Chapter 6

Trace ideal properties of $F_1(\,\cdot\,)(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}F_2(\,\cdot\,)$ and $\langle\,\cdot\,\rangle^{-\delta}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\,\cdot\,\rangle^{-\delta}$

In the first part of this chapter we derive trace ideal properties of operators of the type $F_1(\,\cdot\,)(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}F_2(\,\cdot\,)$, employing results of [26, Section 5.4] in the case $n \geq 3$. In the second part of this chapter we derive trace ideal properties of $\langle\,\cdot\,\rangle^{-\delta}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\,\cdot\,\rangle^{-\delta}$ in the case $n \geq 2$ by a different approach based on a combination of Sobolev's inequality and complex interpolation. These two approaches are independent and complement each other.

The considerations (5.21)–(5.27) readily imply the following facts:

Lemma 6.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and $F, H \in [L^2(\mathbb{R}^n)]^{N\times N}$. Introducing*

$$R_{0,>,F,H}(z;x,y) = F(x)G_{0,>}(z;x-y)H(y), \quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n, \tag{6.1}$$

the integral operator $R_{0,>,F,H}(z)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,>,F,H}(z;\cdot,\cdot)$ satisfies

$$R_{0,>,F,H}(z) \in \mathcal{B}_2\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}, \tag{6.2}$$

and $R_{0,>,F,H}(\,\cdot\,)$ is continuous on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_2([L^2(\mathbb{R}^n)]^N)}$-norm.

In particular, this applies to F, H satisfying for some constant $C \in (0,\infty)$,

$$|F_{j,k}|, |H_{j,k}| \leq C\langle\,\cdot\,\rangle^{-\delta}, \quad \delta > n/2,\ 1 \leq j,k \leq N.$$

Proof. We apply Theorem A.2 (iii) and Lemma A.4.

Let $F, H \in [L^2(\mathbb{R}^n)]^{N\times N}$ and $z \in \overline{\mathbb{C}_+}$ be fixed. To prove (6.2), it suffices to show

$$\big\|R_{0,>,F,H}(z;\cdot,\cdot)\big\|_{\mathcal{B}_2(\mathbb{C}^N)} \in L^2(\mathbb{R}^{2n}; d^n x\, d^n y) \tag{6.3}$$

and apply [27, Theorem 11.6] (in the special case $L^2(\mathbb{R}^n \times \mathbb{R}^n; d^n x\, d^n y)$). To prove (6.3) we recall

$$\|D\|_{\mathcal{B}_2(\mathbb{C}^n)} \leq N^{1/2}\|D\|_{\mathcal{B}(\mathbb{C}^N)}, \quad D \in \mathbb{C}^{N\times N}. \tag{6.4}$$

Then by (5.27) and (6.3),

$$\begin{aligned} &\big\|R_{0,>,F,H}(z;x,y)\big\|_{\mathcal{B}_2(\mathbb{C}^N)} \\ &\quad\leq N^{1/2}\big\|F(x)\big\|_{\mathcal{B}(\mathbb{C}^N)}\big\|G_{0,>}(z;x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)}\big\|H(y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\quad\leq C(z)\big\|F(x)\big\|_{\mathcal{B}(\mathbb{C}^N)}\big\|H(y)\big\|_{\mathcal{B}(\mathbb{C}^N)}, \quad x,y \in \mathbb{R}^n, \end{aligned} \tag{6.5}$$

for an appropriate constant $C(z) > 0$. Since by hypothesis $F, H \in [L^2(\mathbb{R}^n)]^{N\times N}$, and hence,

$$\|F(\cdot)\|^2_{\mathcal{B}(\mathbb{C}^N)} \leq \|F(\cdot)\|^2_{\mathcal{B}_2(\mathbb{C}^N)} = \sum_{j,k=1}^{N} |F_{j,k}(\cdot)|^2 \in L^1(\mathbb{R}^n),$$

and analogously for H, the estimate in (6.5) implies (6.3).

To prove the continuity claim, let $z, z' \in \overline{\mathbb{C}_+}$. One computes (cf. [27, Theorem 11.6])

$$\begin{aligned}
&\|R_{0,>,F,H}(z) - R_{0,>,F,H}(z')\|^2_{\mathcal{B}_2([L^2(\mathbb{R}^n)]^N)} \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} d^n x\, d^n y\, \|R_{0,>,F,H}(z; x, y) - R_{0,>,F,H}(z'; x, y)\|^2_{\mathcal{B}_2(\mathbb{C}^N)} \\
&\leq N \int_{\mathbb{R}^n\times\mathbb{R}^n} d^n x\, d^n y\, \|F(x)\|^2_{\mathcal{B}(\mathbb{C}^N)} \|G_{0,>}(z; x-y) - G_{0,>}(z'; x-y)\|^2_{\mathcal{B}(\mathbb{C}^N)} \\
&\qquad \times \|H(y)\|^2_{\mathcal{B}(\mathbb{C}^N)}.
\end{aligned} \tag{6.6}$$

An application of Lebesgue's dominated convergence theorem, making use of (5.27), $F, H \in [L^2(\mathbb{R}^n)]^{N\times N}$, and the continuity of $G_{0,>}(z; x-y)$ with respect to $(z, x-y)$ in $\mathcal{B}(\mathbb{C}^N)$ (see (5.28)), then yields

$$\lim_{\substack{z \to z' \\ z,z' \in \overline{\mathbb{C}_+}}} \|R_{0,>,F,H}(z) - R_{0,>,F,H}(z')\|_{\mathcal{B}_2([L^2(\mathbb{R}^n)]^N)} = 0. \qquad \blacksquare$$

To improve upon Lemma 6.1, we now recall the following version of Sobolev's inequality (see, e.g., [157, Corollary I.14]).

Theorem 6.2. *Let $n \in \mathbb{N}$, $\lambda \in (0, n)$, $r, s \in (1, \infty)$, $r^{-1} + s^{-1} + \lambda n^{-1} = 2$, $f \in L^r(\mathbb{R}^n)$, $h \in L^s(\mathbb{R}^n)$. Then, there exists $C_{r,s,\lambda,n} \in (0, \infty)$ such that*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} d^n x\, d^n y\, \frac{|f(x)||h(y)|}{|x-y|^\lambda} \leq C_{r,s,\lambda,n} \|f\|_{L^r(\mathbb{R}^n)} \|h\|_{L^s(\mathbb{R}^n)}. \tag{6.7}$$

For subsequent purposes, we also recall some basic facts on L^p-properties of Riesz potentials (see, e.g., [160, Section V.1]):

Theorem 6.3. *Let $n \in \mathbb{N}$, $\alpha \in (0, n)$, and introduce the Riesz potential operator $\mathcal{R}_{\alpha,n}$ as follows:*

$$\begin{aligned}
&(\mathcal{R}_{\alpha,n} f)(x) = \big((-\Delta)^{-\alpha/2} f\big)(x) = \gamma(\alpha, n)^{-1} \int_{\mathbb{R}^n} d^n y\, |x-y|^{\alpha-n} f(y), \\
&\gamma(\alpha, n) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2)/\Gamma\big((n-\alpha)/2\big),
\end{aligned} \tag{6.8}$$

for appropriate functions f (see below).

