Chapter 5

The Green's functions of $H_0(m)$ and H_0

In this chapter, we study the Green's function for H_0, that is, the integral kernel of the resolvent of H_0.

We start, however, with the Green's function of the Laplacian in $L^2(\mathbb{R}^n)$,

$$h_0 = -\Delta, \quad \mathrm{dom}(h_0) = H^2(\mathbb{R}^n).$$

The Green's function of h_0, denoted by $g_0(z;\,\cdot\,,\,\cdot\,)$, is of the form,

$$\begin{aligned} g_0(z;x,y) &:= (h_0 - zI_{L^2(\mathbb{R}^n)})^{-1}(x,y) \\ &= \begin{cases} (i/2)z^{-1/2}e^{iz^{1/2}|x-y|}, & n=1,\ z\in\mathbb{C}\backslash\{0\}, \\ (i/4)\big(2\pi z^{-1/2}|x-y|\big)^{(2-n)/2}H^{(1)}_{(n-2)/2}\big(z^{1/2}|x-y|\big), & n\geqslant 2,\ z\in\mathbb{C}\backslash\{0\}, \end{cases} \\ &\qquad \mathrm{Im}(z^{1/2})>0,\ x,y\in\mathbb{R}^n,\ x\neq y, \end{aligned} \tag{5.1}$$

and for $z = 0$, $n \geq 3$,

$$g_0(0;x,y) = \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, \quad n\geqslant 3,\ x,y\in\mathbb{R}^n,\ x\neq y.$$

Here $H^{(1)}_\nu(\,\cdot\,)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ (cf. [1, Section 9.1]) and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf. [1, Section 6.1]) represents the area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

As $z \to 0$, $g_0(z;\,\cdot\,,\,\cdot\,)$ is continuous on the off-diagonal for $n \geq 3$,

$$\begin{aligned} \lim_{\substack{z\to 0 \\ z\in\mathbb{C}\backslash\{0\}}} g_0(z;x,y) = g_0(0;x,y) &= \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, \\ & x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N},\ n\geqslant 3, \end{aligned} \tag{5.2}$$

but blows up for $n = 1$ as

$$g_0(z;x,y) \underset{\substack{z\to 0 \\ z\in\mathbb{C}\backslash\{0\}}}{=} (i/2)z^{-1/2} - 2^{-1}|x-y| + O\big(z^{1/2}|x-y|^2\big), \quad x,y\in\mathbb{R},$$

and for $n = 2$ as

$$\begin{aligned} g_0(z;x,y) \underset{\substack{z\to 0 \\ z\in\mathbb{C}\backslash\{0\}}}{=} & -\frac{1}{2\pi}\ln\big(z^{1/2}|x-y|/2\big)\big[1+O\big(z|x-y|^2\big)\big] \\ & + \frac{1}{2\pi}\psi(1) + O\big(|z||x-y|^2\big), \quad x,y\in\mathbb{R}^2,\ x\neq y. \end{aligned} \tag{5.3}$$

Here $\psi(w) = \Gamma'(w)/\Gamma(w)$ denotes the digamma function (cf. [1, Section 6.3]).

For reasons of subsequent comparisons with the case of the free massive Dirac operator $H_0(m) = H_0 + m\,\beta$, $m > 0$, we now start with the latter and compute,

$$\begin{aligned}(H_0(m) - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} &= \big(H_0(m) + zI_{[L^2(\mathbb{R}^n)]^N}\big)\big(H_0(m)^2 - z^2 I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} \\ &= (-i\alpha\cdot\nabla + m\,\beta + zI_{[L^2(\mathbb{R}^n)]^N})\big(h_0 - (z^2-m^2)I_{L^2(\mathbb{R}^n)}\big)^{-1} I_N, \quad (5.4)\end{aligned}$$

employing

$$H_0(m)^2 = (h_0 + m^2 I_{L^2(\mathbb{R}^n)})I_N.$$

Assuming

$$\begin{aligned}& m > 0,\ z \in \mathbb{C}\backslash(\mathbb{R}\backslash[-m,m]),\ \operatorname{Im}\big(z^2-m^2\big)^{1/2} > 0, \\ & x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \qquad (5.5)\end{aligned}$$

and exploiting (5.4), one thus obtains for the Green's function $G_0(m,z;\,\cdot\,,\,\cdot\,)$ of $H_0(m)$,

$$\begin{aligned}G_0(m,z;x,y) &:= \big(H_0(m) - zI_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}(x,y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n}(m\,\beta + zI_N) \\ &\quad \times \big[(z^2-m^2)^{1/2}|x-y|\big]^{(n-2)/2} H^{(1)}_{(n-2)/2}\big((z^2-m^2)^{1/2}|x-y|\big) \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}\,\alpha\cdot\frac{(x-y)}{|x-y|} \\ &\quad \times \big[(z^2-m^2)^{1/2}|x-y|\big]^{n/2} H^{(1)}_{n/2}\big((z^2-m^2)^{1/2}|x-y|\big).\end{aligned}$$

