Chapter 10

Analysis of $\mathrm{Im}(F_{H,H_0}(\lambda + i0))$, $\lambda \in \mathbb{R}$

The principal purpose of this chapter is to analyze continuity properties of the function $\mathrm{Im}(F_{H,H_0}(\lambda + i0))$, $\lambda \in \mathbb{R}$.

One recalls (see Lemma 9.4) that

$$F_{H,H_0}(z) = \ln\big(\det_{[L^2(\mathbb{R}^n)]^N, n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big)\big), \quad z \in \mathbb{C}_+.$$

Using a factorization $V = V_1^* V_2$ (see Hypothesis 10.5 for the details of the factorization) and elementary properties of regularized determinants, the analysis of the function $F_{H,H_0}(z)$ reduces to an analysis of

$$\ln\big(\det_{[L^2(\mathbb{R}^n)]^N, n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L_2(\mathbb{R}^n)]^N})^{-1} V_1^*\big)\big), \quad z \in \mathbb{C}_+.$$

Theorem 6.16 then guarantees that the Birman–Schwinger-type operator

$$V_2(H_0 - zI_{[L_2(\mathbb{R}^n)]^N})^{-1} V_1^*, \quad z \in \mathbb{C}_+,$$

extends to a continuous $\mathcal{B}_{n+1}\big([L^2(\mathbb{R}^n)]^N\big)$-valued function for z in the closed upper-half plane $\overline{\mathbb{C}_+}$, provided that V_1 and V_2 are decaying sufficiently fast. In particular, the boundary values of the regularized Fredholm determinant,

$$\det_{[L^2(\mathbb{R}^n)]^N, n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (\lambda + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}\big),$$

exist and are continuous for all $\lambda \in \mathbb{R}$. This means that the function $F_{H,H_0}(\lambda + i0)$ has normal boundary values and is continuous at any point λ in $\mathbb{R}$ such that

$$\det_{[L^2(\mathbb{R}^n)]^N, n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (\lambda + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big) \neq 0.$$

By Theorem 3.4, the latter holds for $\lambda \in \mathbb{R}\backslash\{0\}$ if and only if λ is not an eigenvalue of H. By [103] one can exclude nonzero eigenvalues by assuming (4.2) and (4.3) (see Theorem 4.1). In particular, under these assumptions the function $\mathrm{Im}(F_{H,H_0}(\lambda + i0))$ is continuous for $\lambda \in \mathbb{R}\backslash\{0\}$. Thus, the only point where the behavior $\mathrm{Im}(F_{H,H_0}(\lambda + i0))$, $\lambda \in \mathbb{R}$, remains to be studied is the "threshold point" $\lambda = 0$, and hence the majority of this chapter is devoted to an analysis of the latter.

We start with a series of well-known preliminary results which we state without proof closely following the general outline in the paper by Jensen and Nenciu [99].

Lemma 10.1. (i) (cf. [99]). *Let A be a densely defined closed operator and P a projection in $\mathcal{H}$. Suppose that $(A+P)^{-1} \in \mathcal{B}(\mathcal{H})$ and denote by $a := P - P(A+P)^{-1}P$ an operator in $P\mathcal{H}$. Then*

$$A^{-1} \in \mathcal{B}(\mathcal{H}) \text{ if and only if } a^{-1} \in \mathcal{B}(P\mathcal{H}). \tag{10.1}$$

In particular, if $a^{-1} \in \mathcal{B}(P\mathcal{H})$ then

$$A^{-1} = (A+P)^{-1} + (A+P)^{-1} P a^{-1} P (A+P)^{-1}.$$

(ii) (cf. [76], [108, Section III.6.5]). *Let A be a densely defined closed operator in $\mathcal{H}$ and $\lambda_0 \in \mathbb{C}$ an isolated point in $\sigma(A)$ with P_{λ_0} the Riesz projection in $\mathcal{H}$ associated with A and λ_0. If the quasi-nilpotent operator associated with A and λ_0 vanishes, that is, $D_0 := (A - \lambda_0 I_{\mathcal{H}})P_{\lambda_0} = 0$, then*

$$(A - \lambda_0 I_{\mathcal{H}} + P_{\lambda_0})^{-1} = P_{\lambda_0} + S_{\lambda_0} \in \mathcal{B}(\mathcal{H}),$$

where

$$S_{\lambda_0} = \operatorname*{n-lim}_{\substack{z \to \lambda_0 \\ z \neq \lambda_0}} (A - zI_{\mathcal{H}})^{-1}[I_{\mathcal{H}} - P_{\lambda_0}] \in \mathcal{B}(\mathcal{H}).$$

(iii) (cf. [138]). *Let A be a compact operator in $\mathcal{H}$ and $\lambda_0 \in \mathbb{C}$ an isolated point in $\sigma(A)$ with P_{λ_0} the Riesz projection in $\mathcal{H}$ associated with A and λ_0. Then*

$$(A - \lambda_0 I_{\mathcal{H}} + P_{\lambda_0})^{-1} \in \mathcal{B}(\mathcal{H}).$$

(iv) (cf. [99]). *Suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathfrak{B}$ in $\mathcal{H}$ is the block operator matrix*

$$\mathfrak{B} = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix},$$

where

$$b_{j,j} \text{ are densely defined, closed operators in } \mathcal{H}_j,\ j = 1,2,$$
$$b_{1,2} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1), \quad b_{2,1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2).$$

In addition, assume that $b_{2,2}^{-1} \in \mathcal{B}(\mathcal{H}_2)$. Then

$$\mathfrak{B}^{-1} \in \mathcal{B}(\mathcal{H}) \text{ if and only if } [b_{1,1} - b_{1,2}\, b_{2,2}^{-1}\, b_{2,1}]^{-1} \in \mathcal{B}(\mathcal{H}_1). \tag{10.2}$$

In particular, abbreviating

$$b := [b_{1,1} - b_{1,2}\, b_{2,2}^{-1}\, b_{2,1}],$$

if $b^{-1} \in \mathcal{B}(\mathcal{H}_1)$, then

$$\mathfrak{B}^{-1} = \begin{pmatrix} b^{-1} & -b^{-1}\, b_{1,2}\, b_{2,2}^{-1} \\ -b_{2,2}^{-1}\, b_{2,1}\, b^{-1} & b_{2,2}^{-1} + b_{2,2}^{-1}\, b_{2,1}\, b^{-1}\, b_{1,2}\, b_{2,2}^{-1} \end{pmatrix}. \tag{10.3}$$

We emphasize that b in Lemma 10.1 (iv) is also known as a Schur complement (see, e.g., [170, Section 1.6]) and formula (10.3) is a variant of the so-called Feshbach formula (see, e.g., [50]). In particular, Lemma 10.1 (iv) is especially useful in the context of two-dimensional Schrödinger operators (cf. [99]) as well as two-dimensional massless Dirac operators (cf. [60]).

Lemma 10.2 ([99]). *Suppose that $\Omega \subset \mathbb{C}$ has zero as an accumulation point. Let $A(\zeta) = A_0 + \zeta A_1(\zeta)$, $\zeta \in \Omega$, be a family of $\mathcal{B}(\mathcal{H})$-valued operators, with $A_1(\cdot)$ uniformly bounded for $\zeta \in \Omega$ sufficiently small. Suppose that 0 is an isolated point in $\sigma(A_0)$ and denote by P_0 the Riesz projection in $\mathcal{H}$ associated with A_0 and 0. If $A_0 P_0 = 0$ (i.e., the quasi-nilpotent operator associated with A_0 and 0 vanishes), then for $\zeta \in \Omega$ sufficiently small, the operator $B(\cdot)$ in $P_0\mathcal{H}$, defined by*

$$\begin{aligned} B(\zeta) &:= \zeta^{-1}\{P_0 - P_0[A(\zeta) + P_0]^{-1} P_0\} \\ &= \sum_{j\in\mathbb{N}_0} (-\zeta)^j P_0[A_1(\zeta)(A_0 + P_0)^{-1}]^{j+1} P_0, \end{aligned}$$

is uniformly bounded as $\zeta \to 0$. Moreover, for $\zeta \in \Omega$ sufficiently small,

$$A(\zeta)^{-1} \in \mathcal{B}(\mathcal{H}) \text{ if and only if } B(\zeta)^{-1} \in \mathcal{B}(P_0\mathcal{H}).$$

In particular, if $B(\zeta)^{-1} \in \mathcal{B}(P_0\mathcal{H})$ for $\zeta \in \Omega$ sufficiently small, then

$$\begin{aligned} A(\zeta)^{-1} &= [A(\zeta) + P_0]^{-1} \\ &\quad + \zeta^{-1}[A(\zeta) + P_0]^{-1} P_0 B(\zeta)^{-1} P_0 [A(\zeta) + P_0]^{-1}. \end{aligned} \tag{10.4}$$

Remark 10.3. A combined application of Lemma 10.1 (iv) and Lemma 10.2 can be realized in the following scenario: Suppose

$$\begin{aligned} &b_{1,1}(\zeta) = \zeta^{-1}[b_0 + \beta(\zeta)], \text{ with } b_0^{-1} \in \mathcal{B}(\mathcal{H}_1) \text{ and } \|\beta(\zeta)\|_{\mathcal{B}(\mathcal{H}_1)} \underset{\substack{\zeta\to 0\\ \zeta\in\Omega}}{=} o(1), \\ &b_{2,2}(\zeta)^{-1} \in \mathcal{B}(\mathcal{H}_2) \text{ is uniformly bounded for } \zeta \in \Omega \text{ sufficiently small,} \\ &b_{1,2}(\zeta) \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1),\ b_{2,1}(\zeta) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \text{ are uniformly bounded} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \zeta \in \Omega \text{ sufficiently small.} \end{aligned}$$

Then

$$b_{1,1}(\zeta)^{-1} = \zeta b_0^{-1}[I_{\mathcal{H}_1} + \beta(\zeta) b_0^{-1}]^{-1}, \quad \|b_{1,1}(\zeta)^{-1}\| \underset{\substack{\zeta\to 0\\ \zeta\in\Omega}}{=} O(\zeta),$$

and under these circumstances one then infers, with

$$b(\zeta) := [b_{1,1}(\zeta) - b_{1,2}(\zeta)\, b_{2,2}^{-1}(\zeta)\, b_{2,1}(\zeta)]$$

(cf. (10.2)), that for $\zeta \in \Omega$ sufficiently small,

$$b(\zeta)^{-1} = b_{1,1}(\zeta)^{-1}\big[I_{\mathcal{H}_1} - b_{1,2}(\zeta)\, b_{2,2}(\zeta)^{-1}\, b_{2,1}(\zeta)\, b_{1,1}(\zeta)^{-1}\big]^{-1}. \qquad \diamond$$

At this point one can summarize the strategy in deriving threshold expansions of resolvents described in Jensen and Nenciu [99] (see also Murata [122]), in fact, in our context, expansions of

$$\begin{aligned} &V_2(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^* \\ &\quad = I_{[L^2(\mathbb{R}^n)]^N} - \big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big]^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned} \tag{10.5}$$

in terms of the (symmetrized) Birman–Schwinger-type operator

$$V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^* \tag{10.6}$$

(cf. Theorem 3.4) around $z = 0$ as follows:

(α) One notes upon combining (B.1)–(B.8) and (5.9) that treating even dimensions n is considerably more involved than the case of odd dimensions n due to the presence of the logarithm[1] in (B.4). At any rate, formulas (B.1)–(B.8) and (5.9) permit one to expand the Birman–Schwinger-type operator (10.6) around $z = 0$ assuming sufficient decay of $V_1^*(x)$, $V_2(x)$ as $|x| \to \infty$. This step is cumbersome, but poses no further difficulties. What might cause difficulties is an expansion of the left-hand side of (10.5), or, equivalently, an expansion of the inverse $[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*]^{-1}$ on the right-hand side of (10.5).