(i) *Let* $p \in [1,\infty)$ *and* $f \in L^p(\mathbb{R}^n)$. *Then the integral* $(\mathcal{R}_{\alpha,n} f)(x)$ *converges for (Lebesgue) a.e.* $x \in \mathbb{R}^n$.

(ii) *Let* $1 < p < q < \infty$, $q^{-1} = p^{-1} - \alpha n^{-1}$, *and* $f \in L^p(\mathbb{R}^n)$. *Then there exists* $C_{p,q,\alpha,n} \in (0,\infty)$ *such that*

$$\|\mathcal{R}_{\alpha,n} f\|_{L^q(\mathbb{R}^n)} \leq C_{p,q,\alpha,n} \|f\|_{L^p(\mathbb{R}^n)}. \tag{6.9}$$

We also note the β function-type integral (cf. [160, p. 118]),

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n y\, |e_k - y|^{\alpha-n} |y|^{\beta-n} = \gamma(\alpha,n)\gamma(\beta,n)/\gamma(\alpha+\beta,n), \\ &\qquad\qquad\qquad 0 < \alpha < n,\ 0 < \beta < n,\ \alpha+\beta < n, \\ &e_k = (0,\dots,\underbrace{1}_{k},\dots,0),\ 1 \leq k \leq n. \end{aligned} \tag{6.10}$$

and the Riesz composition formula (see [53, Sections 3.1 and 3.2]),

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n y\, |x_1 - y|^{\alpha-n} |y - x_2|^{\beta-n} \\ &\quad = \big[\gamma(\alpha,n)\gamma(\beta,n)/\gamma(\alpha+\beta,n)\big] |x_1 - x_2|^{\alpha+\beta-n}, \\ &\quad 0 < \alpha < n,\ 0 < \beta < n,\ \alpha+\beta < n,\ x_1, x_2 \in \mathbb{R}^n. \end{aligned} \tag{6.11}$$

For later use in Chapter 10, we recall the following estimate taken from [63, Lemma 6.3].

Lemma 6.4. *Let* $n \in \mathbb{N}$ *and* $x_1, x_2 \in \mathbb{R}^n$. *If* $\alpha, \beta \in (0,n]$, $\varepsilon, \gamma \in (0,\infty)$, *with* $n + \gamma \geq \alpha + \beta$, *and* $\alpha + \beta \neq n$, *then*

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n y\, |x_1 - y|^{\alpha-n} \langle y \rangle^{-\gamma-\varepsilon} |y - x_2|^{\beta-n} \\ &\quad \leq C_{n,\alpha,\beta,\gamma,\varepsilon} \begin{cases} |x_1 - x_2|^{-\max\{0,n-\alpha-\beta\}}, & |x_1 - x_2| \leq 1, \\ |x_1 - x_2|^{-\min\{n-\alpha,n-\beta,n+\gamma-\alpha-\beta\}}, & |x_1 - x_2| \geq 1, \end{cases} \end{aligned}$$

where $C_{n,\alpha,\beta,\gamma,\varepsilon} \in (0,\infty)$ *is an* x_1, x_2*-independent constant.*

Returning to $G_{0,>}(z;\,\cdot)$, we next combine the estimate (5.26) with Theorem 6.2, rather than just using the L^∞-bound (5.27) on $G_{0,>}(z;\,\cdot)$ in Lemma 6.1, yielding a considerable improvement of Lemma 6.1.

Theorem 6.5. *Let* $n \in \mathbb{N}$, $n \geq 2$.

(i) *Let* $z = 0$ *and* $F, H \in [L^{4n/(n+\varepsilon)}(\mathbb{R}^n)]^{N\times N}$ *for some* $\varepsilon > 0$. *Introducing the integral operator* $R_{0,>,F,H}(0)$ *in* $[L^2(\mathbb{R}^n)]^N$ *with integral kernel* $R_{0,>,F,H}(0;\,\cdot\,,\cdot)$ *as in* (6.1), *then*

$$R_{0,>,F,H}(0) \in \mathcal{B}_2\big([L^2(\mathbb{R}^n)]^N\big). \tag{6.12}$$

In particular, this applies to F, H satisfying for some constant $C \in (0,\infty)$,

$$|F_{j,k}|, |H_{j,k}| \leq C\langle\cdot\rangle^{-\delta}, \quad \delta > n/4,\ 1 \leq j,k \leq N.$$

(ii) *Let $z \in \overline{\mathbb{C}_+}$ and $F, H \in [L^{4n/(n+1)}(\mathbb{R}^n)]^{N\times N}$. Introducing the integral operator $R_{0,>,F,H}(z)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,>,F,H}(z;\cdot,\cdot)$ as in (6.1), then*

$$R_{0,>,F,H}(z) \in \mathcal{B}_2\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}, \tag{6.13}$$

and $R_{0,>,F,H}(\cdot)$ is continuous on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_2([L^2(\mathbb{R}^n)]^N)}$-norm.

In particular, this applies to F, H satisfying for some constant $C \in (0,\infty)$,

$$|F_{j,k}|, |H_{j,k}| \leq C\langle\cdot\rangle^{-\delta}, \quad \delta > (n+1)/4,\ 1 \leq j,k \leq N.$$

Proof. Again, we apply Theorem A.2 (iii) and Lemma A.4.

