\ 0_{N/2} & 0_{N/2} \end{pmatrix}, \quad P_+ := \begin{pmatrix} 0_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{pmatrix},$$

and define the transformation

$$\mathcal{F}_{H_0} : [L^2(\mathbb{R}^n)]^N \to L^2\big(\mathbb{R}; d\lambda; [L^2(S^{n-1})]^N\big)$$

according to

$$(\mathcal{F}_{H_0} f)(\lambda, \omega) = \begin{cases} |\lambda|^{(n-1)/2} P_- \widetilde{T}(\omega)^* f^\wedge(|\lambda|\omega), \ \lambda < 0, \\ |\lambda|^{(n-1)/2} P_+ \widetilde{T}(\omega)^* f^\wedge(|\lambda|\omega), \ \lambda \geq 0, \end{cases} \qquad \omega \in S^{n-1}, \ f \in [L^2(\mathbb{R}^n)]^N.$$

The transformation $\mathcal{F}_{H_0}$ is unitary. In fact,

$$\begin{aligned} &\|\mathcal{F}_{H_0} f\|^2_{L^2(\mathbb{R};d\lambda;[L^2(S^{n-1})]^N)} \\ &\quad = \int_0^\infty d\lambda\, |\lambda|^{n-1} \int_{S^{n-1}} d^{n-1}\omega \, \Big\{ \big\| P_- \widetilde{T}(\omega)^* f^\wedge\big(|\lambda|\omega\big) \big\|^2_{\mathbb{C}^N} \\ &\qquad + \big\| P_+ \widetilde{T}(\omega)^* f^\wedge(|\lambda|\omega) \big\|^2_{\mathbb{C}^N} \Big\}, \quad f \in [L^2(\mathbb{R}^n)]^N. \end{aligned} \tag{3.15}$$

Since $(P_-\xi, P_+\eta)_{\mathbb{C}^N} = 0$ for all $\xi, \eta \in \mathbb{C}^N$, an application of the Pythagorean theorem in (3.15) yields

$$\begin{aligned}
&\|\mathcal{F}_{H_0} f\|^2_{L^2(\mathbb{R};d\lambda;[L^2(S^{n-1})]^N)} \\
&\quad = \int_0^\infty d\lambda\, |\lambda|^{n-1} \int_{S^{n-1}} d^{n-1}\omega\, \big\|\widetilde{T}(\omega)^* f^\wedge\big(|\lambda|\omega\big)\big\|^2_{\mathbb{C}^N} \\
&\quad = \|f^\wedge\|^2_{[L^2(\mathbb{R}^n)]^N} = \|f\|^2_{[L^2(\mathbb{R}^n)]^N}, \quad f \in [L^2(\mathbb{R}^n)]^N.
\end{aligned}$$

To check that $\mathcal{F}_{H_0}$ correctly diagonalizes H_0 in the sense that

$$(\mathcal{F}_{H_0} H_0 f)(\lambda, \cdot) = \lambda(\mathcal{F}_{H_0} f)(\lambda, \cdot) \quad \text{for a.e. } \lambda \in \mathbb{R},\ f \in [W^{1,2}(\mathbb{R}^n)]^N, \tag{3.16}$$

one considers separately the cases $\lambda < 0$ and $\lambda \geq 0$. Indeed, for a fixed $f \in [W^{1,2}(\mathbb{R}^n)]^N$, one applies (3.14) to obtain

$$\begin{aligned}
(\mathcal{F}_{H_0} H_0 f)(\lambda, \omega) &= |\lambda|^{(n-1)/2} P_- \widetilde{T}(\omega)^* (H_0 f)^\wedge\big(|\lambda|\omega\big) \\
&= -|\lambda| P_- \widetilde{T}(\omega)^* f^\wedge\big(|\lambda|\omega\big) \\
&= \lambda(\mathcal{F}_{H_0} f)(\lambda, \omega), \quad \lambda < 0,\ \omega \in S^{n-1},
\end{aligned} \tag{3.17}$$

and, similarly,

$$\begin{aligned}
(\mathcal{F}_{H_0} H_0 f)(\lambda, \omega) &= |\lambda|^{(n-1)/2} P_+ \widetilde{T}(\omega)^* (H_0 f)^\wedge\big(|\lambda|\omega\big) \\
&= |\lambda| P_+ \widetilde{T}(\omega)^* f^\wedge\big(|\lambda|\omega\big) \\
&= \lambda(\mathcal{F}_{H_0} f)(\lambda, \omega), \quad \lambda \geq 0,\ \omega \in S^{n-1}.
\end{aligned} \tag{3.18}$$