(β) If this inverse exists boundedly in a sufficiently small neighborhood of $z = 0$, that is, if

$$\big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big]^{-1} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big) \tag{10.7}$$

for $|z|$ sufficiently small, then no difficulty arises and a geometric series argument yields the existence of such an expansion in norm (cf. Chapter 5), given sufficient decay of $V_1^*(x)$, $V_2(x)$ as $|x| \to \infty$ also in appropriate trace ideal norms (cf. the detailed treatment in Chapter 6). This is actually the generic case where H has no zero-energy eigenvalue and no zero-energy resonance (the latter is defined as giving

[1]This is even more pronounced in the case of Schrödinger operators for $n = 2$ due to the logarithmic blowup of the Green's function (5.1) as $z \to 0$. Actually, in the Schrödinger context even the one-dimensional case exhibits a $z^{-1/2}$ singularity at $z = 0$, rendering both cases more involved than $n \geq 3$. Since the Dirac Green's matrix never exhibits a blowup as $z \to 0$ in all dimensions $n \in \mathbb{N}$, $n \geq 2$ (cf. (5.10)), this renders the massless Dirac situation technically a bit simpler than the case of one and two-dimensional Schrödinger operators (considered in great detail in [99]).

rise to an eigenvalue -1 of the Birman–Schwinger-type operator (10.6) but with no associated L^2-eigenfunction in the domain of H). At this point all that remains is a computation of the expansion coefficients, but the latter is of limited urgency in our present context as we will primarily rely on the leading order in all expansions.

(γ) If the inverse in (10.7) does not exist boundedly in a sufficiently small neighborhood of $z = 0$, that is, if the compact operator $V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*$ has an eigenvalue -1, the situation changes drastically. In this case H either has an eigenvalue 0, or zero-energy resonances, or possibly both, a zero-energy eigenvalue and zero-energy resonances (all of them possibly degenerate) in the worst case scenario. In any of these (exceptional) situations the norm of

$$\big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big]^{-1}$$

and hence that of

$$V_2(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*$$

will exhibit a singularity as $z \to 0$. Without going into details in this summary (see, however, Theorem 10.14), we note that the blowup in the case of zero-energy eigenvalue(s) is of the order z^{-1}, and in the presence of zero-energy resonances (but no zero-energy eigenvalues) is of a less singular structure, for instance, like $z^{-1}[\ln(z)]^{-1}$, $z^{-1/2}$, or $\ln(z)$, etc., the details now depending crucially on the space dimension $n \in \mathbb{N}$, $n \geq 2$, and whether Schrödinger or Dirac operators (massive or massless) are considered.

But even though $[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*]$ does not possess a bounded inverse as $z \to 0$, the operator

$$\big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^* + P_0\big],$$

where P_0 is the (finite-dimensional) Riesz projection associated with the operator

$$I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*}, \tag{10.8}$$

and its eigenvalue 0, the norm limit of

$$I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - i\varepsilon I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*$$

as $\varepsilon \downarrow 0$, and its eigenvalue 0, actually has a bounded inverse according to Lemma 10.1 (ii). (Assuming compactness of the operators $\overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}$ as well as $V_2(H_0 - i\varepsilon I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*$, one concludes that $\dim(\operatorname{ran}(P_0)) < \infty$.) Lemma 10.2 then demonstrates the key reduction step where the inverse of $A(\zeta)$ in $\mathcal{H}$ is now reduced to the inverse of $B(\zeta)$ in the finite-dimensional Hilbert space $P_0\mathcal{H}$.

(δ) At this point one iterates the procedure ending up localizing the singularity in subspaces of decreasing dimensions. With each step the singularity is increased. However, since

$$z\big[\overline{V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]$$

stays bounded for $z = i\varepsilon$ as $\varepsilon \downarrow 0$, the reduction process must stop after a finite number of steps, leading to invertibility of a reduced operator so that again a geometric series argument as in step (β) applies. This completes the process resulting in an expansion in appropriate variables involving z, $z^{1/2}$, $\ln(z)$, or $[c + \ln(z)]$ for appropriate $c \in \mathbb{C}\backslash\{0\}$ (again, depending on spatial dimension n and whether Schrödinger or Dirac operators are involved). We refer once more to [99] for the somewhat involved details (and the difficulties associated with expansions involving $\sum_{k=-1}^{\infty}\sum_{\ell=-\infty}^{\infty}\zeta^k[\ln(\zeta)]^{\ell}$ which cannot be asymptotic in nature) in the case of Schrödinger operators and to [60] in the case of two-dimensional massless Dirac operators. Much of the threshold analysis in [60] readily extends to dimensions $n \geq 3$ as we will see later in this chapter.

Remark 10.4. In outlining steps (α)–(δ) above, we deliberately sidestepped verifying the condition $A_0 P_0 = 0$ necessary for Lemma 10.2 to hold. The condition is equivalent to the statement that the algebraic and geometric multiplicities of the eigenvalue 0 of A_0 coincide. Since by (5.10), $G_0(0 + i\,0; x, y)$ is purely imaginary, but also involves the scalar product $\alpha \cdot (x - y)$, employing the polar decomposition for the self-adjoint $N \times N$ matrix $V(\cdot)$ (i.e., $V(\cdot) = U_V(\cdot)|V(\cdot)|$) in the form (cf. [81])

$$\begin{aligned} &V(x) = \big|V(x)\big|^{1/2}U_V(x)\big|V(x)\big|^{1/2} = V_1(x)^*V_2(x) \quad \text{ for a.e. } x \in \mathbb{R}^n, \\ &V_1 = V_1^* = |V|^{1/2}, \quad V_2 = U_V|V|^{1/2} = U_V V_1, \quad U_V^2 = I_N, \end{aligned} \tag{10.9}$$

making the choice that

$$U_V \text{ is unitary and self-adjoint} \tag{10.10}$$

(the choice of U_V is nonunique if V has a kernel and we simply fix U_V to be the identity operator on $\ker(V)$), the matrix-valued integral kernel

$$\big|V(x)\big|^{1/2}(x)G_0(0 + i\,0; x, y)\big|V(y)\big|^{1/2}$$

generates a self-adjoint operator. Hence, the elegant device used in [99] that reduces their analysis to a self-adjoint operator A_0 in Lemma 10.2, so that $A_0 P_0 = 0$ is automatically satisfied, applies also in the massless Dirac operator context. (Naturally, this approach of [99] also applies in the massive case, where $H_0(m)$, $m > 0$, has the spectral gap $(-m, m)$.) In essence, Jensen and Nenciu [99] replace the operator

$$I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{10.11}$$

by its modification

$$U_V I_{[L^2(\mathbb{R}^n)]^N} + V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{10.12}$$

and show that the formalism displayed in (2.6)–(2.10) instantly extends to the setup in (10.12). In particular, the norm limit

$$U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1\big(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} \tag{10.13}$$

is now self-adjoint and hence the analog of the condition

$$A_0 P_0 = 0 \tag{10.14}$$

thus holds automatically. Due to this fact we can, without loss of generality, safely disregard the distinction between (10.11) and (10.12) in much of the remainder of this manuscript.

Finally, by an abuse of notation, we also denote the Riesz projection associated with the self-adjoint operator (10.13) and its eigenvalue 0 by P_0. Assuming compactness of the operator

$$\overline{V_1\big(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*}, \tag{10.15}$$

the fact that $\sigma(U_V) \subseteq \{1, -1\}$ implies that zero is an isolated point in the spectrum of the operator in (10.13) and hence

$$\dim\big(\mathrm{ran}(P_0)\big) < \infty. \tag{10.16}$$

(In the concrete context of (10.9) one has in addition that $V_1 = V_1^*$, but this simplification is not needed to conclude (10.14) and (10.16).) ◇

Applying the resolvent equation (2.7), (2.8) to the pair H, H_0 results in

$$\begin{aligned}(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} &= (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} - \big[V_1(H_0 - \overline{z}I_{[L^2(\mathbb{R}^n)]^N})^{-1}\big]^* \\ &\quad \times \big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big]^{-1} V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}, \\ &\qquad z \in \mathbb{C}\backslash\mathbb{R},\end{aligned}$$

To analyze the possible singularity of $(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}$ as $z \to 0$, we choose arbitrary

$$\psi_j \in C_0^\infty(\mathbb{R}^n) \text{ real-valued}, \ j = 1, 2,$$

and consider

$$\begin{aligned}&\psi_2 I_N (H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_1 I_N \\ &\quad = \psi_2 I_N (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_1 I_N \\ &\qquad - \big[V_1(H_0 - \overline{z}I_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_2 I_N\big]^*\end{aligned}$$

$$\times \big[I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*\big]^{-1}$$
$$\times V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} \psi_1 I_N, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

As long as

$$\big||V|_{\ell,\ell'}^{1/2}(x)\big| = \big|V_{1,\ell,\ell'}(x)\big| \leq C\langle x\rangle^{-1} \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N, \tag{10.17}$$

Theorem 6.7 (iii) implies that

$$V_j(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_{j'} I_N \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+},\ j,j' \in \{1,2\},$$

since obviously

$$\psi_2 I_N(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_1 I_N \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+},$$

(in fact, Theorem 6.13 implies trace ideal properties) one also has

$$\psi_2 I_N(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_1 I_N \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big), \quad z \in \overline{\mathbb{C}_+}.$$

Thus, since $\psi_j \in C_0^\infty(\mathbb{R}^n)$, $j = 1,2$, are arbitrary (apart from being real-valued for simplicity), one thus concludes that

$$\begin{aligned}&\psi_2 I_N(H - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\psi_1 I_N \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big) \quad \text{for } |z| \text{ sufficiently small}\\ &\text{if and only if } \big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]^{-1} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big)\\ &\hfill \text{for } |z| \text{ sufficiently small.}\end{aligned}$$

Given the extensive treatment in [99] in the case of Schrödinger operators in dimensions $n \in \mathbb{N}$ (especially, in the most difficult of cases $n = 1,2$), and in [60] in the case of massless Dirac operators in dimension $n = 2$, and given the fact that dimensions $n \in \mathbb{N}$, $n \geq 3$, subordinate in difficulty to the case $n = 2$ in the massless context, we now briefly discuss the threshold (i.e., $z = 0$) behavior of massless Dirac operator in dimensions $n \geq 2$.