If $z = 0$, then $R_{0,>,F,H}(0;\cdot,\cdot)$ generates a Hilbert–Schmidt operator in $L^2(\mathbb{R}^n)$ upon applying the following modified $z = 0$ part in estimate (5.26),

$$\begin{aligned}
&\big|G_{0,>}(z;x-y)_{j,k}\big| \leq c_{n,\varepsilon}|x-y|^{-(n-\varepsilon)/2},\\
&\quad x,y \in \mathbb{R}^n,\ |x-y| \geq 1,\ 1 \leq j,k \leq N,
\end{aligned}$$

for some constants $c_{n,\varepsilon} \in (0,\infty)$, combined with Sobolev's inequality in the form

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} d^n x\, d^n y\, \frac{|f(x)|^2|h(y)|^2}{|x-y|^{n-\varepsilon}}\chi_{[1,\infty)}\big(|x-y|\big)\\
&\quad \leq C_{n,\varepsilon}\|f^2\|_{L^{2n/(n+\varepsilon)}(\mathbb{R}^n)}\|h^2\|_{L^{2n/(n+\varepsilon)}(\mathbb{R}^n)},
\end{aligned}$$

identifying $r = s = 2n/(n+\varepsilon)$, $\lambda = n - \varepsilon$ in (6.7). One verifies that

$$\langle\cdot\rangle^{-\delta} \in L^{4n/(n+\varepsilon)}(\mathbb{R}^n)$$

if $\delta > (n+\varepsilon)/4$, and, since $\varepsilon > 0$ can be chosen arbitrarily small, if $\delta > n/4$.

The general case $z \in \overline{\mathbb{C}_+}$ follows along the same lines using the modified estimate (5.26),

$$\begin{aligned}
&\big|G_{0,>}(z;x-y)_{j,k}\big| \leq c_n|x-y|^{-(n-1)/2},\\
&\quad x,y \in \mathbb{R}^n,\ |x-y| \geq 1,\ 1 \leq j,k \leq N,
\end{aligned}$$

for some constant $c_n \in (0,\infty)$, again combined with Sobolev's inequality in the form

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} d^n x\, d^n y\, \frac{|f(x)|^2|h(y)|^2}{|x-y|^{n-1}}\chi_{[1,\infty)}\big(|x-y|\big)\\
&\quad \leq C_n\|f^2\|_{L^{2n/(n+1)}(\mathbb{R}^n)}\|h^2\|_{L^{2n/(n+1)}(\mathbb{R}^n)},
\end{aligned} \tag{6.14}$$

identifying $r = s = 2n/(n+1)$, $\lambda = n - 1$ in (6.7). One verifies that $\langle\cdot\rangle^{-\delta} \in L^{4n/(n+1)}(\mathbb{R}^n)$ if $\delta > (n+1)/4$.

Finally, continuity of $R_{0,>,F,H}(\cdot)$ on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_2([L^2(\mathbb{R}^n)]^N)}$-norm follows again by applying Lebesgue's dominated convergence theorem as in the proof of Lemma 6.1. ∎

We recall the following interesting results of McOwen [120] and Nirenberg–Walker [125], which provide necessary and sufficient conditions for the boundedness of certain classes of integral operators in $L^p(\mathbb{R}^n)$:

Theorem 6.6. *Let $n \in \mathbb{N}$, $c, d \in \mathbb{R}$, $c + d > 0$, $p \in (1, \infty)$, and $p' = p/(p-1)$. Then the following items* (i) *and* (ii) *hold.*

(i) *Consider*

$$K_{c,d}(x,y) = |x|^{-c}|x-y|^{(c+d)-n}|y|^{-d}, \quad x, y \in \mathbb{R}^n,\ x \neq x', \tag{6.15}$$

then the integral operator $K_{c,d}$ in $L^p(\mathbb{R}^n)$ with integral kernel $K_{c,d}(\cdot\,,\cdot)$ in (6.15) *is bounded if and only if $c < n/p$ and $d < n/p'$.*

(ii) *Consider*

$$\widetilde{K}_{c,d}(x,y) = \big(1+|x|\big)^{-c}|x-y|^{(c+d)-n}\big(1+|y|\big)^{-d}, \quad x, y \in \mathbb{R}^n,\ x \neq x', \tag{6.16}$$

then the integral operator $\widetilde{K}_{c,d}$ in $L^p(\mathbb{R}^n)$ with integral kernel $\widetilde{K}_{c,d}(\cdot\,,\cdot)$ in (6.16) *is bounded if and only if $c < n/p$ and $d < n/p'$.*

This result implies the following fact.

Theorem 6.7. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(i) *Then the integral operator $R_{0,\delta}$ in $[L^2(\mathbb{R}^n)]^N$ with associated integral kernel $R_{0,\delta}(\cdot\,,\cdot)$ bounded entrywise by*

$$\big|R_{0,\delta}(\cdot\,,\cdot)_{j,k}\big| \leq C\langle\cdot\rangle^{-\delta}\big|G_0(0;\cdot\,,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta}, \quad \delta \geq 1/2,\ 1 \leq j,k \leq N,$$

for some $C \in (0,\infty)$, is bounded,

$$R_{0,\delta} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big). \tag{6.17}$$

(ii) *The integral operator $R_{0,\delta}(z)$ in $[L^2(\mathbb{R}^n)]^N$, with associated integral kernel $R_{0,\delta}(z;\cdot\,,\cdot)$ bounded entrywise by*

$$\begin{aligned}\big|R_{0,\delta}(z;\cdot\,,\cdot)_{j,k}\big| &\leq C\langle\cdot\rangle^{-\delta}|G_0(z;\cdot\,,\cdot)_{j,k}|\langle\cdot\rangle^{-\delta},\\ &\delta \geq (n+1)/4,\ z \in \overline{\mathbb{C}_+},\ 1 \leq j,k \leq N,\end{aligned}$$

for some $C \in (0,\infty)$, is bounded,

$$R_{0,\delta}(z) \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}. \tag{6.18}$$

(iii) *The integral operator* $R_{0,\alpha,\beta}(z)$ *in* $[L^2(\mathbb{R}^n)]^N$*, with associated integral kernel* $R_{0,\alpha,\beta}(z;\,\cdot\,,\,\cdot\,)$ *bounded entrywise by*

$$\begin{aligned}\big|R_{0,\alpha,\beta}(z;\,\cdot\,,\,\cdot\,)_{j,k}\big| \leq C\langle\,\cdot\,\rangle^{-\alpha}\big|G_0(z;\,\cdot\,,\,\cdot\,)_{j,k}\big|\langle\,\cdot\,\rangle^{-\beta},&\\ \alpha \geq (n-1)/2,\ \beta \geq 1,\ z \in \overline{\mathbb{C}_+},\ 1 \leq j,k \leq N,&\end{aligned}$$

for some $C \in (0,\infty)$*, is bounded,*

$$R_{0,\alpha,\beta}(z) \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}. \tag{6.19}$$

Proof. (i) The inclusion (6.17) is then an immediate consequence of (5.10) and hence the estimate $|G_0(0;x,y)_{j,k}| \leq C|x-y|^{1-n}$, $x,y \in \mathbb{R}^n$, $x \neq y$, $1 \leq j,k \leq N$, Theorem 6.6, choosing $c = d = 1/2$ in (6.15), and an application of Theorem A.2 (i) and Lemma A.4.