Equations (3.17) and (3.18) combine to yield (3.16). Of course, (3.16) generalizes to

$$\big(\mathcal{F}_{H_0} \psi(H_0) f\big)(\lambda, \cdot) = \psi(\lambda)(\mathcal{F}_{H_0} f)(\lambda, \cdot) \text{ for a.e. } \lambda \in \mathbb{R},\ f \in \operatorname{dom}\big(\psi(H_0)\big), \tag{3.19}$$

for any measurable function ψ on $\mathbb{R}$.

Consequently, [186, Proposition 2.4.1] applies to H_0, resulting in the following facts:

Proposition 3.3. *Suppose Hypothesis 3.1 (i), (ii) and let $\gamma > 1/2$. Then $\langle Q\rangle^{-\gamma}$ is strongly H_0-smooth on compact subintervals of $\mathbb{R}\backslash\{0\}$ with exponent $\tau > 0$ given by*

$$\tau = \begin{cases} \gamma - (1/2), & \gamma \in \big((1/2), (3/2)\big), \\ 1 - \varepsilon, & \gamma = 3/2,\ \varepsilon \in (0,1), \\ 1, & \gamma \geq 3/2. \end{cases}$$

We note that for $\tau > 1/2$, $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\langle \cdot \rangle^{-\tau} (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} \langle \cdot \rangle^{-\tau} \in \mathcal{B}_\infty\big([L^2(\mathbb{R}^n)]^N\big),$$

a special case of the well-known general fact (cf., e.g., [184, p. 41]),

$$f(Q)g(-i\nabla) \in \mathcal{B}_\infty\big(L^2(\mathbb{R}^n)\big) \quad \text{for any } f,g \in L^\infty(\mathbb{R}^n) \\ \text{with } \lim_{|x|\to\infty} f(x) = 0 = \lim_{|p|\to\infty} g(p).$$

To make the connection with the results collected in Chapter 2, we identify S_0 and H_0 and S with $H = H_0 + V$, and we factorize V according to

$$V = V_1^* V_2, \quad V_1 = V_1^* = \langle\,\cdot\,\rangle^{-\tau} I_N, \quad V_2 = \langle\,\cdot\,\rangle^{\tau} V, \tag{3.20}$$

with V satisfying the conditions in (3.3) for some fixed $\rho > 1$, and hence, with $\tau \in (1/2, \rho)$,

$$\|V_2(\,\cdot\,)\|_{\mathcal{B}(\mathbb{C}^N)} \leq C\langle\,\cdot\,\rangle^{-(\rho-\tau)}. \tag{3.21}$$

In addition,

$$\langle\,\cdot\,\rangle^{-\tau}\big(H - zI_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}\langle\,\cdot\,\rangle^{-\tau} = \langle\,\cdot\,\rangle^{-\tau}\big(H_0 - zI_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}\langle\,\cdot\,\rangle^{-\tau} \\ \times \big[I_{[L^2(\mathbb{R}^n)]^N} + \langle\,\cdot\,\rangle^{\tau} V \left(H_0 - zI_{[L^2(\mathbb{R}^n)]^N}\right)^{-1}\langle\,\cdot\,\rangle^{-\tau}\big]^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R},$$

to mention just a few analogs of the abstract facts collected in (2.5)–(2.9), which all apply in this concrete setting of massless Dirac operators.

