Chapter 7

Powers of resolvents and trace ideals

We now introduce the following considerably strengthened set of assumptions on the short-range potential V:

Hypothesis 7.1. *Let $n \in \mathbb{N}$ and suppose that V satisfies for some constant $C \in (0,\infty)$ and $\rho \in (n,\infty)$,*

$$V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \quad \left|V_{\ell,\ell'}(x)\right| \le C\langle x\rangle^{-\rho} \quad \textit{for a.e. } x \in \mathbb{R}^n,\ 1 \le \ell,\ell' \le N.$$

Given Hypothesis 7.1, the principal purpose of this chapter is to prove that for $k \ge n$,

$$\left[(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} - (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\right] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big),$$
$$z \in \mathbb{C}\backslash\mathbb{R}. \qquad (7.1)$$

Here $H = H_0 + V$ is defined according to (3.4), but we do not assume self-adjointness of the $N \times N$ matrix V in this chapter.

The following arguments are straightforward generalizations of the arguments in [185] in the three-dimensional context $n = 3$. We start with a study of H_0:

Lemma 7.2. *Let $r \in (0,\infty)$, $k \in \mathbb{N}$, and define $p(r,k) := n/\min\{r,k\}$. If $p > p(r,k)$, $p \ge 1$, then*

$$\langle\cdot\rangle^{-r}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

In particular, choosing $r = n+\varepsilon$ for some $\varepsilon > 0$, $k = n+1$, then $p(n+\varepsilon, n+1) < 1$, and hence

$$\langle\cdot\rangle^{-n-\varepsilon}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-n-1} \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R}. \qquad (7.2)$$

Proof. Since

$$\begin{aligned}
&\big(H_0 - zI_{[L_2(\mathbb{R}^n)]^N}\big)^{-k} \\
&\quad = \big(H_0 + zI_{[L_2(\mathbb{R}^n)]^N}\big)^k\big(H_0^2 - z^2 I_{[L_2(\mathbb{R}^n)]^N}\big)^{-k} \\
&\quad = \big(h_0 - z^2 I_{L^2(\mathbb{R}^n)}\big)^{-k/2} I_N \big(H_0 + zI_{[L_2(\mathbb{R}^n)]^N}\big)^k\big(H_0^2 - z^2 I_{[L_2(\mathbb{R}^n)]^N}\big)^{-k/2}, \\
&\qquad\qquad z \in \mathbb{C}\backslash\mathbb{R},
\end{aligned}$$

and the operator $(H_0 + zI_{[L_2(\mathbb{R}^n)]^N})^k\big(H_0^2 - z^2 I_{[L_2(\mathbb{R}^n)]^N}\big)^{-k/2}$ is bounded, it is sufficient to prove the assertion for the operator $\langle\cdot\rangle^{-r}(h_0 - z^2 I_{L^2(\mathbb{R}^n)})^{-k/2}$. The latter follows from [186, p. 145, Lemma 4.3]. ∎

Turning from H_0 to $H = H_0 + V$ then yields the following result.

Lemma 7.3. *Assume that* $V \in [L^\infty(\mathbb{R}^n)]^{N\times N}$*, let* $r \in (0,\infty)$*,* $k \in \mathbb{N}$*, and define* $p(r,k) := n/\min\{r,k\}$*. If* $p > p(r,k)$*,* $p \geq 1$*, then*

$$\langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{7.3}$$

In particular, choosing $r = n+\varepsilon$ *for some* $\varepsilon > 0$*,* $k = n+1$*, then* $p(n+\varepsilon, n+1) < 1$*, and hence*

$$\langle\,\cdot\,\rangle^{-n-\varepsilon}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-n-1} \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Proof. Let $z \in \mathbb{C}\backslash\mathbb{R}$ and $r \in (0,\infty)$. The proof employs induction on $k \in \mathbb{N}$. In the base case, $k = 1$, one writes

$$\begin{aligned}&\langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\\ &=\big[\langle\,\cdot\,\rangle^{-r}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big]\big[(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big].\end{aligned} \tag{7.4}$$

The first factor on the right-hand side in (7.4) belongs to $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ for $p > p(r,1)$, $p \geq 1$ by Lemma 7.2. Since the second factor on the right-hand side in (7.4) is a bounded operator, (7.3) holds with $k = 1$.