We start by making the following assumptions on the matrix-valued potential V.

Hypothesis 10.5. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\varepsilon > 0$. Assume the a.e. self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1\leq \ell,\ell' \leq N}$ satisfies for some fixed $\varepsilon \in (0,1)$, $C \in (0,\infty)$,*

$$\begin{aligned}&V \in [L^\infty(\mathbb{R}^n)]^{N\times N},\\ &\big|V_{\ell,\ell'}(x)\big| \leq C\langle x\rangle^{-2(1+\varepsilon)} \quad \textit{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N.\end{aligned} \tag{10.18}$$

In accordance with the factorization based on the polar decomposition of V discussed in (10.9) we suppose that

$$V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}, \ \textit{where } V_1 = V_1^* = |V|^{1/2},\ V_2 = U_V |V|^{1/2}.$$

We continue with the threshold, that is, the $z = 0$ behavior of H:

Definition 10.6. Assume Hypothesis 10.5 with $\varepsilon = 0$ in (10.18).

(i) The point 0 is called a zero-energy eigenvalue of H if $H\psi = 0$ has a distributional solution ψ satisfying

$$\psi \in \operatorname{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$$

(equivalently, $\ker(H) \supsetneqq \{0\}$).

(ii) The point 0 is called a zero-energy (or threshold) resonance of H if

$$\ker\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big]\big) \supsetneqq \{0\},$$

and if there exists $0 \neq \phi \in \ker([I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}])$ such that ψ defined by

$$\begin{aligned}\psi(x) &= -\big(\big(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^* \phi\big)(x) \\ &= -i2^{-1}\pi^{-n/2}\Gamma(n/2) \int_{\mathbb{R}^n} d^n y\, |x - y|^{-n} \big[\alpha \cdot (x - y)\big] V_1(y)^* \phi(y) \quad (10.19)\end{aligned}$$

(for a.e. $x \in \mathbb{R}^n$, $n \geq 2$) is a distributional solution of $Hu = 0$ satisfying

$$\psi \notin [L^2(\mathbb{R}^n)]^N.$$

(iii) 0 is called a regular point for H if it is neither a zero-energy eigenvalue nor a zero-energy resonance of H.

Additional properties of ψ are isolated in Theorem 10.7.

While the point 0 being regular for H is the generic situation, zero-energy eigenvalues and/or resonances are exceptional cases.

For future purposes we recall the asymptotic Green's matrix expansion as $z \to 0$ in the following form,

$$\begin{aligned}&G_0(z; x, y) \\ &\underset{\substack{z \to 0 \\ z \in \mathbb{C}_+\backslash\{0\}}}{=} i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha \cdot \frac{(x - y)}{|x - y|^n} - \delta_{n,2}(2\pi)^{-1} z \ln(z) I_N \\ &\quad + \delta_{n,2}(2\pi)^{-1}\big[\gamma_{E-M} - i(\pi/2) + \ln\big(|x - y|/2\big)\big] z I_N \\ &\quad + [1 - \delta_{n,2}](n - 2)^{-1} 2^{-1}\pi^{-n/2}\Gamma(n/2)|x - y|^{2-n} z I_N \\ &\quad + \delta_{n,2} O\big(z^2|x - y| \ln(z|x - y|)\big) + \delta_{n,3} O\big(z^2\big) + O\big(z^2|x - y|^2\big) \\ &\underset{\substack{z \to 0 \\ z \in \mathbb{C}_+\backslash\{0\}}}{=} R_{0,0}(x - y) + z R_{1,0}(x - y) \\ &\quad + z\big[-(2\pi)^{-1}\ln(z/2) - (2\pi)^{-1}\gamma_{E-M} + i4^{-1}\big]\delta_{n,2} R_{1,1}(x - y) \\ &\quad + \delta_{n,2} O\big(z^2|x - y| \ln(z|x - y|)\big) + \delta_{n,3} O\big(z^2\big) + O\big(z^2|x - y|^2\big),\end{aligned}$$

where we introduced the following convenient abbreviations (for $x, y \in \mathbb{R}^n$, $x \neq y$):

$$R_{0,0}(x-y) = G_0(0+i0; x, y) = i2^{-1}\pi^{-n/2}\Gamma(n/2)\,\alpha \cdot \frac{(x-y)}{|x-y|^n}$$

$$= \begin{cases} (2\pi)^{-1} i\alpha \cdot \nabla_x \ln\big(|x-y|\big), & n = 2, \\ -i\alpha \cdot \nabla_x g_0(0; x, y), & n \geq 3, \end{cases} \tag{10.20}$$

$$R_{1,0}(x-y) = \begin{cases} -(2\pi)^{-1} \ln\big(|x-y|\big) I_N, & n = 2, \\ g_0(0; x, y) I_N = \frac{1}{(n-2)\omega_{n-1}} |x-y|^{2-n} I_N, & n \geq 3, \end{cases}$$

$$\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2), \tag{10.21}$$

$$R_{1,1}(x-y) = 1, \quad n \geq 2. \tag{10.22}$$

Theorem 10.7. *Assume Hypothesis 10.5 with $\varepsilon = 0$ in (10.18).*

(i) *If $n = 2$, there are precisely four possible cases:*

Case (I)*: 0 is regular for H.*

Case (II)*: 0 is a* (*possibly degenerate*[2]) *resonance of H. In this case the resonance functions ψ satisfy*

$$\psi \in [L^q(\mathbb{R}^2)]^2, \quad q \in (2,\infty) \cup \{\infty\}, \quad \nabla\psi \in [L^2(\mathbb{R}^2)]^{2\times 2},$$
$$\psi \notin [L^2(\mathbb{R}^2)]^2. \tag{10.23}$$

Case (III)*: 0 is a* (*possibly degenerate*) *eigenvalue of H. In this case the corresponding eigenfunctions $\psi \in \mathrm{dom}(H) = [W^{1,2}(\mathbb{R}^2)]^2$ of $H\psi = 0$ also satisfy*

$$\psi \in [L^q(\mathbb{R}^2)]^2, \quad q \in [2,\infty) \cup \{\infty\}. \tag{10.24}$$

Case (IV)*: A possible mixture of Cases* (II) *and* (III).

(ii) *If $n \in \mathbb{N}$, $n \geq 3$, there are precisely two possible cases:*

Case (I)*: 0 is regular for H.*

Case (II)*: 0 is a* (*possibly degenerate*) *eigenvalue of H. In this case, the corresponding eigenfunctions $\psi \in \mathrm{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$ of $H\psi = 0$ also satisfy*

$$\psi \in [L^q(\mathbb{R}^n)]^N, \quad q \in \begin{cases} (3/2,\infty) \cup \{\infty\}, & n = 3, \\ (4/3, 4), & n = 4, \\ \big(2n/(n+2), 2n/(n-2)\big), & n \geq 5. \end{cases} \tag{10.25}$$

In particular, there are no zero-energy resonances of H in dimension $n \geq 3$.

(iii) *The point 0 is regular for H if and only if*

$$\ker\Big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big]\Big) = \{0\}.$$

[2] We will recall in Lemma 10.12 (i) that if $n = 2$, the degeneracy in Case (II) is at most two.

Proof. Since $G_0(0+i0; x, y)$, $x \neq y$, exists for all $n \geq 2$ (cf. (5.10)), the Birman–Schwinger eigenvalue equation (cf. (10.8))

$$\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*}\big]\phi_0 = 0,$$
$$0 \neq \phi_0 \in [L^2(\mathbb{R}^n)]^N, \tag{10.26}$$

gives rise to a distributional zero-energy solution $\psi_0 \in [L^1_{\mathrm{loc}}(\mathbb{R}^n)]^N$ of $H\psi_0 = 0$ in terms of ϕ_0 of the form (for a.e. $x \in \mathbb{R}^n$, $n \geq 2$),

$$\psi_0(x) = -\big(\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*\phi_0\big)(x)$$
$$= -\big[R_{0,0} * (V_1^*\phi_0)\big](x) \tag{10.27}$$
$$= -i2^{-1}\pi^{-n/2}\Gamma(n/2)\int_{\mathbb{R}^n} d^n y\, |x-y|^{-n}\big[\alpha\cdot(x-y)\big]V_1(y)^*\phi_0(y), \tag{10.28}$$
$$= -i2^{-1}\pi^{-n/2}\Gamma(n/2)\int_{\mathbb{R}^n} d^n y\, |x-y|^{-n}\big[\alpha\cdot(x-y)\big]V_1(y)^*V_2(y)\psi_0(y), \tag{10.29}$$
$$\phi_0(x) = (V_2\psi_0)(x). \tag{10.30}$$

In particular, one concludes that $\psi_0 \neq 0$. Thus, one estimates, with $\|V_1(\cdot)\|_{\mathbb{C}^{N\times N}} \leq c\langle\cdot\rangle^{-1}$ and some constant $d_n \in (0,\infty)$,

$$\big\|\psi_0(x)\big\|_{\mathbb{C}^N} \leq d_n \int_{\mathbb{R}^n} d^n y\, |x-y|^{1-n}\langle y\rangle^{-1}\big\|\phi_0(y)\big\|_{\mathbb{C}^N}$$
$$= d_n\mathcal{R}_{1,n}\big(\langle\cdot\rangle^{-1}\big\|\phi_0(\cdot)\big\|_{\mathbb{C}^N}\big)(x), \quad x \in \mathbb{R}^n. \tag{10.31}$$