Here we employed the identity ([1, p. 361]),

$$\big[H^{(1)}_\nu(\zeta)\big]' = -H^{(1)}_{\nu+1}(\zeta) + \nu\,\zeta^{-1}H^{(1)}_\nu(\zeta), \quad \nu,\zeta \in \mathbb{C}.$$

Equations (B.9), (B.10) reveal the facts (still assuming (5.5)),

$$\begin{aligned}\lim_{\substack{z\to\pm m \\ z\in\mathbb{C}\backslash\{\pm m\}}} G_0(m,z;x,y) &= 4^{-1}\pi^{-n/2}\Gamma\big((n-2)/2\big)|x-y|^{2-n}(m\,\beta \pm mI_N) \\ &\quad + i2^{-1}\pi^{-n/2}\Gamma(n/2)\,\alpha\cdot\frac{(x-y)}{|x-y|^n}, \\ & m > 0,\ x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N},\ n\geq 3, \qquad (5.6)\end{aligned}$$

$$\begin{aligned}G_0(m,z;x,y) \underset{\substack{z\to\pm m \\ z\in\mathbb{C}\backslash\{\pm m\}}}{=} & -(4\pi)^{-1}\ln(z^2-m^2)(m\,\beta \pm mI_2) \\ & - (2\pi)^{-1}\ln\big(|x-y|\big)(m\,\beta \pm mI_2) + i(2\pi)^{-1}\,\alpha\cdot\frac{(x-y)}{|x-y|^2} \\ & + O\big((z^2-m^2)\ln(z^2-m^2)\big), \quad m>0,\ x,y\in\mathbb{R}^2,\ x\neq y. \qquad (5.7)\end{aligned}$$

(Here the remainder term $O((z^2-m^2)\ln(z^2-m^2))$ depends on $x, y \in \mathbb{R}^2$, but this is of no concern at this point.) In particular, $G_0(m, z; \cdot, \cdot)$ blows up logarithmically as $z \to \pm m$ in two dimensions, $n = 2$, just as $g_0(z, \cdot, \cdot)$ does as $z \to 0$.

By contrast, the massless case is quite different and assuming

$$z \in \mathbb{C}_+,\ x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \tag{5.8}$$

one computes in the case $m = 0$ for the Green's function $G_0(z; \cdot, \cdot)$ of H_0,

$$\begin{aligned} G_0(z; x, y) &:= (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}(x, y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\left[z|x-y|\right]^{(n-2)/2} H^{(1)}_{(n-2)/2}\big(z|x-y|\big) I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}\big[z|x-y|\big]^{n/2} H^{(1)}_{n/2}\big(z|x-y|\big)\,\alpha \cdot \frac{(x-y)}{|x-y|}. \end{aligned} \tag{5.9}$$

The Green's function $G_0(z; \cdot, \cdot)$ of H_0 continuously extends to $z \in \overline{\mathbb{C}_+}$. In addition, in the massless case $m = 0$, the limit $z \to 0$ exists[1],

$$\lim_{\substack{z \to 0,\\ z \in \overline{\mathbb{C}_+}\backslash\{0\}}} G_0(z; x, y) := G_0(0 + i\,0; x, y) = i2^{-1}\pi^{-n/2}\Gamma(n/2)\,\alpha \cdot \frac{(x-y)}{|x-y|^n},$$
$$x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \tag{5.10}$$

and no blow up occurs for all $n \in \mathbb{N}, n \geq 2$.

Remark 5.1. (i) The observation of an absence of blow up in $G_0(z; \cdot, \cdot)$ as $z \to 0$ is consistent with the sufficient condition for the Dirac operator $H = H_0 + V$ (in dimensions $n \in \mathbb{N}, n \geq 2$), with V an appropriate self-adjoint $N \times N$ matrix-valued potential, having no eigenvalues, as derived in [103, Theorems 2.1 and 2.3].