Thus, temporarily assuming $\rho > 3/2$ in (3.3), Theorem 2.5 applies to $S = H_0 + V$ with

$$\tau_1 = \tau - (1/2) > 1/2, \text{ necessitating } \tau > 1, \tag{3.22}$$

and

$$\tau_2 = (\rho - \tau) - (1/2) > 0, \text{ requiring } \rho > 3/2. \tag{3.23}$$

Actually, as shown in [186, pp. 98–99] in the context of the Laplacian h_0 in $L^2(\mathbb{R}^n)$,

$$h_0 = -\Delta, \quad \operatorname{dom}(h_0) = H^2(\mathbb{R}^n)$$

it suffices to assume just $\rho > 1$ in (3.3) (even though this cannot be inferred directly from abstract results, the latter require $\rho > 3/2$ as outlined in (3.22), (3.23)) and $\tau \in (1/2, \rho - 1/2)$. A closer examination of [186, pp. 98–99] (see also [186, p. 118]) reveals that there is nothing special about h_0 and precisely the same results apply to $H_0 = \alpha \cdot (-i\nabla)$ as we discuss next.

Applying Theorems 2.4–2.6, to the pair (H, H_0) and to a union of compact intervals exhausting $(-\infty, 0) \cup (0, \infty)$, combined with the approach in [186, pp. 98, 99, and 118], thus yield the following result:

Theorem 3.4. *Assume Hypothesis 3.1 and consider H as defined in (3.4). Then*

$$\sigma_{\rm ess}(H) = \sigma_{\rm ac}(H) = \mathbb{R}, \tag{3.24}$$

$$\sigma_{\rm sc}(H) = \emptyset, \tag{3.25}$$

$$\sigma_{\rm s}(H) \cap \big(\mathbb{R}\backslash\{0\}\big) = \sigma_{\rm p}(H) \cap \big(\mathbb{R}\backslash\{0\}\big), \tag{3.26}$$

with the only possible accumulation points of $\sigma_{\rm p}(H)$ *being* 0 *and* $\pm\infty$. *If*

$$\mathcal{N}_0 := \sigma_{\rm p}(H) \cap \big(\mathbb{R}\backslash\{0\}\big) = \sigma_{\rm d}(H) \cap \big(\mathbb{R}\backslash\{0\}\big),$$

then the operators

$$\overline{V_2\big(H_0 - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}, \quad \overline{V_1\big(H_0 - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*} \\ \Big(\textit{resp.}, \ \overline{V_1\big(H - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\Big) \tag{3.27}$$

are Hölder continuous in $\mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big)$*-norm with respect to* λ *varying in compact subintervals of* $\mathbb{R}\backslash\{0\}$ (*resp.*, $\mathbb{R}\backslash(\{0\} \cup \mathcal{N}_0)$). *In particular, with* $\mathcal{N}_\pm$ *defined in analogy to* (2.11) *by*

$$\mathcal{N}_\pm = \big\{\lambda \in \mathbb{R}\backslash\{0\} \mid \textit{there exists } 0 \neq f \in [L^2(\mathbb{R}^n)]^N \textit{ s.t.} \\ -f = \overline{V_2\big(H_0 - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*} f\big\}, \tag{3.28}$$

one obtains

$$\mathcal{N}_+ = \mathcal{N}_- = \mathcal{N}_0,$$

and the (*geometric*) *multiplicities of the eigenvalue* $\lambda_0 \in \mathbb{R}\backslash\{0\}$ *of* H *and the eigenvalue* -1 *of* $\overline{V_2(H_0 - (\lambda_0 \pm i0) I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}$ *coincide and are finite. Finally, the global wave operators*

$$W_\pm(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0}, \tag{3.29}$$

exist and are complete, that is,

$$\ker\big(W_\pm(H, H_0)\big) = \{0\}, \quad \operatorname{ran}\big(W_\pm(H, H_0)\big) = E_{H,\rm ac}\mathcal{H}, \tag{3.30}$$

with $E_{H,\rm ac}$ *the projection onto the absolutely continuous subspace of* H.

Proof. As discussed above, Theorems 2.5 and 2.6 apply to $S_0 = H_0$ and $S = H$ and a union of closed intervals Λ_0 exhausting $(-\infty, 0) \cup (0, \infty)$ under the additional assumption that $\rho > 3/2$ (and $\tau \in (1, \rho - 1/2)$). Hence, Theorem 3.4 is proved subject to $\rho > 3/2$.