Invoking the Riesz potential $\mathcal{R}_{1,n}$ (cf. Theorem 6.3), one obtains (for some constant $\tilde{d}_n \in (0,\infty)$)

$$\big\|\psi_0(x)\big\|_{\mathbb{C}^N} \leq d_n \int_{\mathbb{R}^n} d^n y\, |x-y|^{1-n}\langle y\rangle^{-1}\big\|\phi_0(y)\big\|_{\mathbb{C}^N}$$
$$\leq \tilde{d}_n\mathcal{R}_{1,n}\big(\langle\cdot\rangle^{-1}\big\|\phi_0(\cdot)\big\|_{\mathbb{C}^N}\big)(x), \quad x \in \mathbb{R}^n, \tag{10.32}$$

and hence (6.9) implies (for some constant $\widetilde{C}_{p,q,n} \in (0,\infty)$)

$$\begin{aligned}
\|\psi_0\|_{[L^q(\mathbb{R}^n)]^N} &\leq \tilde{d}_n\big\|\mathcal{R}_{1,n}\big(\langle\cdot\rangle^{-1}\|\phi_0(\cdot)\|_{\mathbb{C}^N}\big)\big\|_{L^q(\mathbb{R}^n)}\\
&\leq \widetilde{C}_{p,q,n}\big\|\langle\cdot\rangle^{-1}\|\phi_0(\cdot)\|_{\mathbb{C}^N}\big\|_{L^p(\mathbb{R}^n)}\\
&\leq \widetilde{C}_{p,q,n}\big\|\langle\cdot\rangle^{-1}\big\|_{L^s(\mathbb{R}^n)}\big\|\|\phi_0(\cdot)\|_{\mathbb{C}^N}\big\|_{L^2(\mathbb{R}^n)}\\
&= \widetilde{C}_{p,q,n}\big\|\langle\cdot\rangle^{-1}\big\|_{L^s(\mathbb{R}^n)}\|\phi_0\|_{[L^2(\mathbb{R}^n)]^N},
\end{aligned} \tag{10.33}$$
$$1 < p < q < \infty,\ p^{-1} = q^{-1} + n^{-1},\ s = 2qn[2n+2q-qn]^{-1} \geq 1.$$

In particular,

$$p = qn/(n+q), \quad 2n + 2q - qn > 0.$$

(a) The case $n = 2$: Then one can choose $q \in (2, \infty)$, hence $p = 2q/(q+2) \in (1, 2)$, and $s = q > 2$. Thus, (10.33) and $\|\langle \cdot \rangle^{-1}\|_{L^s(\mathbb{R}^2)} < \infty$ imply

$$\psi_0 \in [L^q(\mathbb{R}^2)]^N, \quad q \in (2, \infty).$$

Recalling $R_{0,0}(x-y)$ in (10.20), this implies

$$-i\alpha \cdot \nabla_x R_{0,0}(x-y) = -\Delta_x g_0(0; x, y) I_N = \delta(x-y) I_N,$$
$$x, y \in \mathbb{R}^n, \ x \neq y, \ n \geq 2, \qquad (10.34)$$

in the sense of distributions. Here we abused notation a bit and denoted also in the case $n = 2$,

$$g_0(0; x, y) = -(2\pi)^{-1} \ln\big(|x-y|\big), \quad x, y \in \mathbb{R}^2, \ x \neq y, \ n = 2. \qquad (10.35)$$

Thus, one obtains

$$\begin{aligned} i\alpha \cdot (\nabla \psi_0)(x) &= -i\alpha \cdot \nabla_x \big[R_{0,0} * (V_1^* \phi_0)\big](x) \\ &= -i\alpha \cdot \nabla_x \big[-i(\alpha \cdot \nabla_x g_0) * (V_1^* \phi_0)\big](x) \\ &= \big[(-\Delta_x g_0 I_N) * (V_1^* \phi_0)\big](x) \\ &= (V_1^* \phi_0)(x) \in [L^2(\mathbb{R}^2)]^2, \end{aligned} \qquad (10.36)$$

proving $\nabla \psi_0 \in [L^2(\mathbb{R}^2)]^{2\times 2}$, upon employing the fact that $[\alpha \cdot p]^2 = I_N |p|^2$, $p \in \mathbb{R}^n$.

To prove that $\psi_0 \in [L^\infty(\mathbb{R}^2)]^2$ in (10.23) and (10.24), one applies (10.30) to the inequality in (10.31), and then employs the condition $\|V_2(\cdot)\|_{\mathbb{C}^{2\times 2}} \leq C\langle \cdot \rangle^{-1}$ for some constant $C \in (0, \infty)$ to obtain

$$\|\psi_0(x)\|_{\mathbb{C}^2} \leq \tilde{d}_2 \int_{\mathbb{R}^2} d^2y \, |x-y|^{-1} \langle y \rangle^{-2} \|\psi_0(y)\|_{\mathbb{C}^2}, \quad x \in \mathbb{R}^2,$$

where $\tilde{d}_2 \in (0, \infty)$ is an appropriate x-independent constant. By Hölder's inequality,

$$\|\psi_0(x)\|_{\mathbb{C}^2} \leq \tilde{d}_2 \bigg(\int_{\mathbb{R}^2} d^2y \, |x-y|^{-3/2} \langle y \rangle^{-3} \bigg)^{2/3} \bigg(\int_{\mathbb{R}^2} d^2y \, \|\psi_0(y)\|_{\mathbb{C}^2}^3 \bigg)^{1/3},$$
$$x \in \mathbb{R}^2. \qquad (10.37)$$

The second integral on the right-hand side in (10.37) is finite since $\psi_0 \in [L^3(\mathbb{R}^2)]^2$. Choosing $x_1 = x$, $\alpha = n - (3/2)$, $\beta = n$, $\gamma = 2$, and $\varepsilon = 1$ in Lemma 6.4, one infers that

$$\int_{\mathbb{R}^2} d^2y \, |x-y|^{-3/2} \langle y \rangle^{-3} \leq C_{2,3/2,0,2,1}, \quad x \in \mathbb{R}^2. \qquad (10.38)$$

Hence, the containment $\psi_0 \in [L^\infty(\mathbb{R}^2)]^2$ follows from (10.37) and (10.38).

(b) The case $n \geq 3$: An application of Theorem 6.6 (ii) with $c = 0$, $d = 1$, $p = p' = 2$, and the inequality $1 < n/2$, combined with $\|\phi_0(\cdot)\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n)$, yield

$$\big\|\psi_0(\cdot)\big\|_{\mathbb{C}^N} \in L^2(\mathbb{R}^n) \quad \text{and hence,} \quad \psi_0 \in [L^2(\mathbb{R}^n)]^N, \quad n \geq 3. \tag{10.39}$$

To prove that actually $\psi_0 \in \mathrm{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$, it suffices to argue as follows:

$$i\alpha \cdot \nabla \psi_0 = -V\psi_0 \in [L^2(\mathbb{R}^n)]^N \tag{10.40}$$

in the sense of distributions since $V \in [L^\infty(\mathbb{R}^n)]^{N\times N}$ and $\psi_0 \in [L^2(\mathbb{R}^n)]^N$. Given the fact $\mathrm{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N$ (cf. (3.2)), one concludes

$$\psi_0 \in [W^{1,2}(\mathbb{R}^n)]^N, \quad n \geq 3. \tag{10.41}$$

By (10.41) we know that $\psi_0 \in [W^{1,2}(\mathbb{R}^n)]^N$. Employing the fact that $\phi_0 = V_2\psi_0$ in the first line of (10.32), one obtains

$$\begin{aligned} \big\|\psi_0(x)\big\|_{\mathbb{C}^N} &\leq \widetilde{D}_n \int_{\mathbb{R}^n} d^n y\, |x-y|^{1-n} \langle y\rangle^{-2} \big\|\psi_0(y)\big\|_{\mathbb{C}^N} \\ &= D_n \mathcal{R}_{1,n}\big(\langle\cdot\rangle^{-2}\big\|\psi_0(\cdot)\big\|_{\mathbb{C}^N}\big)(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{10.42}$$

for some constants $\widetilde{D}_n, D_n \in (0,\infty)$. Thus, as in (10.33), (6.9) implies for $n \geq 3$,

$$\begin{aligned} \|\psi_0\|_{[L^q(\mathbb{R}^n)]^N} &\leq D_n \big\|\mathcal{R}_{1,n}\big(\langle\cdot\rangle^{-2}\|\psi_0(\cdot)\|_{\mathbb{C}^N}\big)\big\|_{L^q(\mathbb{R}^n)} \\ &\leq \widetilde{D}_{p,q,n} \big\|\langle\cdot\rangle^{-2}\|\psi_0(\cdot)\|_{\mathbb{C}^N}\big\|_{L^p(\mathbb{R}^n)} \\ &\leq \widetilde{D}_{p,q,n} \big\|\langle\cdot\rangle^{-2}\big\|_{L^s(\mathbb{R}^n)} \big\|\|\psi_0(\cdot)\|_{\mathbb{C}^N}\big\|_{L^2(\mathbb{R}^n)} \\ &= \widetilde{D}_{p,q,n} \big\|\langle\cdot\rangle^{-2}\big\|_{L^s(\mathbb{R}^n)} \|\psi_0\|_{[L^2(\mathbb{R}^n)]^N}, \\ 1 < p < q < \infty,\ p^{-1} &= q^{-1} + n^{-1},\ s = 2qn[2n+2q-qn]^{-1} \geq 1, \end{aligned} \tag{10.43}$$

for some constant $\widetilde{D}_{p,q,n} \in (0,\infty)$. In particular, one again has $p = qn/(n+q)$ and $2n + 2q - qn > 0$. The latter condition implies $q < 2n/(n-2)$. The requirement $p > 1$ results in $q > n/(n-1)$, and the condition $s \geq 1$ yields $q \geq 2n/(3n-2)$ which, however, is superseded by $q > p > 1$. Moreover, the requirement $\big\|\langle\cdot\rangle^{-2}\big\|_{L^s(\mathbb{R}^n)} < \infty$ yields $q > 2n/(n+2)$. Putting it all together implies (10.25).

To prove the containment $\psi_0 \in [L^\infty(\mathbb{R}^3)]^4$ in (10.25), one invokes the inequality in (10.42) with $n = 3$. Indeed, applying Hölder's inequality (with conjugate exponents $q' = 27/20$ and $q = 27/7$) to the integral on the right-hand side of the inequality in (10.42), one infers that

$$\begin{aligned} &\big\|\psi_0(x)\big\|_{\mathbb{C}^4} \\ &\quad \leq d_3\bigg(\int_{\mathbb{R}^3} d^3y\, |x-y|^{-27/10}\langle y\rangle^{-27/10}\bigg)^{20/27}\bigg(\int_{\mathbb{R}^3} d^3y\, \big\|\psi_0(y)\big\|_{\mathbb{C}^4}^{2/77}\bigg)^{7/27}, \\ &\hspace{20em} x \in \mathbb{R}^3. \end{aligned} \tag{10.44}$$

The second integral in (10.44) is finite since $\psi_0 \in [L^{27/7}(\mathbb{R}^3)]^4$, and the first integral in (10.44) may be estimated by taking $x_1 = x$, $\alpha = n - (27/10)$, $\beta = n$, $\gamma = 2$, and $\varepsilon = 7/10$ in Lemma 6.4,

$$\int_{\mathbb{R}^3} d^3y\, |x-y|^{-\frac{27}{10}} \langle y\rangle^{-\frac{27}{10}} \leq C_{3,\frac{27}{10},0,2,\frac{7}{10}}, \quad x \in \mathbb{R}^3 \backslash \{0\}. \tag{10.45}$$

Hence, the containment $\psi_0 \in [L^{\infty}(\mathbb{R}^3)]^4$ follows from (10.44) and (10.45).