(ii) The asymptotic behavior, for some $d_n \in (0, \infty)$,

$$\big\|G_0(0 + i\,0; x, y)\big\|_{\mathbb{C}^N} \underset{\substack{z \to 0,\\ z \in \overline{\mathbb{C}_+}\backslash\{0\}}}{=} d_n|x-y|^{1-n}, \quad x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2,$$

implies the absence of zero-energy resonances (cf. Chapter 10 for a detailed discussion) of H for $n \in \mathbb{N}, n \geq 3$, for sufficiently fast decaying short-range potentials V at infinity, as $|\cdot|^{1-n}$ lies in $L^2(\mathbb{R}^n)$ near infinity if and only if $n \geq 3$. This is consistent with observations in [8], [16, Section 4.4], [17, 28, 150, 151, 190] for $n = 3$ (see also Remark 10.8 (ii)). This should be contrasted with the behavior of Schrödinger

[1] Our choice of notation $0 + i\,0$ in $G_0(0 + i\,0; x, y)$ indicates that the limit $\lim_{z \to 0}$ is performed in the closed upper half-plane $\overline{\mathbb{C}_+}$.

operators where

$$\lim_{\substack{z\to 0\\ z\in\mathbb{C}\backslash\{0\}}} g_0(z;x,y) = g_0(0;x,y) = \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n},$$
$$x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N},\ n\geqslant 3,$$

implies the absence of zero-energy resonances of $h = h_0 + w$ for $n \in \mathbb{N}$, $n \geq 5$, again for sufficiently fast decaying short-range potentials w at infinity, as $|\cdot|^{2-n}$ lies in $L^2(\mathbb{R}^n)$ near infinity if and only if $n \geq 5$, as observed in [96]. ◇

Remark 5.2. In the special case $n = 3$, the identities

$$H^{(1)}_{1/2}(\zeta) = -i\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{i\zeta}}{\zeta^{1/2}},$$
$$H^{(1)}_{3/2}(\zeta) = -\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{i\zeta}(\zeta+i)}{\zeta^{3/2}}, \quad \zeta\in\mathbb{C}\backslash\{0\},$$

combine in (5.9) to yield

$$G_0(z;x,y) = \frac{e^{iz|x-y|}}{4\pi|x-y|}\left[z\,I_N + z\,\alpha\cdot\frac{(x-y)}{|x-y|} + i\,\alpha\cdot\frac{(x-y)}{|x-y|^2}\right],$$
$$x,y\in\mathbb{R}^3,\ x\neq y,\ z\in\mathbb{C}_+.$$ ◇

Remark 5.3. It is possible to expand the massless Dirac Green's function $G_0(z;\cdot,\cdot)$ in powers of z in such a way that several coefficients in the expansion vanish (the precise number of vanishing coefficients depending on the dimension n) for odd dimensions $n \geq 5$. This observation relies on the following connection between the modified Bessel and spherical Bessel functions (cf., e.g., [1, Section 10.1.1]):

$$H^{(1)}_{j+(1/2)}(\zeta) = (2\pi^{-1}\zeta)^{1/2}h^{(1)}_j(\zeta), \quad \zeta\in\mathbb{C}\backslash\{0\},\ j\in\mathbb{N}. \tag{5.11}$$

Moreover, by [1, Equation 10.1.16],

$$h^{(1)}_j(\zeta) = i^{-(j+1)}\zeta^{-1}e^{i\zeta}\sum_{k=0}^{j}\frac{(j+k)!}{k!(j-k)!}(-2i\zeta)^{-k}, \quad \zeta\in\mathbb{C}\backslash\{0\},\ j\in\mathbb{N}. \tag{5.12}$$

Upon combining (5.11) and (5.12), one obtains for odd dimensions $n \geq 3$,

$$\begin{aligned} H^{(1)}_{(n/2)-1}(\zeta) &= H^{(1)}_{[(n-3)/2]+(1/2)}(\zeta) \\ &= 2^{1/2}\pi^{-1/2}i^{(1-n)/2}\zeta^{-1/2}e^{i\zeta}\sum_{k=0}^{(n-3)/2}\frac{\big([(n-3)/2]+k\big)!}{k!\big([(n-3)/2]-k\big)!}(-2i\zeta)^{-k}, \\ &\qquad \zeta\in\mathbb{C}\backslash\{0\}, \end{aligned} \tag{5.13}$$

and

$$H^{(1)}_{(n/2)}(\zeta) = H^{(1)}_{[(n-1)/2]+(1/2)}(\zeta)$$
$$= 2^{1/2}\pi^{-1/2} i^{-(n+1)/2}\zeta^{-1/2} e^{i\zeta} \sum_{k=0}^{(n-1)/2} \frac{([(n-1)/2]+k)!}{k!([(n-1)/2]-k)!}(-2i\zeta)^{-k},$$
$$\zeta \in \mathbb{C}\backslash\{0\}. \quad (5.14)$$