Suppose that (7.3) holds for $k \in \mathbb{N}$. Multiplying throughout the commutator identity

$$(H - zI_{[L^2(\mathbb{R}^n)]^N})\langle\,\cdot\,\rangle^{-r} - \langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N}) = \big[H_0, \langle\,\cdot\,\rangle^{-r}\big]$$

from the left by $(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}$ and right by $(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k-1}$, one obtains

$$\begin{aligned}&\langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k-1}\\ &\quad = (H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\\ &\qquad + (H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big[H_0, \langle\,\cdot\,\rangle^{-r}\big](H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k-1}.\end{aligned} \tag{7.5}$$

One has

$$\begin{aligned}&(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\,\cdot\,\rangle^{-r}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\\ &=\big[(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\cdot\rangle^{-r(k+1)^{-1}}\big]\big[\langle\cdot\rangle^{-kr(k+1)^{-1}}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\big].\end{aligned} \tag{7.6}$$

Now, by the base case,

$$\begin{aligned}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\langle\,\cdot\,\rangle^{-r(k+1)^{-1}} &\in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big),\\ &\quad p > p\big(r(k+1)^{-1}, 1\big),\end{aligned}$$

and by the induction step

$$\langle\,\cdot\,\rangle^{-kr(k+1)^{-1}}\big(H - zI_{[L^2(\mathbb{R}^n)]^N}\big)^{-k} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big),$$
$$p > p\big(kr(k+1)^{-1}, k\big).$$

Therefore, the product on the right-hand side in (7.6) belongs to the trace ideal $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ for $p \geq 1$, with

$$p^{-1} < p\big(r(k+1)^{-1}, 1\big)^{-1} + p\big(kr(k+1)^{-1}, k\big)^{-1}. \tag{7.7}$$

To compute the right-hand side of (7.7), one distinguishes the two possible cases: (i) $r(k+1)^{-1} < 1$ or (ii) $r(k+1)^{-1} \geq 1$.

In case (i), $r < k+1$, and

$$p\big(kr(k+1)^{-1}, k\big) = n(k+1)r^{-1},$$
$$p\big(kr(k+1)^{-1}, k\big)^{-1} = n(k+1)r^{-1}k^{-1}.$$

Hence, the right-hand side of (7.7) equals

$$n^{-1}(k+1)^{-1}r + n^{-1}(k+1)^{-1}kr$$
$$= n^{-1}r = n^{-1}\min\{r, k+1\} = p(r, k+1)^{-1},$$

so the right-hand side in (7.6) belongs to $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ for all indices $p > p(r, k+1)$.

In case (ii), $r \geq k+1$, and

$$p\big(kr(k+1)^{-1}, k\big) = n, \quad p\big(kr(k+1)^{-1}, k\big)^{-1} = nk^{-1}.$$

Hence the right-hand side of (7.7) equals

$$n^{-1}(k+1) = p(r, k+1)^{-1},$$

so the right-hand side of (7.6), and hence the first term on the right-hand side in (7.5), belongs to $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ for all indices $p > p(r, k+1)$. To treat the second term in (7.5), one uses

$$\big[H_0, \langle\,\cdot\,\rangle^{-r}\big] = V_0, \tag{7.8}$$

where

$$V_0(x) = -r\langle x\rangle^{-(r+2)}(\alpha\cdot x), \quad x \in \mathbb{R}^n,$$

so that

$$\|V_0(x)\|_{\mathcal{B}(\mathbb{C}^N)} \leq C\langle x\rangle^{-(r+1)}, \quad x \in \mathbb{R}^n, \tag{7.9}$$

for an x-independent constant $C > 0$. Thus, the second term on the right-hand side in (7.5) belongs to $\mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big)$ for all indices $p > p(r, k+1)$ by the same argument used to treat the first term. ■

Given these preparations, the principal result of this chapter reads as follows.