(ii) To prove the inclusion (6.18) we employ the estimates (B.9)–(B.11) (cf. also (5.20)) to obtain

$$\begin{aligned}&\big|G_0(z;x,y)_{j,k}\big|\\ &\quad \leq C(z)|x-y|^{1-n}\chi_{[0,1]}\big(|x-y|\big) + D(z)|x-y|^{(1-n)/2}\chi_{[1,\infty)}\big(|x-y|\big),\\ &\qquad\qquad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y,\ 1 \leq j,k \leq N,\end{aligned} \tag{6.20}$$

for some $C, D(z) \in (0,\infty)$, and apply Theorems 6.6 (parts (i) or (ii)) and A.2 (i) (cf. also Lemma A.4) to both terms on the right-hand sides of (6.20). The part $0 \leq |x-y| \leq 1$ in (6.20) leads to $\delta \geq 1/2$, whereas the part $|x-y| \geq 1$ in (6.20) yields $\delta \geq (n+1)/4$, implying (6.18).

(iii) Again we employ the estimate (6.20) and argue as in item (ii) for the part where $|x-y| \leq 1$. For the part $|x-y| \geq 1$ in (6.20) one employs Theorem 6.6 with $c = \alpha \geq (n-1)/2$ and $d = \beta \geq 1$. ■

Given the fact (6.13), we will now focus on $G_{0,<}(z;\,\cdot\,)$, $z \in \overline{\mathbb{C}_+}$. We begin by recalling that $a(-i\nabla)$ is a convolution-type operator of the form,

$$\begin{aligned}\big(a(-i\nabla)\varphi\big)(x) &:= \big(\big(\mathcal{F}^{-1}a\big) * \varphi\big)(x)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} d^n y\, a^{\vee}(x-y)\varphi(y), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),\end{aligned} \tag{6.21}$$

given $a \in \mathcal{S}'(\mathbb{R}^n)$, $n \in \mathbb{N}$. We are particularly interested in operators of the type

$$b(Q)a(-i\nabla),$$

with Q abbreviating the operator of multiplication by the independent variable x, such that $b(Q)a(-i\nabla)$ extends to a bounded, actually, compact operator in $L^2(\mathbb{R}^n)$, in fact, we will focus on its membership in certain Schatten–von Neumann classes. The prime result we will employ from [26, Section 5.4] in this context can be formulated as follows:

Theorem 6.8 ([26, p. 103, Section 5.4]). *Let $2 < r < s$, and suppose that $a, \psi \in L^r_{\text{weak}}(\mathbb{R}^n)$, $\psi > 0$, $\|\psi\|_{L^r_{\text{weak}}(\mathbb{R}^n)} \leq 1$, and let b be a measurable function such that $b/\psi \in L^s_{\text{weak}}(\mathbb{R}^n; \psi^r d^n x)$. Then*

$$b(Q)a(-i\nabla) \in \mathcal{B}_s(L^2(\mathbb{R}^n)),$$

and for some constant $C(r,s) \in (0,\infty)$,

$$\begin{aligned}&\|b(Q)a(-i\nabla)\|_{\mathcal{B}_s(L^2(\mathbb{R}^n))}\\&\quad\leq C(r,s)\|a\|_{L^r_{\text{weak}}(\mathbb{R}^n)}\bigg(\int_{\mathbb{R}^n} d^n x\, b(x)^s \psi(x)^{r-s}\bigg)^{1/s}.\end{aligned}$$

(*See, e.g.,* [26, Section 1] *for the notion of $L^r_{\text{weak}}(\cdot)$.*)

Next, we recall (with $n \in \mathbb{N}$, $n \geq 2$) that

$$\begin{aligned}&|\cdot|^{\gamma-n} \in L^{n/(n-\gamma)}_{\text{weak}}(\mathbb{R}^n), \quad 0<\gamma<n,\\&\big((|\cdot|^{\gamma-n})^\wedge\big)(\xi) = c_n|\xi|^{-\gamma}, \quad (|\cdot|^{\gamma-n})^\wedge \in L^{n/\gamma}_{\text{weak}}(\mathbb{R}^n), \quad 0<\gamma<n,\\&G_{0,<}(z;\cdot)_{j,k} = |\cdot|^{1-n} f_n(z,\cdot)_{j,k}\chi_{[0,1]}(|\cdot|) \in L^{n/(n-1)}_{\text{weak}}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),\\&\hspace{6cm} 1 \leq j,k \leq N,\ p \in (0, n/(n-1)),\end{aligned}$$

where $f_n(z,\cdot)$ and $G_{0,<}(z;\cdot)$ are defined by (5.22) and (5.24), respectively. In addition, we recall the Hausdorff–Young inequality and its weak analog (cf., e.g., [139, p. 32]),

$$\|f^\wedge\|_{L^{p/(p-1)}(\mathbb{R}^n)} \leq D_{p,n}\|f\|_{L^p(\mathbb{R}^n)}, \qquad p \in [1,2], \tag{6.22}$$

$$\|f^\wedge\|_{L^{p/(p-1)}_{\text{weak}}(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p_{\text{weak}}(\mathbb{R}^n)}, \quad p \in (1,2), \tag{6.23}$$

noting that $p/(p-1) \in (2,\infty)$ if $p \in (1,2)$. In particular, since $p = n/(n-1) \in (1,2)$ for $n \geq 3$,

$$\begin{aligned}&[G_{0,<}(z;\cdot)_{j,k}]^\wedge = [|\cdot|^{1-n} f_n(z,\cdot)_{j,k}\chi_{[0,1]}(|\cdot|)]^\wedge \in L^n_{\text{weak}}(\mathbb{R}^n),\\&\hspace{6cm} 1 \leq j,k \leq N,\ n \in \mathbb{N},\ n \geq 3.\end{aligned} \tag{6.24}$$

Thus, an application of Theorem 6.8 and yields the following result.