Returning to arbitrary $n \geq 2$, we show (following the proof of [60, Lemma 7.4]) that if $\ker(H) \supsetneqq \{0\}$ then also

$$\ker\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big]\big) \supsetneqq \{0\}.$$

Indeed, if $0 \neq \psi_0 \in \ker(H)$, then $\phi_0 := V_2\psi_0 = U_V V_1 \psi_0 \in [L^2(\mathbb{R}^n)]^N$ and hence $V_1^*\phi_0 \in [L^2(\mathbb{R}^n)]^N$. Then, $H\psi_0 = 0$ yields $i\alpha\cdot\nabla\psi_0 = V\psi_0 = V_1^* V_2 \psi_0 = V_1^*\phi_0$.

Thus, applying (10.20), (10.34)–(10.35) once again, one obtains for all $n \geq 2$,

$$\begin{aligned}
& -i\alpha\cdot\nabla\big[\psi_0 + \big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*\phi_0\big](x) \\
&\quad = -i\,[\alpha\cdot\nabla\psi_0](x) - i\alpha\cdot\nabla_x\big[R_{0,0} * (V_1^*\phi_0)\big](x) \\
&\quad = -i\,[\alpha\cdot\nabla\psi_0](x) - i\alpha\cdot\nabla_x\big[-i(\alpha\cdot\nabla_x g_0) * (V_1^*\phi_0)\big](x) \\
&\quad = -i\,[\alpha\cdot\nabla\psi_0](x) + \big[(-\Delta_x g_0 I_N) * (V_1^*\phi_0)\big](x) \\
&\quad = -i\,[\alpha\cdot\nabla\psi_0](x) + (V_1^*\phi_0)(x) \\
&\quad = -V(x)\psi_0(x) + V(x)\psi_0(x) = 0.
\end{aligned}$$

Consequently,

$$-i\alpha\cdot\nabla\big[\psi_0 + \big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*\phi_0\big] = 0,$$

implying

$$\psi_0 + \big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*\phi_0 = c^{\top}$$

for some $c \in \mathbb{C}^N$. Since $\psi_0 \in [L^2(\mathbb{R}^n)]^N$, and by exactly the same arguments employed in (10.31)–(10.39), also $R_{0,0} * (V_1^*\phi_0) \in [L^2(\mathbb{R}^n)]^N$, one concludes that $c = 0$ and hence

$$\psi_0 = -\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*\phi_0.$$

Thus, $\phi_0 \neq 0$, and

$$\begin{aligned}
0 &= V_2\psi_0 + V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*\phi_0 \\
&= \big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big]\phi_0,
\end{aligned}$$

that is,

$$0 \neq \phi_0 \in \ker\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}\big]\big).$$

This concludes the proof. ∎

Recalling results of [60], we will revisit the basic elements in the proof of item (i) of Theorem 10.7 in Lemma 10.12.

Remark 10.8. (i) In physical notation, the zero-energy resonances in Cases (II) and (IV) for $n = 2$ correspond to eigenvalues $\pm 1/2$ of the spin-orbit operator (cf. the operator S in [103, 106]) when V is spherically symmetric, see the discussion in [60].

(ii) For basics on the Birman–Schwinger principle in an abstract context, especially, if $0 \in \rho(H_0)$, we refer to [79] (cf. also [20, 77]) and the extensive literature cited therein. In the concrete case of Schrödinger operators, relations (10.28), (10.30) are discussed at length in [10, 29, 30, 58, 61, 62, 64, 68, 69, 76, 96–99, 122], [124, Section 10.3.2], [169] (see also the list of references quoted therein), and in [59, 60, 65–67] in the case of (massive and massless) Dirac operators.

(iii) As mentioned in Remark 5.1 (ii), the absence of zero-energy resonances is well known in the three-dimensional case $n = 3$, see [8], [16, Section 4.4], [17, 28, 150, 151, 190]. In fact, for $n = 3$ the absence of zero-energy resonances has been shown under the weaker decay $|V_{j,k}| \leq C\langle x\rangle^{-1-\varepsilon}$, $x \in \mathbb{R}^3$, in [8]. The absence of zero-energy resonances for massless Dirac operators in dimensions $n \geq 4$ as contained in Theorem 10.7 (ii) appears to have gone unnoticed in the literature and was only recently observed in [82]. ◇

To determine the leading order behavior of

$$\left[U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1 (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}\right]^{-1} \quad \text{as } z \to 0,\ z \in \overline{\mathbb{C}_+},$$

in all possible cases discussed in Theorem 10.7, it is convenient to introduce some more notation:

$$T(z) := U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1 (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}, \quad z \in \mathbb{C}_+, \tag{10.46}$$

$$T(\lambda) := U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1 \big(H_0 - (\lambda + i0) I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1} V_1^*}, \quad \lambda \in \mathbb{R}. \tag{10.47}$$

Next, we split P_0 in (10.16) according to all possible cases in Theorem 10.7 as follows: If $n = 2$, we write

$$P_0 = P_{0,1} \oplus P_{0,2}, \tag{10.48}$$

where $P_{0,1}$ represents case (II), $P_{0,2}$ represents case (III), and if $P_{0,1}$ and $P_{0,2}$ are both nonzero, P_0 represents case (IV). Similarly, if $n \geq 3$, $P_0 \neq 0$ represents case (II). (Again, we remark that we will discuss in Lemma 10.12 (i) that $\dim(\operatorname{ran}(P_{0,1})) \leq 2$.)

In the following we denote the integral operators in $[L^2(\mathbb{R}^n)]^N$ generated by the integral kernels $R_{j,k}(\,\cdot\,,\,\cdot\,)$ in (10.20)–(10.22) by $R_{j,k}$, $j, k \in \{0, 1\}$. In particular,

$$T(0) = U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1 R_{0,0} V_1^*}.$$

In order to study asymptotics as $z \to 0$ of the Birman–Schwinger-type operators, we strengthen Hypothesis 10.5 as follows.

Hypothesis 10.9. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\varepsilon > 0$. Assume the a.e. self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1\leq \ell,\ell' \leq N}$ satisfies for some fixed $\varepsilon \in (0,1)$, $C \in (0,\infty)$,*

$$V \in [L^\infty(\mathbb{R}^n)]^{N\times N},$$
$$\left|V_{\ell,\ell'}(x)\right| \leq C\langle x\rangle^{-n(1+\varepsilon)} \quad \textit{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N. \tag{10.49}$$

In accordance with the factorization based on the polar decomposition of V discussed in (10.9) *we suppose that*

$$V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}, \textit{ where } V_1 = V_1^* = |V|^{1/2},\ V_2 = U_V |V|^{1/2}.$$

We note that, in accordance with (10.49), the entries of $V_1(\cdot)$ satisfy

$$\left|(V_1)_{\ell,\ell'}(x)\right| \leq \widetilde{C}\langle x\rangle^{-n(1+\varepsilon)/2} \qquad \text{for a.e. } x \in \mathbb{R}^n, 1 \leq \ell,\ell' \leq N,$$

for a constant $\widetilde{C} \in (0,\infty)$.

Lemma 10.10. *Assume Hypothesis* 10.9. *Then (cf.* (10.20)–(10.22)*)*

$$\overline{V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*} \underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} \overline{V_1 R_{0,0} V_1^*} + z\overline{V_1 R_{1,0} V_1^*}$$
$$+ z\big[-(2\pi)^{-1}\ln(z/2) - (2\pi)^{-1}\gamma_{E-M} + i4^{-1}\big]\delta_{n,2}\overline{V_1 R_{1,1} V_1^*} + E(z), \tag{10.50}$$

where

$$\|E(z)\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} \underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{1+\varepsilon}\big) \tag{10.51}$$

(*with $0 < \varepsilon$ taken as in Hypothesis* 10.9).

Proof. In order to prove (10.50) and (10.51) it suffices to show

$$\begin{aligned}\big\|V_1(x)G_0(z;x,y)V_1^*(y) &- V_1(x)R_{0,0}(x-y)V_1^*(y)\\ &- zV_1(x)R_{1,0}(x-y)V_1^*(y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \leq c_0|z|^{1+\varepsilon}k(x,y),\\ &x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\end{aligned} \tag{10.52}$$

for some positive (z,x,y)-independent constant c_0 and for some z-independent function $k(\cdot\,,\cdot)$ which generates a bounded integral operator in $L^2(\mathbb{R}^n)$. In the following we treat separately Cases (I) n odd and (II) n even.

Case (I): n odd. In order to prove (10.52), we estimate

$$\begin{aligned}&G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y),\\ &\quad x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\end{aligned}$$

separately in the regimes $|z||x-y| \leq 1$ and $|z||x-y| > 1$.

The expansion (C.5) implies

$$\begin{aligned}\big\|G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)} &\leq c_1\big[|z|^2 + |z|^2|z-y|^{3-n}\big] \\ &\leq c_1\big[|z|^2 + |z|^{1+\varepsilon}|z-y|^{(2+\varepsilon)-n}\big] \leq c_1|z|^{1+\varepsilon}\big[1 + |x-y|^{(2+\varepsilon)-n}\big], \\ &x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\ |z||x-y| \leq 1, \quad (10.53)\end{aligned}$$

for some (z,x,y)-independent constant $c_1 \in (0,\infty)$. By Lemma B.6,

$$\begin{aligned}&G_0(z;x,y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{(2-n)/2}z^{n/2}e^{iz|x-y|}\omega_{\frac{n-2}{2}}(z|x-y|)I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{(2-n)/2}z^{n/2}e^{iz|x-y|}\omega_{\frac{n}{2}}(z|x-y|)\alpha\cdot\frac{(x-y)}{|x-y|}, \\ &\qquad x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\}, \quad (10.54)\end{aligned}$$

with

$$\begin{aligned}|x-y|^{(2-n)/2}|z|^{n/2}\big|\omega_\nu(z|x-y|)\big| &\leq c_2|x-y|^{1-\frac{n}{2}}|z|^{n/2}\big(1+|z||x-y|\big)^{-1/2} \\ &\leq c_2|z|^{n-1}\big(1+|z||x-y|\big)^{-1/2} \leq c_2|z|^2\big(1+|z||x-y|\big)^{-1/2}, \\ &x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\ |z||x-y| \geq 1, \quad (10.55)\end{aligned}$$

for some (z,x,y)-independent constant $c_2 \in (0,\infty)$. The representation (10.54) and the estimate (10.55) combine to yield

$$\begin{aligned}&\big\|G_0(z;x,y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \leq c_3|z|^2\big(1+|z||x-y|\big)^{-1/2}, \\ &x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\ |z||x-y| \geq 1, \quad (10.56)\end{aligned}$$

for some (z,x,y)-independent constant $c_3 \in (0,\infty)$, and it follows that

$$\begin{aligned}&\big\|G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\quad \leq c_4\Big(|z|^2\big(1+|z||x-y|\big)^{-1/2} + |x-y|^{1-n} + |z||x-y|^{2-n}\Big) \leq c_4|z|^2, \\ &\qquad \text{for a.e. } x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\ |z||x-y| \geq 1, \quad (10.57)\end{aligned}$$

for some (z,x,y)-independent constant $c_4 \in (0,\infty)$.