Theorem 7.4. *Let $k \in \mathbb{N}$ with $k \geq n$ and suppose that V satisfies Hypothesis 7.1. Then*

$$\big[(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} - (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-k}\big] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{7.10}$$

Proof. Let $k \geq n$. By the second resolvent equation,

$$\begin{aligned}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} &- (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} \\ &= -(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} V (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}.\end{aligned} \tag{7.11}$$

Differentiation of (7.11) with respect to z yields

$$\begin{aligned}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} &- (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-k} \\ &= -\sum_{j=1}^{k} (H - zI_{[L^2(\mathbb{R}^n)]^N})^{-j} V (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{j-k-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}.\end{aligned} \tag{7.12}$$

From this point on, let $z \in \mathbb{C}\backslash\mathbb{R}$ be fixed and write

$$\begin{aligned}&(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-j} V (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{j-k-1} \\ &\quad = \big[(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-j} \langle x\rangle^{-j\rho(k+1)^{-1}}\big]\big[\langle x\rangle^{\rho} V\big] \\ &\qquad \times \big[\langle x\rangle^{-(k+1-j)\rho(k+1)^{-1}} (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{j-k-1}\big], \\ &\qquad\qquad j \in \mathbb{N},\ 1 \leq j \leq k.\end{aligned} \tag{7.13}$$

By Lemma 7.3, for a fixed $j \in \mathbb{N}$, $1 \leq j \leq k$,

$$(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-j} \langle x\rangle^{-j\rho(k+1)^{-1}} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > p\big(j\rho(k+1)^{-1}, j\big),$$

and by Lemma 7.2,

$$\langle x\rangle^{-(k+1-j)\rho(k+1)^{-1}} (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{j-k-1} \in \mathcal{B}_p\big([L^2(\mathbb{R}^n)]^N\big), \quad p > p\big((k+1-j)\rho(k+1)^{-1}, k+1-j\big).$$

One distinguishes the two possible cases: (i) $\rho(k+1)^{-1} < 1$, or (ii) $\rho(k+1)^{-1} \geq 1$.

In case (i) with $\rho(k+1)^{-1} < 1$, one computes

$$\begin{aligned} p\big((k+1-j)\rho(k+1)^{-1}, k+1-j\big) &= \frac{n}{(k+1-j)\rho(k+1)^{-1}}, \\ p\big(j\rho(k+1)^{-1}, j\big) &= \frac{n}{j\rho(k+1)^{-1}},\end{aligned}$$

so that

$$p\big((k+1-j)\rho(k+1)^{-1}, k+1-j\big)^{-1} + p\big(j\rho(k+1)^{-1}, j\big)^{-1} = \rho/n > 1.$$

Hence, the right-hand side of (7.13) belongs to $\mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big)$.

In case (ii) with $\rho(k+1)^{-1} \geq 1$, one computes

$$\begin{aligned} p\big((k+1-j)\rho(k+1)^{-1}, k+1-j\big) &= n/(k+1-j), \\ p\big(j\rho(k+1)^{-1}, j\big) &= n/j, \end{aligned}$$

so that

$$\begin{aligned} &p\big((k+1-j)\rho(k+1)^{-1}, k+1-j\big)^{-1} + p\big(j\rho(k+1)^{-1}, j\big)^{-1} \\ &\quad = (k+1)/n = (k/n) + (1/n) \geq 1 + (1/n) > 1. \end{aligned}$$

Hence, the right-hand side of (7.13) belongs to $\mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big)$.

In either case, the right-hand side of (7.13) belongs to $\mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big)$. Since $j \in \mathbb{N}$, $1 \leq j \leq k$, was arbitrary, it follows that every term in the summation on the right-hand side in (7.12) belongs to $\mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big)$, and then (7.10) follows from the vector space properties of the trace class. ■

We conclude this chapter by recalling a well-known result:

Lemma 7.5. *Suppose $p > n/\min(\tau, 2\kappa)$, $p \geq 1$, with $\tau > 0$, $\kappa > 0$. Then*

$$\langle\,\cdot\,\rangle^{-\tau}(h_0 + I_{L^2(\mathbb{R}^n)})^{-\kappa} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n)\big). \tag{7.14}$$

In particular, if V satisfies Hypothesis 7.1 and $\kappa > n/2$,

$$V(H_0^2 + I_{[L^2(\mathbb{R}^n)]^N})^{-\kappa} \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big). \tag{7.15}$$

Proof. While (7.14) is a special case of [186, Proposition 3.1.5 and Lemma 3.4.3] (see also [83], [159, Chapter 4]), (7.15) follows from combining (3.5) and (7.14). ■