Theorem 6.9. *Let $n \in \mathbb{N}$, $n \geq 3$, $\varepsilon > 0$, and assume that for some $q \in (n,\infty)$, $F \in [L^q(\mathbb{R}^n; (1+|x|)^{(q-n)(1+\varepsilon)} d^n x)]^{N\times N}$. Then,*

$$F(Q)G_{0,<}(z; -i\nabla)^\wedge \in \mathcal{B}_q\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}, \tag{6.25}$$

and

$$\begin{aligned} &\big\| F(Q)_{j,\ell}\big[G_{0,<}(z;-i\nabla)_{\ell,k}\big]^{\wedge}\big\|_{\mathcal{B}_q(L^2(\mathbb{R}^n))} \\ &\leq C_{\ell,k}(n,q)\big\|\big[G_{0,<}(z,\cdot)_{\ell,k}\big]^{\wedge}\big\|_{L^n_{\text{weak}}(\mathbb{R}^n)}\bigg(\int_{\mathbb{R}^n}(1+|x|)^{(q-n)(1+\varepsilon)}d^nx\,\big|F(x)_{j,\ell}\big|^q\bigg)^{1/q}, \\ &\hspace{8cm} 1 \leq j,k,\ell \leq N. \quad (6.26) \end{aligned}$$

In addition, the operator $F(Q)G_{0,<}(z;-i\nabla)^{\wedge}$ is continuous with respect to $z \in \overline{\mathbb{C}_+}$ in the $\mathcal{B}_q\big([L^2(\mathbb{R}^n)]^N\big)$-norm.

Proof. Pick $k, \ell \in \{1, \dots, N\}$. Introducing $\psi(x) = c(1+|x|)^{-1-\varepsilon}$, $x \in \mathbb{R}^n$, $c > 0$, one infers that $\psi \in L^n(\mathbb{R}^n)$ and choosing $c = \|(1+|\cdot|)^{-1-\varepsilon}\|^{-1}_{L^n_{\text{weak}}(\mathbb{R}^n)}$ yields $\|\psi\|_{L^n_{\text{weak}}(\mathbb{R}^n)} \leq 1$. Identifying $a^{\vee}$ in Theorem 6.8 and

$$G_{0,<}(z;\cdot)_{\ell,k} \in L^{n/(n-1)}_{\text{weak}}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \quad p \in \big(0, n/(n-1)\big),$$

the inclusion (6.24) yields

$$a = (a^{\vee})^{\wedge} = \big[G_{0,<}(z;\cdot)_{\ell,k}\big]^{\wedge} \in L^n_{\text{weak}}(\mathbb{R}^n).$$

Identifying b in Theorem 6.8 with $F_{j,\ell} \in L^q(\mathbb{R}^n;(1+|x|)^{(q-n)(1+\varepsilon)}d^nx)$, r with $n \geq 3$, and s with $q > n$, one verifies that $F_{j,\ell}/\psi \in L^n(\mathbb{R}^n;\psi^n d^nx)$ and all hypotheses of Theorem 6.8 are satisfied. Hence, the inclusion (6.25) and the estimate (6.26) hold.

Continuity of $F(Q)G_{0,<}(z;-i\nabla)^{\wedge}$ with respect to $z \in \overline{\mathbb{C}_+}$ in the $\mathcal{B}_q([L^2(\mathbb{R}^n)]^N)$-norm follows from the estimate (6.26) (replacing $G_{0,<}(z;-i\nabla)$ by $G_{0,<}(z;-i\nabla) - G_{0,<}(z';-i\nabla)$), the explicit structure of $G_{0,<}(z;\cdot)$ in (5.21), (5.23), and the continuity of $f_n(\cdot,\cdot)$, combined with the weak Hausdorff–Young inequality (6.23) and the fact that $L^q(\mathbb{R}^n;d\rho) \subset L^q_{\text{weak}}(\mathbb{R}^n;d\rho)$ with $\|g\|_{L^q_{\text{weak}}(\mathbb{R}^n;d\rho)} \leq \|g\|_{L^q(\mathbb{R}^n;d\rho)}$, $g \in L^q(\mathbb{R}^n;d\rho)$, $q \in (0,\infty)$. Indeed, with $C_n \in (0,\infty)$ some universal constant,

$$\begin{aligned} &\big\|\big[G_{0,<}(z;\cdot)_{\ell,k}\big]^{\wedge} - [G_{0,<}(z';\cdot)_{\ell,k}]^{\wedge}\big\|_{L^n_{\text{weak}}(\mathbb{R}^n)} \\ &\quad \leq C_n\|G_{0,<}(z;\cdot)_{\ell,k} - G_{0,<}(z';\cdot)_{\ell,k}\|_{L^{n/(n-1)}_{\text{weak}}(\mathbb{R}^n)} \\ &\quad \leq C_n\|G_{0,<}(z;\cdot)_{\ell,k} - G_{0,<}(z';\cdot)_{\ell,k}\|_{L^{n/(n-1)}(\mathbb{R}^n)} \underset{\substack{z \to z' \\ z,z' \in \overline{\mathbb{C}_+}}}{\longrightarrow} 0, \end{aligned}$$

applying the dominated convergence theorem. ∎

A combination of Theorems 6.5 and 6.9 then yields the first principal result of this chapter, which strengthens a part of Theorem 3.4 (see (3.27)) and shows that the Birman–Schwinger operators $\overline{V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}$ are continuous in the closed upper half-plane in an appropriate Schatten norm, provided that V_j, $j = 1,2$, satisfy appropriate boundedness and decay hypotheses.

Theorem 6.10. *Let $n \in \mathbb{N}$, $n \geq 3$, $\varepsilon > 0$, and suppose that*

$$F_1 \in \big[L^q\big(\mathbb{R}^n; \big(1+|x|\big)^{(q-n)(1+\varepsilon)} d^n x\big)\big]^{N\times N} \quad \textit{for some } q \in (n,\infty),$$

and

$$F_\ell \in [L^{4n/(n+1)}(\mathbb{R}^n)]^{N\times N} \cap [L^\infty(\mathbb{R}^n)]^{N\times N}, \quad \ell = 1,2.$$

Introducing

$$R_{0,F_1,F_2}(z,x,y) = F_1(x)G_0(z;x,y)F_2(y), \quad z \in \overline{\mathbb{C}_+},\ x,y \in \mathbb{R}^n,\ x \neq y, \tag{6.27}$$

the integral operator $R_{0,F_1,F_2}(z)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,F_1,F_2}(z,\cdot,\cdot)$ satisfies

$$R_{0,F_1,F_2}(z) \in \mathcal{B}_q\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}, \tag{6.28}$$

and $R_{0,F_1,F_2}(\cdot)$ is continuous on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_q([L^2(\mathbb{R}^n)]^N)}$-norm.

