Chapter 4

On the absence of eigenvalues for interacting, massless Dirac operators

In this chapter, we briefly comment on results concerning the absence of eigenvalues of massless Dirac operators. Since we are particularly interested in potentials vanishing at $\pm\infty$, implying the absence of spectral gaps of H,

$$\sigma_{\text{ess}}(H) = \mathbb{R},$$

the absence of eigenvalues is equivalent to the absence of eigenvalues embedded into the essential spectrum of H (a somewhat unusual situation from a quantum mechanical point of view).

In the context of massive Dirac operators $H(m) = H + m\beta$, with mass parameter $m > 0$ and vanishing potentials at $\pm\infty$, there exists a fair number of papers describing the absence of embedded eigenvalues in the essential spectrum of $H(m)$,

$$\sigma_{\text{ess}}\big(H(m)\big) = (-\infty, -m] \cup [m, \infty)$$

(or in certain regions of the essential spectrum), especially in the three-dimensional case, $n = 3$. Relevant references in this context are, for instance, [9, 25, 100, 101, 103, 126, 145] (however, this reference is imprecise w.r.t. implicit smoothness assumptions on the electromagnetic potential coefficients), [168, 179].

The existence of threshold eigenvalues (and/or resonances) at $\pm m$ are discussed, for instance, in [56, 152].

In the massless case, $m = 0$, zero eigenvalues and/or zero-energy resonances (as well as the absence zero-energy resonances) are treated in [2–5, 7, 8, 13–15], [16, Chapter 4], [17, 24, 52, 55, 57, 70, 73, 103, 116, 129, 144, 146, 150–156, 190]. A fair number of these references consider the case of Pauli operators in three dimensions, $[\sigma \cdot (-i\nabla - A)]^2$, with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the standard Pauli matrices.

The subject of absence of zero modes, especially, zero-energy eigenvalues, for massless Dirac operators has hardly been studied. Exceptions are [48] (see also [105]), [102, 103], and properties of the corresponding (generalized) eigenfunctions are discussed in [8], [16, Section 4.4], [17, 150, 151, 190].

Here we recall the following special cases of results in [103, Theorems 2.1 and 2.3]:

Theorem 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(i) *Assume that* $V\colon \mathbb{R}^n \to \mathbb{C}^{N\times N}$ *is Lebesgue measurable and self-adjoint a.e. on* $\mathbb{R}^n$, *satisfying*

$$\operatorname*{ess.\,sup}_{x\in\mathbb{R}^n} |x| \|V(x)\|_{\mathcal{B}(\mathbb{C}^N)} \leq C \quad \text{for some } C \in \big(0,(n-1)/2\big), \tag{4.1}$$

with $\|\cdot\|_{\mathcal{B}(\mathbb{C}^N)}$ *denoting the operator norm of an* $N\times N$ *matrix in* $\mathbb{C}^N$. *Then any distributional solution* $u \in [W^{1,2}_{\rm loc}(\mathbb{R}^n)]^N \cap [L^2(\mathbb{R}^n;|x|^{-1}d^nx)]^N$ *of* $(H_0+V)u=0$ *vanishes identically.*

(ii) *Suppose that*

$$\begin{aligned} &V:\mathbb{R}^n \to \mathbb{C}^{N\times N} \text{ is Lebesgue measurable and self-adjoint a.e. on } \mathbb{R}^n, \text{ and that}\\ &\text{for some } R>0,\ V\in [C^1(E_R)]^{N\times N}, \text{ where } E_R=\big\{x\in\mathbb{R}^n \,\big|\, |x|>R\big\}, \end{aligned} \tag{4.2}$$

and

$$|x|^{1/2}V_{\ell,\ell'}(x) \underset{|x|\to\infty}{=} o(1), \quad (x\cdot\nabla V_{\ell,\ell'})(x) \underset{|x|\to\infty}{=} o(1), \quad 1\leq \ell,\ell'\leq N, \\ \text{uniformly with respect to directions}. \tag{4.3}$$

Then if for some $\lambda\in\mathbb{R}\backslash\{0\}$, $u\in[L^2(E_R)]^N$ *satisfies* $(H_0+V)u=\lambda u$ *on* E_R *in the distributional sense, then* u *vanishes identically on* $\mathbb{R}^n$.

(iii) *The self-adjoint realization* H *of* H_0+V *satisfying*

$$\int_{\mathbb{R}^n} d^nx\, |x|^{-1}\|f(x)\|^2_{\mathbb{C}^N} < \infty, \quad f\in \operatorname{dom}(H), \tag{4.4}$$

has no eigenvalue zero in case (i) *and no eigenvalue* $\lambda\in\mathbb{R}\backslash\{0\}$ *in case* (ii).

Remark 4.2. We note that Theorem 4.1 due to [103, Theorems 2.1 and 2.3] appears to have been the first, and up to now, the only result available proving absence of eigenvalues of H in the massless case. The results on global H_0-smoothness of $\langle Q\rangle^{-1}$ proven in [39] now yields a second such result (and unitary equivalence of H and H_0) for $\|V\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)}$ sufficiently small. ◇

Remark 4.3. In the context of Theorem 4.1 (iii) we note that if V satisfies (3.3) and hence $\operatorname{dom}(H)=[W^{1,2}(\mathbb{R}^n)]^N$, then Kato's inequality (cf., e.g., [16, pp. 19–20], [92]),

$$\int_{\mathbb{R}^n} d^nx\,|x|^{-1}\big|f(x)\big|^2 \leq C_n \int_{\mathbb{R}^n} d^np\,|p|\big|\hat f(p)\big|^2, \quad f\in\mathcal{S}(\mathbb{R}^n),\ n\in\mathbb{N},\ n\geq 2,$$

for appropriate constants $C_n\in(0,\infty)$, $n\geq 2$ (Kato's inequality extends to the homogeneous Sobolev space $D^{1/2}(\mathbb{R}^n)$ of order $1/2$), yields, in particular,

$$\int_{\mathbb{R}^n} d^nx\,|x|^{-1}\big|f(x)\big|^2 \leq C_n \int_{\mathbb{R}^n} d^np\,|p|\big|\hat f(p)\big|^2 \leq C_n \int_{\mathbb{R}^n} d^np\,\big[1+|p|^2\big]\big|\hat f(p)\big|^2, \\ f\in W^{1,2}(\mathbb{R}^n),\ n\in\mathbb{N},\ n\geq 2.$$

Thus, under assumption (3.3) on V, condition (4.4) holds automatically. ◇

We summarize the discussion on absence of eigenvalues in this chapter as follows:

Corollary 4.4. (i) *In addition to Hypothesis* 3.1 *assume that* V *satisfies conditions* (4.2)*,* (4.3)*. Then*

$$\sigma_{\mathrm{p}}(H) \subseteq \{0\}. \tag{4.5}$$

(ii) *In addition to Hypothesis* 3.1 *assume that* V *satisfies conditions* (4.1)–(4.3)*. Then*

$$\sigma_{\mathrm{p}}(H) = \emptyset. \tag{4.6}$$

Moreover, H *and* H_0 *are unitarily equivalent.*

Proof. The inclusion (4.5) follows from Theorem 4.1 (ii), (iii) and Remark 4.3. The fact (4.6) follows from Theorem 4.1 (i), Remark 4.3, and (4.5). Unitary equivalence of H and H_0 is a consequence of (3.25), (3.29), (3.30), and (4.6). ∎

Note added in proof: In connection with the absence of zero-energy eigenvalues of massless Dirac operators with vector potentials we also refer to [72].