Thus, using the expansions (5.13) and (5.14) in (5.9), one obtains the following expansion for the massless Dirac Green's function in odd dimensions $n \geq 3$:

$$G_0(z; x, y) = i(-1)^{(1-n)/2} 2^{-(n+1)/2} \pi^{(1-n)/2} e^{iz|x-y|} I_N$$
$$\times \sum_{k=0}^{(n-3)/2} \frac{([(n-3)/2]+k)!}{k!([(n-3)/2]-k)!}(-2)^{-k}(iz)^{-k+[(n-1)/2]}|x-y|^{-k-[(n-1)/2]}$$
$$+ i(-1)^{(1-n)/2} 2^{-(n+1)/2} \pi^{(1-n)/2} e^{iz|x-y|} \alpha \cdot \frac{(x-y)}{|x-y|}$$
$$\times \sum_{k=0}^{(n-1)/2} \frac{([(n-1)/2]+k)!}{k!([(n-1)/2]-k)!}(-2)^{-k}(iz)^{-k+[(n-1)/2]}|x-y|^{-k-[(n-1)/2]},$$
$$x, y \in \mathbb{R}^n,\ x \neq y,\ z \in \mathbb{C}\backslash\mathbb{R}. \quad (5.15)$$

Introducing the power series for the exponential in (5.15) and reordering the series to combine like powers of iz, one obtains

$$G_0(z; x, y) = (-1)^{(3-n)/2} 2^{-(n+1)/2} \pi^{(1-n)/2} z|x-y| \sum_{j=0}^{\infty} d_j (iz)^j |x-y|^{-(n-1-j)} I_N$$
$$+ i(-1)^{(1-n)/2} 2^{-(n+1)/2} \pi^{(1-n)/2} \sum_{j=0}^{\infty} d'_j (iz)^j |x-y|^{-(n-1-j)} \alpha \cdot \frac{(x-y)}{|x-y|},$$
$$x, y \in \mathbb{R}^n,\ x \neq y,\ z \in \mathbb{C}\backslash\mathbb{R}, \quad (5.16)$$

where for each $j \in \mathbb{N}_0$, the numerical coefficients d_j and d'_j are given by

$$d_j = \sum_{\substack{k=0 \\ k \geq [(n-3)/2]-j}}^{(n-3)/2} \frac{([(n-3)/2]+k)!}{k!([(n-3)/2]-k)!}(-2)^{-k} \frac{1}{(j+k-[(n-3)/2])!}, \quad (5.17)$$

$$d'_j = \sum_{\substack{k=0 \\ k \geq [(n-1)/2]-j}}^{(n-1)/2} \frac{([(n-1)/2]+k)!}{k!([(n-1)/2]-k)!}(-2)^{-k} \frac{1}{(j+k-[(n-1)/2])!}. \quad (5.18)$$

In odd dimensions $n \geq 5$, certain of the coefficients d_j and d_j' in (5.17) and (5.18) vanish based on the combinatorial identity in Proposition 5.4 below. Applying Proposition 5.4 with $m = (n-3)/2$ and $m = (n-1)/2$, one infers that for $n \geq 5$ is odd, the free massless Green's function $G_0(z, \cdot, \cdot)$ is given by (5.16)–(5.18) and

$$\begin{aligned} d_j &= 0 \quad \text{for all odd } j \in \mathbb{N} \text{ satisfying } 1 \leq j \leq n-4, \\ d_j' &= 0 \quad \text{for all odd } j \in \mathbb{N} \text{ satisfying } 1 \leq j \leq n-2. \end{aligned} \qquad \diamond$$

Proposition 5.4 ([96, Lemma 3.3]). *If $m \in \mathbb{N}$ and*

$$c_j := \sum_{\substack{k=0 \\ k \geq m-j}}^{m} \frac{(m+k)!}{k!(m-k)!}(-2)^{-k}\frac{1}{(k+j-m)!}, \quad j \in \mathbb{N}_0,$$

then $c_j = 0$ for $j = 1, 3, \dots, 2m-1$.

Since H_0 has no spectral gap, $\sigma(H_0) = \mathbb{R}$, but h_0 has the half-line $(-\infty, 0)$ in its resolvent set, a comparison of h_0 with the massive free Dirac operator $H_0(m) = H_0 + m\beta$, $m > 0$, with spectral gap $(-m, m)$, replacing the energy $z = 0$ by $z = \pm m$, is quite natural and then exhibits a similar logarithmic blowup behavior as $z \to 0$ in dimensions $n = 2$.