In particular, this applies to F_ℓ, $\ell = 1,2$, satisfying for some constant $C \in (0,\infty)$,

$$|F_{\ell,j,k}| \leq C\langle\cdot\rangle^{-\delta}, \quad \delta > (n+1)/4,\ 1 \leq j,k \leq N,\ \ell = 1,2.$$

Proof. Recalling the decomposition (5.23),

$$G_0(z;x,y) = G_{0,<}(z;x-y) + G_{0,>}(z;x-y), \quad x,y \in \mathbb{R}^n,\ x \neq y, \tag{6.29}$$

(now employed for $n \in \mathbb{N}$, $n \geq 3$), one applies Theorem 6.5 to $G_{0,>}(z;\cdot)$ and Theorem 6.9 to $G_{0,<}(z;\cdot)$.

One readily verifies that if $\delta > (n+1)/4$, then $\langle\cdot\rangle^{-\delta} I_N$ satisfies the conditions assumed on F_ℓ, $\ell = 1,2$, ∎

This handles the case $n \geq 3$. Due to the condition $s > r > 2$ (in the underlying concrete case, $r = n$) in Theorem 6.8, the special case $n = 2$ in connection with $G_{0,<}(z;\cdot)$ does not subordinate to these techniques and hence will be treated using an alternative approach next (which actually applies to all dimensions $n \geq 2$). While Theorem 6.10 only handles the case $n \geq 3$, it has the advantage that it yields continuity of $R_{0,F_1,F_2}(\cdot)$ on $\overline{\mathbb{C}_+}$ (and hence, particularly along the real axis) in a straightforward manner.

To describe an alternative approach to this circle of ideas, we start with some preparatory material on the following trace ideal interpolation result, see, for instance, [85, Theorem III.13.1], [186, Theorem 0.2.6] (see also [80], [86, Theorem III.5.1]).

Theorem 6.11. *Let $p_j \in [1,\infty) \cup \{\infty\}$, $\Sigma = \{\zeta \in \mathbb{C} \mid \mathrm{Re}(\zeta) \in (\xi_1,\xi_2)\}$, $\xi_j \in \mathbb{R}$, $\xi_1 < \xi_2$, $j = 1,2$. Suppose that $A(\zeta) \in \mathcal{B}(\mathcal{H})$, $\zeta \in \overline{\Sigma}$ and that $A(\cdot)$ is analytic on Σ, continuous up to $\partial\Sigma$, and that $\|A(\cdot)\|_{\mathcal{B}(\mathcal{H})}$ is bounded on $\overline{\Sigma}$. Assume that for some $C_j \in (0,\infty)$,*

$$\sup_{\eta\in\mathbb{R}} \big\|A(\xi_j + i\eta)\big\|_{\mathcal{B}_{p_j}(\mathcal{H})} \leq C_j, \quad j = 1,2. \tag{6.30}$$

Then

$$A(\zeta) \in \mathcal{B}_{p(\mathrm{Re}(\zeta))}(\mathcal{H}), \quad \frac{1}{p\big(\mathrm{Re}(\zeta)\big)} = \frac{1}{p_1} + \frac{\mathrm{Re}(\zeta) - \xi_1}{\xi_2 - \xi_1}\left[\frac{1}{p_2} - \frac{1}{p_1}\right], \quad \zeta \in \overline{\Sigma}, \tag{6.31}$$

and

$$\|A(\zeta)\|_{\mathcal{B}_{p(\mathrm{Re}(\zeta))}(\mathcal{H})} \leq C_1^{(\xi_2 - \mathrm{Re}(\zeta))/(\xi_2-\xi_1)} C_2^{(\mathrm{Re}(\zeta)-\xi_1)/(\xi_2-\xi_1)}, \quad \zeta \in \overline{\Sigma}. \tag{6.32}$$

In case $p_j = \infty$, $\mathcal{B}_\infty(\mathcal{H})$ can be replaced by $\mathcal{B}(\mathcal{H})$.

A combination of Theorems 6.6, 6.2 and 6.11 then yields the following fact (cf. [83]).

Theorem 6.12. *Let $n \in \mathbb{N}$, $n \geq 2$, $0 < 2\gamma < n$, $\delta > \gamma$, and suppose that $T_{\gamma,\delta}$ is an integral operator in $L^2(\mathbb{R}^n)$ whose integral kernel $T_{\gamma,\delta}(\cdot\,,\cdot)$ satisfies the estimate*

$$\big|T_{\gamma,\delta}(x,y)\big| \leq C\langle x\rangle^{-\delta}|x-y|^{2\gamma-n}\langle y\rangle^{-\delta}, \quad x,y \in \mathbb{R}^n,\ x \neq y$$

for some $C \in (0,\infty)$. Then,

$$T_{\gamma,\delta} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n)\big), \quad p > n/(2\gamma),\ p \geq 2, \tag{6.33}$$

and

$$\begin{aligned}
&\|T_{\gamma,\delta}\|_{\mathcal{B}_{n/(2\gamma-\varepsilon)}(L^2(\mathbb{R}^n))} \\
&\quad \leq \sup_{\eta\in\mathbb{R}} \Big[\big\|T_{\gamma,\delta}(-2\gamma+\varepsilon+i\eta)\big\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\Big]^{2[-2\gamma+(n/2)+\varepsilon]/n} \\
&\qquad \times \sup_{\eta\in\mathbb{R}} \Big[\big\|T_{\gamma,\delta}(-2\gamma+(n/2)+\varepsilon+i\eta)\big\|_{\mathcal{B}_2(L^2(\mathbb{R}^n))}\Big]^{2(2\gamma-\varepsilon)/n}
\end{aligned} \tag{6.34}$$

for $0 < \varepsilon$ sufficiently small.

Proof. Following the idea behind Yafaev's proof of [186, Lemma 0.13.4], we introduce the analytic family of integral operators $T_{\gamma,\delta}(\cdot)$ in $L^2(\mathbb{R}^n)$ generated by the integral kernel

$$T_{\gamma,\delta}(\zeta; x,y) = T_{\gamma,\delta}(x,y)\,\langle x\rangle^{-(\zeta/2)}|x-y|^{\zeta}\langle y\rangle^{-(\zeta/2)}, \quad x,y \in \mathbb{R}^n,\ x \neq y,$$

noting $T_{\gamma,\delta}(0) = T_{\gamma,\delta}$.