To improve this to $\rho > 1$ (and $\tau \in (1/2, \rho - 1/2)$) we now follow [186, pp. 98, 99, and 118]. First, one notes that if $\lambda \in \mathbb{R}\backslash\{0\}$ is an eigenvalue of H with a corresponding eigenvector $\psi \in [L^2(\mathbb{R}^n)]^N$, then

$$\big\|(\mathcal{F}_{H_0}\psi)(\mu, \cdot)\big\|_{[L^2(S^{n-1})]^N} \leq C_1 |\mu - \lambda|^{\rho-(3/2)}, \quad \mu \in \mathbb{R}, \tag{3.31}$$

for some $C_1 \in (0,\infty)$. In fact, by (3.3), $g := -V\psi \in [L^2_\rho(\mathbb{R}^n)]^N$, and since $\langle Q\rangle^{-\rho}$ is strongly H_0-continuous with exponent $\rho - (1/2) > (1/2)$ by Proposition 3.3, the function $\tilde{g} := \mathcal{F}_{H_0} g$ is Hölder continuous:

$$\big\|\tilde{g}(\mu,\cdot) - \tilde{g}(\lambda,\cdot)\big\|_{[L^2(S^{n-1})]^N} \leq C|\mu - \lambda|^{\rho-(1/2)}, \quad \mu \in \mathbb{R}, \tag{3.32}$$

for some constant $C \in (0,\infty)$, which is independent of λ and $\mu \in \mathbb{R}$. In addition, since $g = H_0\psi - \lambda\psi$, the spectral representation in (3.19) yields

$$\tilde{g}(\mu,\cdot) = (\mu - \lambda)(\mathcal{F}_{H_0}\psi)(\mu,\cdot), \quad \mu \in \mathbb{R}, \tag{3.33}$$

which implies $\tilde{g}(\lambda,\cdot) = 0$. Therefore, (3.32) reduces to

$$\big\|\tilde{g}(\mu,\cdot)\big\|_{[L^2(S^{n-1})]^N} \leq C|\mu - \lambda|^{\rho-(1/2)}, \quad \mu \in \mathbb{R},$$

and then (3.33) yields (3.31) with $C_1 = C$. In addition, one also notes that the equation $(H_0 - \lambda I_{[L^2(\mathbb{R}^n)]^N})\psi = -V\psi$ implies

$$g = -V\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} g, \tag{3.34}$$

since $\psi = (H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} g$ and $\psi = 0$ if $g = 0$.

To prove that non-zero eigenvalues of H have finite multiplicity and may only accumulate at 0 and $\pm\infty$, one may follow the proof of [186, Proposition 1.9.2] essentially verbatim; one only needs to replace $\mathbb{R}_+$ by $\mathbb{R}\backslash\{0\}$.

Next, one proves that for any $\tau \in (0,1/2]$ and $p < 2(1-2\tau)^{-1}$, and for any compact set $X \subset \mathbb{R}$,

$$\int_X d\lambda \big\|(\mathcal{F}_{H_0} f)(\lambda,\cdot)\big\|^2_{[L^2(S^{n-1})]^N} \leq C_2 \|f\|^p_{[L^2_\tau(\mathbb{R}^n)]^N}, \quad f \in [L^2_\tau(\mathbb{R}^n)]^N, \tag{3.35}$$

for some $C_2 = C_2(\alpha, p, X) \in (0,\infty)$. To prove (3.35), one can follow, with minor modifications, the proof of [186, Proposition 1.9.3]. Indeed, for an arbitrary compact set $X \subset \mathbb{R}$, one introduces the family of spaces

$$L^p\big(X; d\lambda; [L^2(S^{n-1})]^N\big), \quad p \in [1,\infty) \cup \{\infty\},$$

and observes that by [186, Theorem 1.1.4],

$$\mathcal{F}_{H_0} f \in L^\infty\big(X; d\lambda; [L^2(S^{n-1})]^N\big), \quad f \in [L^2_\tau(\mathbb{R}^n)]^N, \ \tau > 1/2.$$