Returning to our analysis of the resolvent of H_0, the asymptotic behavior (B.9)–(B.11) implies for some $c_n \in (0, \infty)$,

$$\begin{aligned} \|G_0(0 + i\,0; x, y)\|_{\mathcal{B}(\mathbb{C}^N)} &\leq c_n |x-y|^{1-n}, \\ & x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \end{aligned} \tag{5.19}$$

and for given $R \geq 1$,

$$\begin{aligned} \|G_0(z; x, y)\|_{\mathcal{B}(\mathbb{C}^N)} \leq c_{n,R}(z) e^{-\operatorname{Im}(z)|x-y|} &\begin{cases} |x-y|^{1-n}, & |x-y| \leq 1,\ x \neq y, \\ 1, & 1 \leq |x-y| \leq R, \\ |x-y|^{(1-n)/2}, & |x-y| \geq R, \end{cases} \\ z \in \overline{\mathbb{C}_+},\ x, y \in \mathbb{R}^n,&\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \end{aligned} \tag{5.20}$$

for some $c_{n,R}(\cdot) \in (0, \infty)$ continuous and locally bounded on $\overline{\mathbb{C}_+}$.

For future purposes we now rewrite $G_0(z; \cdot, \cdot)$ as follows:

$$\begin{aligned} G_0(z; x, y) &= i\,4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\,[z|x-y|]^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2} H^{(1)}_{n/2}(z|x-y|)\,\alpha \cdot \frac{(x-y)}{|x-y|} \\ &= |x-y|^{1-n} f_n(z, x-y), \quad z \in \overline{\mathbb{C}_+},\ x, y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2, \end{aligned} \tag{5.21}$$

where f_n is continuous and locally bounded on $\overline{\mathbb{C}_+} \times \mathbb{R}^n$, in addition,

$$\begin{aligned} &\| f_n(z,x)\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\quad \leq c_n(z) e^{-\operatorname{Im}(z)|x|} \begin{cases} 1, & 0 \leq |x| \leq 1, \\ |x|^{(n-1)/2}, & |x| \geq 1, \end{cases} \quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n, \end{aligned} \tag{5.22}$$

for some constant $c_n(\cdot) \in (0,\infty)$ continuous and locally bounded on $\overline{\mathbb{C}_+}$. In particular, decomposing $G_0(z;\cdot,\cdot)$ into

$$\begin{aligned} G_0(z;x,y) &= G_0(z;x,y)\chi_{[0,1]}\big(|x-y|\big) + G_0(z;x,y)\chi_{[1,\infty)}\big(|x-y|\big) \\ &= G_{0,<}(z;x-y) + G_{0,>}(z;x-y), \\ &\quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N}, n \geq 2, \end{aligned} \tag{5.23}$$

where

$$G_{0,<}(z;x-y) := G_0(z;x,y)\chi_{[0,1]}\big(|x-y|\big), \tag{5.24}$$

$$G_{0,>}(z;x-y) := G_0(z;x,y)\chi_{[1,\infty)}\big(|x-y|\big), \tag{5.25}$$

$$z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2,$$

one verifies that

$$\big|G_{0,>}(z;x-y)_{j,k}\big| \leq \begin{cases} C_n|x-y|^{-(n-1)}, & z = 0, \\ C_n(z)|x-y|^{-(n-1)/2}, & z \in \overline{\mathbb{C}_+}, \end{cases} \quad x,y \in \mathbb{R}^n,\ |x-y| \geq 1,\ 1 \leq j,k \leq N, \tag{5.26}$$

for some constants $C_n, C_n(\cdot) \in (0,\infty)$, in particular,

$$G_{0,>}(z;\cdot) \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \quad z \in \overline{\mathbb{C}_+}, \tag{5.27}$$

and that

$$G_{0,>}(\cdot\,;\cdot) \text{ is continuous on } \overline{\mathbb{C}_+} \times \mathbb{R}^n. \tag{5.28}$$

In the next chapter, we will use the decomposition (5.23) to derive trace ideal properties of operators of the type $F_1(\cdot)(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}F_2(\cdot)$, employing results of [26, Section 5.4] in the case $n \geq 3$. We also derive trace ideal properties of $\langle\cdot\rangle^{-\delta}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\cdot\rangle^{-\delta}$ in the case $n \geq 2$ using a different approach based on a combination of Sobolev's inequality and complex interpolation.