By Theorems 6.6 (ii) and A.2 (i) (for $N = 1$),

$$T_{\gamma,\delta}(\zeta) \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big), \quad 0 < \mathrm{Re}(\zeta) + 2\gamma < n,\ \delta \geq \gamma.$$

To check the Hilbert–Schmidt property of $T_{\gamma,\delta}(\cdot)$ one estimates for the square of $|T_{\gamma,\delta}(\cdot\,;\cdot\,,\cdot)|$,

$$\begin{aligned}
\big|T_{\gamma,\delta}(\zeta; x,y)\big|^2 \leq \langle x\rangle^{-2\delta-\mathrm{Re}(\zeta)}|x-y|^{2\,\mathrm{Re}(\zeta)+4\gamma-2n}\langle y\rangle^{-2\delta-\mathrm{Re}(\zeta)},& \\
x,y \in \mathbb{R}^n,\ x \neq y,&
\end{aligned}$$

and hence one can apply Theorem 6.2 upon identifying $\lambda = 2n - 4\gamma - 2\,\mathrm{Re}(z)$, $r = s = n/[\mathrm{Re}(\zeta) + 2\gamma]$, and $f = h = \langle\cdot\rangle^{-[2\delta+\mathrm{Re}(\zeta)]}$, to verify that $0 < \lambda < n$ translates into $n/2 < \mathrm{Re}(\zeta) + 2\gamma < n$, and $f \in L^r(\mathbb{R}^n)$ holds with $r \in (1,2)$ if $\delta > \gamma$. Hence,

$$T_{\gamma,\delta}(\zeta) \in \mathcal{B}_2\big(L^2(\mathbb{R}^n)\big), \quad n/2 < \mathrm{Re}(\zeta) + 2\gamma < n, \ \delta > \gamma.$$

It remains to interpolate between the $\mathcal{B}\big(L^2(\mathbb{R}^n)\big)$ and $\mathcal{B}_2\big(L^2(\mathbb{R}^n)\big)$ property, employing Theorem 6.11 as follows. Choosing $0 < \varepsilon$ sufficiently small, one identifies $\xi_1 = -2\gamma + \varepsilon$, $\xi_2 = -2\gamma + (n/2) + \varepsilon$, $p_1 = \infty$, $p_2 = 2$, and hence obtains

$$p\big(\mathrm{Re}(\zeta)\big) = n/\big[\,\mathrm{Re}(\zeta) + 2\gamma - \varepsilon\big], \tag{6.35}$$

in particular, $p(0) > n/(2\gamma)$ (and of course, $p(0) \geq 2$). Since ε may be taken arbitrarily small, (6.33) follows from (6.35) and (6.34) is a direct consequence of (6.32). ∎

One notes that while subordination in general only applies to $\mathcal{B}_p$-ideals with p even (see the discussion in [159, p. 24 and Addendum E]), the use of complex interpolation in Theorem 6.12 (and the focus on bounded and Hilbert–Schmidt operators) permits one to avoid this restriction.

Combining Theorems 6.2, 6.6 (ii), 6.11, and 6.12 then yields the second principal result of this chapter.

Theorem 6.13. *Let $n \in \mathbb{N}$, $n \geq 2$. Then the integral operator $R_{0,\delta}$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,\delta}(\cdot\,,\cdot)$ permitting the entrywise bound*

$$\big|R_{0,\delta}(\cdot\,,\cdot)_{j,k}\big| \leq C\langle\cdot\rangle^{-\delta}\big|G_0(0+i\,0;\cdot\,,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta}, \quad \delta > 1/2, \ 1 \leq j,k \leq N,$$

for some $C \in (0,\infty)$, satisfies

$$R_{0,\delta} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > n. \tag{6.36}$$

In a similar fashion, the integral operator $R_{0,\delta}(z)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,\delta}(z;\cdot\,,\cdot)$ permitting the entrywise bound

$$\begin{aligned}\big|R_{0,\delta}(z;\cdot\,,\cdot)_{j,k}\big| &\leq C\langle\cdot\rangle^{-\delta}\big|G_0(z;\cdot\,,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta},\\ &\quad z \in \overline{\mathbb{C}_+}, \ \delta > (n+1)/4, \ 1 \leq j,k \leq N,\end{aligned}$$

for some $C \in (0,\infty)$, satisfies

$$R_{0,\delta}(z) \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > n, \ z \in \overline{\mathbb{C}_+}. \tag{6.37}$$

Proof. We will apply the fact (A.5).

The inclusion (6.36) is immediate from (5.10) (employing the elementary estimate $|G_0(0;x,y)_{j,k}| \leq C|x-y|^{1-n}$, $x,y \in \mathbb{R}^n$, $x \neq y$, $1 \leq j,k \leq N$) and Theorem 6.12 (with $\gamma = 1/2$).

To prove the inclusion (6.37) we again employ the estimate (6.20). An application of Theorem 6.12 to both terms in (6.20), then yields for the part where $0 \le |x-y| \le 1$ that $\gamma = 1/2$ and hence $\delta > 1/2$ and $p > n$. Similarly, for the part where $|x-y| \ge 1$ one infers $\gamma = (n+1)/4$ and hence $\delta > (n+1)/4$ and $p > 2n/(n+1)$, $p \ge 2$, and thus one concludes $\delta > (n+1)/4$ and $p > n$. ∎

Remark 6.14. Continuity of $R_{0,\delta}(\cdot)$ on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_p([L^2(\mathbb{R}^n)]^N)}$-norm, $p > n$, appears to be more difficult to prove within this complex interpolation approach. In this context the first approach described in this chapter is by far simpler to apply, but in turn it is restricted to the case $n \ge 3$. In fact, as recorded in Theorem 6.10, if $\delta > (n+1)/4$, then $\langle\cdot\rangle^{-\delta} I_N$ satisfies the conditions assumed on F_ℓ, $\ell = 1, 2$, in Theorem 6.10, implying the fact,

$$\text{For } n \ge 3,\ \delta > (n+1)/4,\ R_{0,\delta}(\cdot) \text{ is continuous on } \overline{\mathbb{C}_+}$$
$$\text{with respect to the } \|\cdot\|_{\mathcal{B}_p([L^2(\mathbb{R}^n)]^N)}\text{-norm, } p > n.$$

Fortunately, the remaining case $n = 2$ can easily be handled directly as we demonstrate next. ◇

Corollary 6.15. *Let $n = 2$, $\delta > 3/4$, and $z_0 \in \overline{\mathbb{C}_+}$. Then $R_{0,\delta}(\cdot)$, as introduced in Theorem 6.13, satisfies*

$$\big[R_{0,\delta}(z_1) - R_{0,\delta}(z_2)\big] \in \mathcal{B}_2\big([L^2(\mathbb{R}^2)]^N\big), \quad z_j \in \overline{\mathbb{C}_+},\ j = 1, 2, \tag{6.38}$$

and

$$\lim_{\substack{z \to z_0 \\ z \in \overline{\mathbb{C}_+}\backslash\{z_0\}}} \big\|R_{0,\delta}(z) - R_{0,\delta}(z_0)\big\|_{\mathcal{B}_2([L^2(\mathbb{R}^2)]^N)} = 0.$$

Proof. Once more we will apply the fact (A.5) (for $p = 2$).