The formula

$$\begin{gathered}\big(\mathcal{T}(f_1,f_2)\big)(\lambda) = \big((\mathcal{F}_{H_0} f_1)(\lambda,\cdot), (\mathcal{F}_{H_0} f_2)(\lambda,\cdot)\big)_{[L^2(S^{n-1})]^N} \\ \text{for a.e. } \lambda \in X \text{ and } (f_1,f_2) \in [L^2_\tau(\mathbb{R}^n)]^N \times [L^2_\tau(\mathbb{R}^n)]^N,\end{gathered}$$

defines a bilinear map for each $q \geq 1$ and $\tau \geq 0$:

$$\mathcal{T} : [L^2_\tau(\mathbb{R}^n)]^N \times [L^2_\tau(\mathbb{R}^n)]^N \to L^q(X, d\lambda).$$

The map $\mathcal{T}$ is continuous for $\tau_0 = 0, q_0 = 1$ and $\tau_1 > 1/2, q_1 = \infty$, so by Calderón's complex bilinear interpolation theorem (cf., e.g., [171, Section 1.19.5], [37]), for any $s \in [0, 1]$ the map $\mathcal{T}$ is continuous for

$$\tau = \tau(s) = s\tau_0 + (1-s)\tau_1, \quad q^{-1} = q(s)^{-1} = sq_0^{-1} + (1-s)q_1^{-1},$$

and

$$\begin{aligned} &\|\mathcal{T}(f_1, f_2)\|_{L^q(X,d\lambda)} \\ &\quad \leq C(\tau, X)\|f_1\|_{[L^2_\tau(\mathbb{R}^n)]^N}\|f_2\|_{[L^2_\tau(\mathbb{R}^n)]^N}, \quad f_1, f_2 \in [L^2_\tau(\mathbb{R}^n)]^N. \end{aligned} \tag{3.36}$$

Taking $f_1 = f_2 = f \in [L^2_\tau(\mathbb{R}^n)]^N$ and $q = p/2$ in (3.36) yields (3.35).

In analogy with [186, Lemma 1.9.4], if $h \in [L^2_\tau(\mathbb{R}^n)]^N$ for some $\tau \in (1/2, 1]$ and $(\mathcal{F}_{H_0}h)(\lambda, \cdot) = 0$ for some $\lambda \in \mathbb{R}\backslash\{0\}$, then

$$\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}h \in [L^2_{-\tilde{\tau}}(\mathbb{R}^n)]^N, \quad \tilde{\tau} > 1 - \tau. \tag{3.37}$$

To prove (3.37), it suffices to show

$$\begin{aligned} &\Big|\Big(\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)N]}\big)^{-1}h, g\Big)_{[L^2(\mathbb{R}^n)]^N}\Big| \\ &\quad \leq C_3\|h\|_{[L^2_\tau(\mathbb{R}^n)]^N}\|g\|_{[L^2_{\tilde{\tau}}(\mathbb{R}^n)]^N}, \quad g \in [\mathcal{S}(\mathbb{R}^n)]^N, \end{aligned}$$

for some $C_3 = C_3(\lambda) \in (0, \infty)$. Using the spectral representation for H_0, one infers that

$$\begin{aligned} &\Big(\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}h, g\Big)_{[L^2(\mathbb{R}^n)]^N} \\ &= \int_{\mathbb{R}} d\mu\, (\mu - \lambda \mp i0)^{-1}\big(\tilde{h}(\mu, \cdot), \tilde{g}(\mu, \cdot)\big)_{[L^2(S^{n-1})]^N}, \quad g \in [\mathcal{S}(\mathbb{R}^n)]^N. \end{aligned} \tag{3.38}$$