By combining (10.53) and (10.57), one obtains

$$\begin{aligned}&\big\|G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\quad \leq c_5\big[|z|^2 + |z|^{1+\varepsilon}|x-y|^{(2+\varepsilon)-n}\big] \leq c_5|z|^{1+\varepsilon}\big[1+|x-y|^{(2+\varepsilon)-n}\big], \\ &\qquad \text{for a.e. } x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1, \quad (10.58)\end{aligned}$$

for some (z,x,y)-independent constant $c_5 \in (0,\infty)$. Hence, (10.52) holds for some constant $c_0 \in (0,\infty)$ and

$$\begin{aligned} k(x,y) = {} & \langle x\rangle^{-n(1+\varepsilon)/2}\langle y\rangle^{-n(1+\varepsilon)/2} \\ & + \big[1+|x|\big]^{-n(1+\varepsilon)/2}|x-y|^{(2+\varepsilon)-n}\big[1+|y|\big]^{-n(1+\varepsilon)/2}, \\ & \hspace{6cm} x,y \in \mathbb{R}^n,\ x \neq y. \end{aligned} \tag{10.59}$$

In deducing the form of $k(\cdot\,,\cdot)$ in (10.59), one uses

$$\big\|V_1(x)\big\|_{\mathcal{B}(\mathbb{C}^N)} \leq C'\langle x\rangle^{-n(1+\varepsilon)/2} \leq C''\big[1+|x|\big]^{-1-(\varepsilon/2)} \quad \text{for a.e. } x \in \mathbb{R}^n \tag{10.60}$$

for appropriate x-independent constants $C', C'' \in (0,\infty)$.

The first term on the right-hand side in (10.59) generates a Hilbert–Schmidt integral operator in $L^2(\mathbb{R}^n)$, since $\langle\cdot\rangle^{-n(1+\varepsilon)/2} \in L^2(\mathbb{R}^n)$. The second term on the right-hand side in (10.59) generates a bounded integral operator in $L^2(\mathbb{R}^n)$ as a consequence of Theorem 6.6 (ii) with the choices $c = d = 1 + (\varepsilon/2)$ and $p = p' = 2$, since $1 + (\varepsilon/2) < 3/2 \leq n/2$. Thus, $k(\cdot\,,\cdot)$ generates a bounded integral operator in $L^2(\mathbb{R}^n)$.

Case (II): n even. The case $n = 2$ is treated in detail in [60, Lemma 5.1], so we consider $n \geq 4$ here. It suffices to verify the inequality in (10.52). The expansion (C.15) implies

$$\begin{aligned} & \big\|G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\ & \quad \leq c_1\big[|z|^{n-1} + |z|^2|x-y|^{3-n} + |z|^{n-1}\big|\ln\big(z|x-y|\big)\big|\big] \\ & \quad \leq c_1\big[|z|^{n-1} + |z|^{1+\varepsilon}|x-y|^{(2+\varepsilon)-n} + |z|^{n-1}\big|\ln\big(z|x-y|\big)\big|\big] \\ & \quad \leq c_1|z|^{1+\varepsilon}\big[1 + |x-y|^{(2+\varepsilon)-n} + |x-y|^{-1}\big], \\ & x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,\ |z||x-y| \leq 1, \end{aligned} \tag{10.61}$$

for some (z,x,y)-independent constant $c_1(\varepsilon) \in (0,\infty)$, and an argument entirely analogous to (10.54)–(10.56) shows that (10.57) extends to the current case where n is even. Combining (10.57) and (10.61), one obtains

$$\begin{aligned} & \big\|G_0(z;x,y) - R_{0,0}(x-y) - zR_{1,0}(x-y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\ & \quad \leq c_3|z|^{1+\varepsilon}\big[1 + |x-y|^{(2+\varepsilon)-n} + |x-y|^{-1}\big], \\ & \text{for a.e. } x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1, \end{aligned}$$

Hence, (10.52) holds for some constant $c_0 \in (0,\infty)$ and

$$\begin{aligned} k(x,y) = {} & \langle x\rangle^{-n(1+\varepsilon)/2}\langle y\rangle^{-n(1+\varepsilon)/2} \\ & + \big[1+|x|\big]^{-n(1+\varepsilon)/2}|x-y|^{(2+\varepsilon)-n}\big[1+|y|\big]^{-n(1+\varepsilon)/2} \\ & + \big[1+|x|\big]^{(1-n)/2}|x-y|^{-1}\big[1+|y|\big]^{(1-n)/2}, \quad x,y \in \mathbb{R}^n,\ x \neq y. \end{aligned} \tag{10.62}$$

In deducing the form of $k(\cdot,\cdot)$ in (10.62), we used (10.60) and the elementary bound $[1+|x|]^{-n(1+\varepsilon)/2} \leq [1+|x|]^{-(n-1)/2}$, $x \in \mathbb{R}^n$. The fact that the first two terms on the right-hand side in (10.62) generate bounded integral operators in $L^2(\mathbb{R}^n)$ was established in Case (I) above. The third term on the right-hand side in (10.62) generates a bounded integral operator in $L^2(\mathbb{R}^n)$ by Theorem 6.6 (ii) with the choices $c = d = (n-1)/2$ and $p = p' = 2$, since $(n-1)/2 < n/2$, $c + d = n - 1 > 0$, and $n - (c+d) = 1$. ■

Lemma 10.11. *Assume Hypothesis 10.9. If $T(\cdot)$ is defined by (10.46) and P_0 denotes the (finite-dimensional) Riesz projection associated to the operator (10.8), then*

$$\begin{aligned}
&\big[T(z)+P_0\big]^{-1} \\
&\underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} \big[T(0)+P_0\big]^{-1} - z\big[-(2\pi)^{-1}\ln(z/2) - (2\pi)^{-1}\gamma_{E-M} + i4^{-1}\big] \\
&\qquad \times \delta_{n,2}[T(0)+P_0]^{-1}\overline{V_1 R_{1,1} V_1^*}\big[T(0)+P_0\big]^{-1} \\
&\qquad - z\big[T(0)+P_0\big]^{-1}\overline{V_1 R_{1,0} V_1^*}\big[T(0)+P_0\big]^{-1} + E_1(z),
\end{aligned} \tag{10.63}$$

where

$$\begin{aligned}
&\big\|E_1(z)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} \\
&\underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} \begin{cases} O\big(|z|^{1+k}\big) \text{ for any } 0 < k < \min\{1,\varepsilon\}, & n = 2, \\ O\big(|z|\big), & n \geq 3 \end{cases}
\end{aligned} \tag{10.64}$$

(*with $0 < \varepsilon$ taken as in Hypothesis 10.9*).

Proof. The case $n = 2$ is treated in detail in [60, Lemma 5.2], so we consider $n \geq 3$ here. By Lemma 10.10,

$$\begin{aligned}
&\big[T(z)+P_0\big]^{-1} \\
&\quad = \big[T(0)+P_0 + z\overline{V_1 R_{1,0} V_1^*} + E(z)\big]^{-1} \\
&\quad = \big(I_{[L^2(\mathbb{R}^n)]^N} + \big[T(0)+P_0\big]^{-1} z\overline{V_1 R_{1,0} V_1^*} + \big[T(0)+P_0\big]^{-1}E(z)\big)^{-1}\big[T(0)+P_0\big]^{-1}, \\
&\qquad\qquad z \in \overline{\mathbb{C}_+}\backslash\{0\},\ 0 < |z| \ll 1,
\end{aligned} \tag{10.65}$$

where $E(\cdot)$ satisfies (10.51). By (10.21) and (10.58),

$$\begin{aligned}
&\big\|V_1(x)G_0(z;x,y)V_1^*(y) - V_1(x)R_{0,0}(x-y)V_1^*(y)\big\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\quad \leq c_1|z|\big[V_1(x)V_1^*(y) + \big[1+|x|\big]^{(1-n)/2}|x-y|^{-1}\big[1+|y|\big]^{(1-n)/2} \\
&\qquad + \big[1+|x|\big]^{-1}|x-y|^{2-n}\big[1+|y|\big]^{-1}\big], \\
&\qquad \text{for a.e. } x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ |z| \leq 1,
\end{aligned} \tag{10.66}$$

for some (z, x, y)-independent constant $c_1 \in (0, \infty)$. The kernel

$$k(x, y) = V_1(x)V_1^*(y) + \big[1 + |x|\big]^{(1-n)/2}|x - y|^{-1}\big[1 + |y|\big]^{(1-n)/2} \\ + \big[1 + |x|\big]^{-1}|x - y|^{2-n}\big[1 + |y|\big]^{-1}, \quad x, y \in \mathbb{R}^n,\ x \neq y, \tag{10.67}$$

generates a bounded integral operator in $L^2(\mathbb{R}^n)$. The first term on the right-hand side in (10.67) generates a Hilbert–Schmidt operator due to the containment

$$\big\|V_1(\cdot)\big\|_{\mathcal{B}(\mathbb{C}^N)} \in L^2(\mathbb{R}^n).$$

The fact that the second term generates a bounded operator is explained in the proof of Lemma 10.10 in connection with (10.59). Finally, the third term in (10.67) generates a bounded integral operator by an application of Theorem 6.6 (ii) with $a = b = 1$. It follows that

$$\big\|T(z) - T(0)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} \leq c_2|z|, \quad z \in \overline{\mathbb{C}_+},\ |z| \leq 1, \tag{10.68}$$

for some z-independent constant $c_2 \in (0, \infty)$. The estimate in (10.68) implies that, for $z \in \overline{\mathbb{C}_+}$ with $0 < |z| \ll 1$, a Neumann series may be used to obtain

$$\begin{aligned} &\big[T(z) + P_0\big]^{-1} \\ &\quad = \big[T(0) + P_0\big]^{-1} - z\big[T(0) + P_0\big]^{-1}\overline{V_1 R_{1,0} V_1^*}\big[T(0) + P_0\big]^{-1} \\ &\qquad - \big[T(0) + P_0\big]^{-1}E(z)\big[T(0) + P_0\big]^{-1} + \sum_{n=2}^{\infty}(-1)^n A(z)^n\big[T(0) + P_0\big]^{-1}, \\ &\qquad\qquad z \in \overline{\mathbb{C}_+}\backslash\{0\},\ 0 < |z| \ll 1, \end{aligned} \tag{10.69}$$

where

$$A(z) := z\big[T(0)+P_0\big]^{-1}\overline{V_1 R_{1,0} V_1^*} + \big[T(0)+P_0\big]^{-1}E(z) \underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|\big), \tag{10.70}$$

applying (10.51). In particular,

$$\Bigg\|\sum_{n=2}^{\infty}(-1)^n A(z)^n\big[T(0) + P_0\big]^{-1}\Bigg\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} \underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|\big). \tag{10.71}$$