By Theorem 6.5 (ii) it suffices to focus on $G_{0,<}(z;\,\cdot)$. The explicit formula (see (C.23), (C.24)),

$$\begin{aligned} G_0(z;x,y) = {} & i4^{-1} z\, H_0^{(1)}\big(z|x-y|\big) I_N \\ & - 4^{-1}|x-y|^{-1}\big[z|x-y|\big] H_1^{(1)}\big(z|x-y|\big)\, \alpha \cdot \frac{(x-y)}{|x-y|}, \\ & z \in \overline{\mathbb{C}_+},\ x, y \in \mathbb{R}^2,\ x \ne y, \end{aligned} \tag{6.39}$$

together with the $z \to 0$, $z \in \overline{\mathbb{C}_+}\backslash\{0\}$ limit (C.25), then permit the following conclusions: Only if $z \to 0$ ($z \in \mathbb{C}_+\backslash\{0\}$) and/or if $|x-y| \to 0$, can $G_0(z;x,y)$ develop a singularity which then is of the form $\ln(z|x-y|)$ and $|x-y|^{-1}$. (In all other circumstances G_0 is continuous on $\mathbb{C}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$.) However, the $|x-y|^{-1}$-singularity is z-independent and hence drops out in differences of the form $R_{0,\delta}(z_1) - R_{0,\delta}(z_2)$, $z_j \in \overline{\mathbb{C}_+}$, $j = 1, 2$. Thus one can safely ignore the $|x-y|^{-1}$-singularity. Consequently, this only leaves the $\ln(z_j|x-y|)$-singularity, $j = 1,2$, when considering $G_{0,<}(z_1;x,y) - G_{0,<}(z_2;x,y)$. This then yields the estimate (see also (5.21) and (5.22)),

$$\begin{aligned}
&\big|G_{0,<}(z;x,y)_{j,k} - G_{0,<}(z_0;x,y)_{j,k}\big| \\
&\quad \leq \begin{cases} C\big[|z|+|z_0|\big]\big|\ln\big(|x-y|\big)\big| + D(z,z_0), & z,z_0 \in \overline{\mathbb{C}_+}\backslash\{0\}, \\ C|z|\big|\ln\big(|x-y|\big)\big| + D(z), & z\in\overline{\mathbb{C}_+}\backslash\{0\},\ z_0=0, \end{cases} \\
&\qquad\qquad x,y\in\mathbb{R}^2,\ 0<|x-y|\leq 1,\ 1\leq j,k\leq N,
\end{aligned} \tag{6.40}$$

with $C \in (0,\infty)$, and $D(\cdot\,,z_0), D(\cdot) \in (0,\infty)$ continuous and locally bounded on $\overline{\mathbb{C}_+}$. The logarithmic-type integral kernel in (6.39) can now be handled as in [186, Proposition 7.1.17] (upon multiplying $R_0(z)$ by a factor of z if $z_0 = 0$, and choosing $|z| = 1$ in equation (7.1.25) in [186, p. 272]) if $z_0 \in \overline{\mathbb{C}_+}\backslash\{0\}$, implying the asserted Hilbert–Schmidt property. Alternatively, one can use the very rough estimate (for some $c_0 \in (0,\infty)$)

$$\big|\ln\big(|x-y|\big)\big|^2 \leq c_0|x-y|^{-1}, \quad 0<|x-y|\leq 1,$$

and apply the Sobolev inequality in the form of (6.7) with $n = 2$, $\lambda = 1$, $r = s = 4/3$, recalling that $\langle\cdot\rangle^{-2\delta} \in L^{4/3}(\mathbb{R}^2)$ if $\delta > 3/4$. ■

Combining Theorems 6.10, 6.13, Remark 6.14, and Corollary 6.15, we finally summarize the principal results of this chapter as follows:

Theorem 6.16. *Let $n \in \mathbb{N}$, $n \geq 2$ and consider the integral operator $R_{0,\delta}(z)$ in $[L^2(\mathbb{R}^n)]^N$ with integral kernel $R_{0,\delta}(z;\cdot\,,\cdot)$ permitting the entrywise bound*

$$\begin{aligned}
&\big|R_{0,\delta}(z;\cdot\,,\cdot)_{j,k}\big| \leq C\langle\cdot\rangle^{-\delta}\big|G_0(z;\cdot\,,\cdot)_{j,k}\big|\langle\cdot\rangle^{-\delta}, \\
&\qquad z\in\overline{\mathbb{C}_+},\ \delta>(n+1)/4,\ 1\leq j,k\leq N,
\end{aligned}$$

for some $C \in (0,\infty)$. Then $R_{0,\delta}(z)$ satisfies

$$R_{0,\delta}(z) \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p>n,\ z\in\overline{\mathbb{C}_+}. \tag{6.41}$$

Moreover, if $n \geq 3$, $\delta > (n+1)/4$, then $R_{0,\delta}(\cdot)$ is continuous on $\overline{\mathbb{C}_+}$ with respect to the $\|\cdot\|_{\mathcal{B}_p([L^2(\mathbb{R}^n)]^N)}$-norm for $p > n$. Finally, if $n = 2$, $\delta > 3/4$, then

$$\big[R_{0,\delta}(z_1) - R_{0,\delta}(z_2)\big] \in \mathcal{B}_2\big([L^2(\mathbb{R}^2)]^N\big), \quad z_j\in\overline{\mathbb{C}_+},\ j=1,2, \tag{6.42}$$

and

$$\lim_{\substack{z\to z_0 \\ z\in\overline{\mathbb{C}_+}\backslash\{z_0\}}} \big\|R_{0,\delta}(z) - R_{0,\delta}(z_0)\big\|_{\mathcal{B}_2([L^2(\mathbb{R}^2)]^N)} = 0. \tag{6.43}$$