Since

$$\tilde{h}, \tilde{g} \in L^2\big([0, \infty); d\lambda; L^2(S^{n-1})^{N/2}\big) \oplus L^2\big((-\infty, 0]; d\lambda; L^2(S^{n-1})^{N/2}\big),$$

it suffices to estimate the integral in (3.38) over a compact neighborhood, say X_λ, of the point λ. By Proposition 3.3,

$$\begin{aligned} \|\tilde{h}(\mu, \cdot)\|_{[L^2(S^{n-1})]^N} &= \|\tilde{h}(\lambda, \cdot) - \tilde{h}(\mu, \cdot)\|_{[L^2(S^{n-1})]^N} \\ &\leq C_0|\lambda - \mu|^{\tau-(1/2)}\|h\|_{[L^2_\tau(\mathbb{R}^n)]^N}, \end{aligned}$$

for some $C_0 \in (0,\infty)$, and consequently,

$$\bigg| \int_{X_\lambda} d\mu\, (\mu - \lambda \mp i0)^{-1} \big(\tilde{h}(\mu,\cdot), \tilde{g}(\mu,\cdot)\big)_{[L^2(S^{n-1})]^N} \bigg| \\ \leq C_0 \|h\|_{[L^2_{\tilde{\tau}}(\mathbb{R}^n)]^N} \int_{X_\lambda} d\mu\, |\lambda - \mu|^{\tau-(3/2)} \big\| \tilde{g}(\mu,\cdot)\big\|_{[L^2(S^{n-1})]^N}, \\ g \in [\mathcal{S}(\mathbb{R}^n)]^N. \tag{3.39}$$

An application of Hölder's inequality yields for any conjugate pair $p^{-1} + q^{-1} = 1$ the estimate

$$\int_{X_\lambda} d\mu\, |\lambda - \mu|^{\tau-(3/2)} \big\| \tilde{g}(\mu,\cdot)\big\|_{[L^2(S^{n-1})]^N} \\ \leq \bigg(\int_{X_\lambda} d\mu\, |\lambda - \mu|^{-q((3/2)-\tau)} \bigg)^{q^{-1}} \bigg(\int_{X_\lambda} d\mu\, \big\| \tilde{g}(\mu,\cdot)\big\|^p_{[L^2(S^{n-1})]^N} \bigg)^{p^{-1}}. \tag{3.40}$$

By (3.35) with $C_2 = C_2(\alpha, p, X_\lambda)$,

$$\int_{X_\lambda} d\mu\, \big\| \tilde{g}(\mu,\cdot)\big\|^p_{[L^2(S^{n-1})]^N} \leq C_2 \|g\|^p_{[L^2_{\tilde{\tau}}(\mathbb{R}^n)]^N}, \tag{3.41}$$

where $p < 2(1-2\tilde{\tau})^{-1}$. Therefore, the conjugate exponent satisfies $q > 2(1+2\tilde{\tau})^{-1}$, and consequently $q((3/2)-\tau) > (3-2\tau)(1+2\tilde{\tau})^{-1}$. Thus, if $\tilde{\tau} > 1-\tau$, that is, if $\tau + \tilde{\tau} > 1$, then $(3-2\tau)(1+2\tilde{\tau})^{-1} < 1$, so $q((3/2)-\tau)$ may be chosen to be smaller than 1, rendering the first integral on the right-hand side in (3.40) finite. In conclusion, (3.39), (3.40), and (3.41) combine to yield the desired estimate.

Finally, we turn to the issue of absence of singular continuous spectrum for H. Introducing the set $\mathcal{N} := \mathcal{N}_+ \cup \mathcal{N}_-$, so that

$$\sigma_s(H)\backslash\{0\} \subseteq \mathcal{N},$$

to prove that $\sigma_{sc}(H) = \emptyset$, it suffices to show that any $\lambda \in \mathcal{N}$ must be an eigenvalue of H. To this end, let $\lambda \in \mathcal{N}$, so that there exists an $f \in [L^2(\mathbb{R}^n)]^N\backslash\{0\}$ such that

$$-f = \overline{V_2 \big(H_0 - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\, f, \tag{3.42}$$

with V_1 and V_2 taken as in (3.20) with $\tau \in (1/2, \rho - (1/2))$. Introducing

$$g = \langle\cdot\rangle^{-\tau} f \in [L^2_\tau(\mathbb{R}^n)]^N, \tag{3.43}$$

the equations for f in (3.42) may be recast as (3.34). In view of (3.43) and the fact that $\tilde{g}(\lambda) = 0$, the estimate in (3.37) applies to $h = g$:

$$\big(H_0 - (\lambda \pm i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} g \in [L^2_{-\tilde{\tau}}(\mathbb{R}^n)]^N, \quad \tilde{\tau} > 1 - \tau.$$