Hence, (10.63) follows from (10.69) with

$$E_1(z) := -\big[T(0) + P_0\big]^{-1}E(z)\big[T(0) + P_0\big]^{-1} + \sum_{n=2}^{\infty}(-1)^n A(z)^n\big[T(0) + P_0\big]^{-1}, \\ z \in \overline{\mathbb{C}_+},\ 0 < |z| \ll 1.$$

Thus, the $O(|z|)$ relation in (10.64) for $n \geq 3$ follows from (10.51) and (10.71). ∎

Lemma 10.12 ([60, Lemmas 5.2, 7.1–7.6]). *Assume Hypothesis 10.9 and $n = 2$. The following statements* (i)–(iv) *hold.*

(i) *If $\phi_0 \in \ker(T(0))$, then $\phi_0 = U_V V_1 \psi_0$, with ψ_0 a distributional solution of $H\psi_0 = 0$ satisfying $\psi_0 \in [L^p(\mathbb{R}^2)]^2$ for all $p \in (2,\infty) \cup \{\infty\}$. Moreover,*

$$\psi_0(x) = -i\alpha \cdot x\big[2\pi \langle x\rangle^2\big]^{-1}(R_{1,1}V_1^*\phi_0) + \psi_1(x),$$

where

$$(R_{1,1}V_1^*\phi_0) = \int_{\mathbb{R}^2} d^2y\, V_1^*(y)\phi_0(y) \quad \textit{and} \quad \psi_1 \in \big[W^{1,2}(\mathbb{R}^2)\big]^2.$$

In particular,

$$\psi_0 \in \big[W^{1,2}(\mathbb{R}^2)\big]^2 \textit{ if and only if } (R_{1,1}V_1^*\phi_0) = \int_{\mathbb{R}^2} d^2y\, V_1^*(y)\phi_0(y) = 0.$$

Moreover, the rank of P_0 is at most two plus the dimension of the eigenspace of H at energy zero, that is,

$$P_0 = P_{0,1} \oplus P_{0,2}, \quad \textit{with } \dim\big(\mathrm{ran}(P_{0,1})\big) \le 2$$

in (10.48).

(ii) *If $\psi_0 \in [L^2(\mathbb{R}^2)]^2 + \bigcap_{p\in(2,\infty)\cup\{\infty\}}[L^p(\mathbb{R}^2)]^2$, then*

$$\phi_0 = U_V V_1 \psi_0 \in \ker\big(T(0)\big).$$

(iii) *If $\phi_0 = U_V V_1 \psi_0 \in \ker(T(0))$, then $\phi_0 \in \mathrm{ran}(P_{0,2})$*

$$\textit{if and only if} \quad \psi_0 \in \big[W^{1,2}\big(L^2(\mathbb{R}^2)\big)\big]^2.$$

Thus, $\phi_0 \in \mathrm{ran}(P_{0,2})$ if and only if $\phi_0 \in \ker(P_0 V_1 R_{1,1} V_1^* P_0)$.

(iv) *If $\phi_0 \in \mathrm{ran}(P_{0,2})$, then*

$$(R_{0,0}V_1^*\phi_0, R_{0,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^2)]^2} = (V_1^*\phi_0, R_{1,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^2)]^2} \tag{10.72}$$

and

$$\ker(P_{0,2}V_1 R_{1,0} V_1^* P_{0,2}) = \{0\}. \tag{10.73}$$

Lemma 10.13. *Assume Hypothesis 10.9 and $n \ge 3$. The following statements* (i) *and* (ii) *hold.*

(i) $\phi_0 = U_V V_1 \psi_0 \in \ker(T(0))$ *(i.e., $\phi_0 \in \mathrm{ran}(P_0)$)*

$$\textit{if and only if} \quad \psi_0 \in \big[W^{1,2}\big(L^2(\mathbb{R}^n)\big)\big]^N.$$

Thus, $\phi_0 \in \mathrm{ran}(P_0)$ if and only if $\phi_0 \in \ker(P_0 V_1 R_{1,1} V_1^* P_0)$.

(ii) $\phi_0 \in \operatorname{ran}(P_0)$, *then*

$$(R_{0,0}V_1^*\phi_0, R_{0,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^n)]^N} = (V_1^*\phi_0, R_{1,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^n)]^N} \tag{10.74}$$

and

$$\ker(P_0 V_1 R_{1,0} V_1^* P_0) = \{0\}. \tag{10.75}$$

Proof. Item (i) is just a rephrasing of the proof of Theorem 10.7 for $n \geq 3$. Item (ii) is proved exactly along the lines of [60, Lemma 7.6]; we briefly sketch the argument. By item (i), $\psi_0 = -R_{0,0}V_1^*\phi_0 \in [L^2(\mathbb{R}^2)]^2$ and hence, applying Fourier transforms,

$$\begin{aligned}
&(R_{0,0}V_1^*\phi_0, R_{0,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^n)]^N} \\
&\quad = \int_{\mathbb{R}^n} d^n p\, |p|^{-4}\big((\alpha\cdot p)(V_1^*\phi_0)^\wedge, (\alpha\cdot p)(V_1^*\phi_0)^\wedge\big)_{\mathbb{C}^N} \\
&\quad = \int_{\mathbb{R}^n} d^n p\, |p|^{-2}\big((V_1^*\phi_0)^\wedge, (V_1^*\phi_0)^\wedge\big)_{\mathbb{C}^N}.
\end{aligned} \tag{10.76}$$

On the other hand, employing the monotone convergence theorem,

$$\begin{aligned}
&(V_1^*\phi_0, R_{1,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^n)]^N} = \big(V_1^*\phi_0, (-\Delta)V_1^*\phi_0\big)_{[L^2(\mathbb{R}^n)]^N} \\
&\quad = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} d^n p\, \big[|p|^2+\varepsilon^2\big]^{-1}\big((V_1^*\phi_0)^\wedge, (V_1^*\phi_0)^\wedge\big)_{\mathbb{C}^N} \\
&\quad = \int_{\mathbb{R}^n} d^n p\, |p|^{-2}\big((V_1^*\phi_0)^\wedge, (V_1^*\phi_0)^\wedge\big)_{\mathbb{C}^N},
\end{aligned}$$

proving (10.74). Finally, assume that $\phi_0 \in \ker(P_0 V_1 R_{1,0} V_1^* P_0)$. Then (10.74) yields

$$\|\psi_0\|_{[L^2(\mathbb{R}^n)]^N} = (R_{0,0}V_1^*\phi_0, R_{0,0}V_1^*\phi_0)_{[L^2(\mathbb{R}^n)]^N} = 0,$$

implying $\psi_0 = 0$ and thus $\phi_0 = U_V V_1 \psi_0 = 0$. ∎

One of the principal results of this chapter then reads as follows:

Theorem 10.14. *Assume Hypothesis 10.5.*

(i) *Suppose $n = 2$. Then*

$$\begin{aligned}
T(z)^{-1} &= \big[U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]^{-1} \\
&\underset{\substack{z\to 0\\ z\in\mathbb{C}_+\backslash\{0\}}}{=} \begin{cases}
T(0)^{-1} - T(0)^{-1}\big[O\big(\big|z\ln(z)\big|\big)\big]T(0)^{-1} \text{ in Case (I)},\\
\big[z\ln(z)\big]^{-1}P_{0,1}AP_{0,1}\\
\qquad + P_{0,1}\big[O\big(|z|^{-1}|\ln(z)|^{-2}\big)\big]P_{0,1} \text{ in Case (II)},\\
z^{-1}P_{0,2}[P_{0,2}V_1R_{1,0}V_1^*P_{0,2}]P_{0,2}\\
\qquad + P_{0,2}\big[O\big(|z|^{-1+\varepsilon}\big)\big]P_{0,2} \text{ in Case (III)},\\
z^{-1}P_0\begin{pmatrix}0 & 0\\ 0 & P_{0,2}V_1R_{1,0}V_1^*P_{0,2}\end{pmatrix}P_0\\
\qquad + P_0\big[O\big(|z\ln(z)|^{-1}\big)\big]P_0 \text{ in Case (IV)},
\end{cases}
\end{aligned}$$

where

$$T(0) = U_V + V_1 R_{0,0} V_1^*, \quad T(0)^{-1} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big) \quad \textit{in Case (I)},$$
$$A \in \mathbb{R}\backslash\{0\} \quad \textit{if } \dim\big(\mathrm{ran}(P_{0,1})\big) = 1 \quad \textit{in Case (II)},$$
$$\det_{\mathbb{C}^2}(A) \neq 0 \quad \textit{if } \dim\big(\mathrm{ran}(P_{0,1})\big) = 2 \quad \textit{in Case (II)}.$$

(ii) *Suppose* $n \in \mathbb{N}$, $n \geq 3$. *Then*

$$\begin{aligned} T(z)^{-1} &= \big[U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]^{-1} \\ &\underset{\substack{z\to 0\\ z\in\mathbb{C}_+\backslash\{0\}}}{=} \begin{cases} T(0)^{-1} - T(0)^{-1}\big[O\big(\big|z\ln(z)\big|\big)\big]T(0)^{-1} & \textit{in Case (I)}, \\ z^{-1}P_0[P_0V_1R_{1,0}V_1^*P_0]P_0 + P_0\big[O\big(|z|^{-1+\varepsilon}\big)\big]P_0 & \textit{in Case (II)}, \end{cases} \end{aligned}$$

where, again,

$$T(0) = U_V + V_1 R_{0,0} V_1^*, \quad T(0)^{-1} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big) \textit{ in Case (I)}.$$

Moreover, in both items (i) *and* (ii), *the coefficients of all singular terms in the expansion of* $T(z)^{-1}$ *at* $z = 0$ (*i.e., in cases different from* (I)) *are finite-rank operators acting in* (*subspaces of*) $P_0[L^2(\mathbb{R}^n)]^N$.