Then the condition in (3.3) and the identity in (3.34) combine to yield

$$g = -V\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} g \in [L^2_\nu(\mathbb{R}^n)]^N, \quad \nu < \tau + \rho - 1.$$

Iterating the same argument $\ell \in \mathbb{N}$ times yields

$$g \in [L^2_\nu(\mathbb{R}^n)]^N, \quad \nu < \tau + \ell(\rho - 1). \tag{3.44}$$

If ℓ is chosen so that $\ell > (1-\tau)/(\rho-1)$, then $\tau + \ell(\rho - 1) > 1$. Therefore, (3.44) implies, in particular, that $g \in [L^2_{\widetilde{\tau}}(\mathbb{R}^n)]^N$ for some $\widetilde{\tau} \in (1, 3/2)$. Consequently, by Proposition 3.3, $\tilde{g}$ is Hölder continuous of order $\widetilde{\tau} - (1/2)$. Next, the function $\psi := (H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} g$ belongs to $\operatorname{dom}(H) = \operatorname{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N$ since $\widetilde{\psi}(\mu) = \tilde{g}(\mu)(\mu - \lambda)^{-1}$, $\tilde{g}(\lambda) = 0$, and $\widetilde{\tau} - (1/2) > 1/2$. By (3.34), ψ satisfies the Dirac equation $H\psi = \lambda\psi$. Moreover, ψ is a nontrivial solution. Indeed, if $\psi = 0$, then $g = -V\psi = 0$. Of course, one then obtains $f = \langle\,\cdot\,\rangle^{\tau} g = 0$, which contradicts the assumption $f \in [L^2(\mathbb{R}^n)]^N \backslash \{0\}$. Therefore, ψ is an eigenfunction and λ is a corresponding eigenvalue. As a result, $\mathcal{N} \subseteq \sigma_{\rm p}(H)$ and $\sigma_{\rm sc}(H) = \emptyset$. ■

Remark 3.5. The fact that $\|\overline{V_2(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^3)]^4})^{-1} V_1^*}\|_{\mathcal{B}(L^2(\mathbb{R}^3))}$ does not decay as $\lambda \to \pm\infty$ shows that in principle one cannot rule out eigenvalues of H running off to ∞ and/or $-\infty$. In fact, it has been shown in [104] that for all $\tau > 1/2$, there exists a constant $C_\tau \in (0, \infty)$ such that

$$\sup_{z \in \mathbb{C}\backslash\mathbb{R}} \big\|\langle\,\cdot\,\rangle^{-\tau}\big(H_0 - zI_{[L^2(\mathbb{R}^3)]^4}\big)^{-1}\langle\,\cdot\,\rangle^{-\tau}\big\|_{\mathcal{B}([L^2(\mathbb{R}^3)]^4)} \leq C_\tau \tag{3.45}$$

and that

$$\big\|\langle\,\cdot\,\rangle^{-\tau}\big(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^3)]^4}\big)^{-1}\langle\,\cdot\,\rangle^{-\tau}\big\|_{\mathcal{B}([L^2(\mathbb{R}^3)]^4)} \text{ does not decay as } |\lambda| \to \infty \text{ for any } \tau > 1/2. \tag{3.46}$$

In the case of massive Dirac operators (i.e., with H_0 replaced by $H_0(m)$), the condition $\tau > 1/2$ needs to be replaced by $\tau \geq 1$. For results in this direction we also refer to [131–133, 189]. This contrasts sharply with the case of Schrödinger operators where a Riemann–Lebesgue-type argument yields decay of the underlying Birman–Schwinger operator (see, e.g., [157, Theorem III.13]). ◇

Remark 3.6. The transformation in (3.9) employed to diagonalize $\alpha \cdot p$ is similar to the celebrated Foldy–Wouthuysen transformation (see, e.g., [46, 162], [165, Section 5.6]). The latter is well known to diagonalize H_0. In fact, introducing the unitary $N \times N$ block operator matrix U_N in $[L^2(\mathbb{R}^n)]^N$, $n \in \mathbb{N}$, $n \geq 2$, via

$$U_N = 2^{-1/2}\big[I_N + \beta\big(\alpha \cdot (-i\nabla)\big)|-i\nabla|^{-1}\big],$$
$$U_N^{-1} = 2^{-1/2}\big[I_N - \beta\big(\alpha \cdot (-i\nabla)\big)|-i\nabla|^{-1}\big],$$

one infers that

$$\widetilde{H}_0 = U_N H_0 U_N^{-1} = \begin{pmatrix} I_{N/2}(-\Delta)^{1/2} & 0_{N/2} \\ 0_{N/2} & -I_{N/2}(-\Delta)^{1/2} \end{pmatrix},$$
$$\operatorname{dom}\big(\widetilde{H}_0\big) = [W^{1,2}(\mathbb{R}^n)]^N. \tag{3.47}$$

It is worth pointing out that every result in this chapter has a verbatim analog for operators of the type

$$\widetilde{H}_0 + V, \quad \text{and} \quad I_N(-\Delta + m^2 I_{L^2(\mathbb{R}^n)})^{1/2} + V,\ m \geq 0,$$

in $[L^2(\mathbb{R}^n)]^N$, with V satisfying (3.3). More generally, $(-\Delta)^{1/2}$ can be replaced by general fractional powers $(-\Delta)^{\gamma}$, $\gamma > 0$, and even by more general functions $h(-\Delta)$ (cf. [22]). This comment is of some significance as a large body of work went into studying $I_N(-\Delta)^{1/2} + V$ (especially, in the scalar case $N = 1$) over the past two decades. We refer, for instance, to [22, 32, 93, 109, 110, 117, 143, 147], [158, p. 124], [172–177, 182]. ◇

In the following chapter we will recall conditions on V that yield the absence of eigenvalues of H (implying unitary equivalence of H and H_0 via the wave operators $W_{\pm}(H, H_0)$ in Theorem 3.4, see Remark 4.3).

We conclude this chapter with some hints at additional literature (beyond [186, Sections 1.11, 2.1, and 2.2]) concerning the absence of singular continuous spectrum and proofs of limiting absorption principles for operators of the form $H_0 + V$.

In the case of three-dimensional massless Dirac operators, the absence of singular continuous spectrum of H with scalar potentials (i.e., $V = v\, I_N$), including the case of long-range interactions v, was proved in [47]. The limiting absorption principle for H_0 in three dimensions was derived in [151]. For the proof of existence of absolutely continuous spectrum of massless Dirac operators for $n = 3$, where $V = v\,\beta$, see [49]. To the best of our knowledge, these references in the special case $n = 3$ comprise all explicit statements about the absence of the singular continuous spectrum of H and/or the limiting absorption principle for H_0. So Theorem 3.4 is new for $n \in \mathbb{N}\backslash\{3\}$, $n \geq 2$, which is particularly interesting in the case $n = 2$ as the latter is related to applications involving graphene. On the other hand, we emphasize that Theorem 3.4 is a direct consequence of the material presented by Yafaev in [186, Section 2.4]. In the context of massless Dirac operators in dimension $n = 2$ we also refer to [60] (see also [59]). We also note that a global limiting absorption principle for H_0 on $\mathbb{R}$ for all $n \in \mathbb{N}$, $n \geq 2$, was proved in [23, 39, 104].

For the case of massive Dirac operators $H(m) = H + m\,\beta$, $m > 0$, we also refer to [18, 33, 35, 65–67, 74, 94, 95, 121, 127, 133, 134, 149, 167, 180], [186, Section 1.12], [187–189].

Finally, for scattering theory for Dirac operators we refer, for instance, to [47, 54, 74, 89, 95, 115, 123, 133, 134, 136, 148, 158, 161, 163, 164], [165, Chapter 8], [166, 167, 178], [186, Section 1.12].