Here, $O(|\zeta|^a)$, $a \in \mathbb{R}$, *abbreviate estimates with respect to the operator norm.*

Proof. Item (i) for $n = 2$ has been treated in detail [60, Section 5] on the basis of the Jensen and Nenciu method [99] outlined in Lemmas 10.1, 10.2, Remarks 10.3, 10.4, and our summary in items (α)–(δ) following Remark 10.3. Item (ii) for $n \geq 3$ parallels Cases (I) and (III) for $n = 2$. ∎

Remark 10.15. A comparison of the threshold behavior of massless Dirac operators [60, Theorem 9.10 (i)] and Schrödinger operators [29, 30, 99, 122] demonstrates that in both situations zero-energy resonances produce a logarithmically weaker singularity of the order $O(|z\ln(z)|^{-1})$ than the zero-energy eigenvalues which produce the expected $O(|z|^{-1})$ singularity. ◇

Finally, returning to F_{H,H_0}, we again introduce the strengthened assumptions made in Hypothesis 7.1 and Corollary 4.4 (ii).

Hypothesis 10.16. *Let* $n \in \mathbb{N}$ *and suppose that* $V = \{V_{\ell,\ell'}\}_{1\leq\ell,\ell'\leq N}$ *satisfies for some constants* $C \in (0,\infty)$ *and* $\varepsilon > 0$,

$$\begin{aligned} &V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \\ &\big|V_{\ell,\ell'}(x)\big| \leq C\langle x\rangle^{-n-\varepsilon} \quad \textit{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N. \end{aligned} \tag{10.77}$$

In addition, assume that $V(x) = \{V_{\ell,\ell'}(x)\}_{1\leq\ell,\ell'\leq N}$ *is self-adjoint for a.e.* $x \in \mathbb{R}^n$. *In accordance with the factorization based on the polar decomposition of* V *discussed in* (10.9) *we suppose that* $V = V_1^*V_2 = |V|^{1/2}U_V|V|^{1/2}$, *where* $V_1 = V_1^* = |V|^{1/2}$, $V_2 = U_V|V|^{1/2}$.

In addition, we assume that V satisfies (4.2) *and* (4.3)[3].

According to Remark 9.7, we now use the symmetrized version of the Birman–Schwinger operator in connection with (9.13) and hence write

$$
\begin{aligned}
&F_{H,H_0}(z) \\
&= \ln\big(\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big((H - zI_{[L^2(\mathbb{R}^n)]^N})(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big)\big) \\
&= \ln\big(\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}\big)\big) \\
&= \ln\big(\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big)\big) \\
&= \ln\big(\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(U_V\big\{U_V I_{[L^2(\mathbb{R}^n)]^N} + V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\big\}\big)\big) \\
&= \ln(\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(U_V T(z)\big)\big), \quad z \in \mathbb{C}_\pm,
\end{aligned}
\tag{10.78}
$$

employing $U_V^2 = I_N$.

Next, we briefly recall a few facts on continuous (resp., analytic) logarithms and continuous arguments of complex-valued functions (see [12, pp. 40–46] for details):

If $S \subseteq \mathbb{C}$ and $f: S \to \mathbb{C}\backslash\{0\}$, then g is called a *continuous logarithm* of f on S if g is continuous on S and $f(z) = e^{g(z)}$, $z \in S$. Similarly, $\theta: S \to \mathbb{R}$ is called a *continuous argument* of f on S if θ is continuous on S and $f(z) = |f(z)|e^{i\theta(z)}$, $z \in S$.

• If g is a continuous logarithm of f on S, then $\mathrm{Im}(g)$ is a continuous argument of f on S.

• If θ is a continuous argument of f, then $\ln(|f|) + i\theta$ is a continuous logarithm of f on S.

• Thus, f has a continuous logarithm on S if and only if f has a continuous argument on S.

If $\Omega \subseteq \mathbb{C}$ is open and $f: \Omega \to \mathbb{C}\backslash\{0\}$ is analytic, then $g: \Omega \to \mathbb{C}$ is called an analytic logarithm of f on Ω if g is analytic on Ω and $f(z) = e^{g(z)}$, $z \in \Omega$.

• If $\Omega \subseteq \mathbb{C}$ is open and starlike and $f: \Omega\backslash\{0\}$ is analytic, then f has an analytic logarithm on Ω.

• Suppose Ω is open and $f: \Omega \to \mathbb{C}\backslash\{0\}$ is analytic with g a continuous logarithm of f on Ω. Then g is analytic on Ω.

• Let $a, b, c, d \in \mathbb{R}$, $R = \{z = x + iy \,|\, a \le x \le b,\ c \le y \le d\}$, and $f: R \to \mathbb{C}\backslash\{0\}$ continuous. Then f has a continuous logarithm on R.

• $f: \mathbb{C}_+ \to \mathbb{C}\backslash\{0\}$ analytic, $f: \overline{\mathbb{C}_+} \to \mathbb{C}\backslash\{0\}$ continuous, then f has an analytic logarithm on $\mathbb{C}_+$ which is continuous on $\overline{\mathbb{C}_+}$. More generally, $f: \mathbb{C}_+ \to \mathbb{C}\backslash\{0\}$ analytic, $f: \overline{\mathbb{C}_+} \to \mathbb{C}$ continuous, then f has an analytic logarithm on $\mathbb{C}_+$ which is continuous at $x_0 \in \mathbb{R}$ if $f(x_0) \neq 0$.

This yields the final and principal result of this chapter.

[3]The first condition in (4.3) is superseded by assumption (10.77).

Theorem 10.17. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume Hypothesis 10.16. Then F_{H,H_0}, $z \in \mathbb{C}_\pm$, has normal boundary values on $\mathbb{R}\backslash\{0\}$. In addition, the boundary values to $\mathbb{R}$ of the function $\operatorname{Im}(F_{H,H_0}(z))$, $z \in \mathbb{C}_+$, are continuous on $(-\infty, 0) \cup (0, \infty)$,*

$$\operatorname{Im}\big(F_{H,H_0}(\lambda + i0)\big) \in C\big((-\infty, 0) \cup (0, \infty)\big), \tag{10.79}$$

and the left and right limits at zero,

$$\operatorname{Im}\big(F_{H,H_0}(0_\pm + i0)\big) = \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(F_{H,H_0}(\pm\varepsilon + i0)\big), \tag{10.80}$$

exist. In particular, if 0 is a regular point for H according to Definition 10.6 (iii) *and Theorem 10.7* (iii) (*this corresponds to case* (I) *in Theorem 10.7* (i), (ii)), *then*

$$\operatorname{Im}\big(F_{H,H_0}(\lambda + i0)\big) \in C(\mathbb{R}). \tag{10.81}$$

Proof. Applying Theorem 3.4, Corollary 4.4 (i), and Theorem 6.16, the function $\det_{[L^2(\mathbb{R}^n)]^N,n+1}(U_V T(z))$, $z \in \mathbb{C}_\pm$, in (10.78) continuously extends to $z \in \overline{\mathbb{C}_\pm}\backslash\{0\}$ and does not vanish there. In particular, F_{H,H_0} has normal boundary values on $\mathbb{R}\backslash\{0\}$. Moreover, combining Theorem 6.16 and [12, Theorem 3.1.7], and especially, by [12, p. 46, Exercise 3.2.6], the function

$$\begin{aligned}
&\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(U_V T(z)\big)\\
&\quad = \det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(U_V\big\{U_V I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_1(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big\}\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in \overline{\mathbb{C}_+}\backslash\{0\},
\end{aligned}$$

has a continuous argument in any rectangle of the form

$$\big\{z = x + iy \mid x \in [a,b] \subset (-\infty, 0) \cup (0, \infty),\ y \in [0,c]\big\}, \quad c > 0,$$

in $\overline{\mathbb{C}_+}\backslash\{0\}$, proving (10.79). Thus $\lambda = 0$ is the only possible exception to continuity of $\operatorname{Im}(F_{H,H_0}(\cdot + i0))$ on $\mathbb{R}$.

If 0 is a regular point for H, that is, if

$$\ker\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]\big) = \{0\}, \tag{10.82}$$

then

$$\det_{[L^2(\mathbb{R}^n)]^N,n+1}\big(U_V T(0)\big) \neq 0$$

and hence $\det_{[L^2(\mathbb{R}^n)]^N,n+1}(U_V T(z))$ has a continuous argument in any rectangle of the form

$$\big\{z = x + iy \mid x \in [a,b] \subset \mathbb{R},\ y \in [0,c]\big\}, \quad c > 0,$$

proving (10.81).

If

$$\ker\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big]\big) \supsetneqq \{0\}, \tag{10.83}$$

denote by $P_{0,+}$ the projection onto the (finite-dimensional) eigenspace of the compact operator $\overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}$ corresponding to the eigenvalue -1. By Lemma 10.1 (iii),

$$\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*} + P_{0,+}\big)^{-1} \in \mathcal{B}\big([L^2(\mathbb{R}^n)]^N\big)$$

and hence,

$$\det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*} + P_{0,+}\big) \neq 0$$

and

$$\begin{aligned}
&\det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\big)\\
&\quad= \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0-(0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+} - P_{0,+}\big)\\
&\quad= \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*} + P_{0,+}\big]\\
&\quad\times\big\{I_{[L^2(\mathbb{R}^n)]^N} - \big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0-(0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}\big\}P_{0,+}\big).
\end{aligned} \tag{10.84}$$

Applying Theorem D.1 in (10.84) one obtains

$$\begin{aligned}
&\det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*}\big)\\
&\quad= \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big)\\
&\qquad\times \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N}\\
&\qquad - \big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\big)\\
&\qquad\times e^{\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}(X_{n+1})}\\
&\quad= \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big)\\
&\qquad\times \det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\big(I_{[L^2(\mathbb{R}^n)]^N}\\
&\qquad - P_{0,+}\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\big)\\
&\qquad\times \exp\big(\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}(X_{n+1})\big).
\end{aligned}$$

Here $\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}(X_{n+1})$ is a finite sum of traces of products of the operators

$$\big[\overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]$$

and

$$-\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}$$

of at least $n + 1$ factors (in various orders) as described in detail in Appendix D, in particular,

$$\exp\big(\mathrm{tr}_{[L^2(\mathbb{R}^n)]^N}(X_{n+1})\big) \neq 0.$$

Thus, the structure of the zero of the modified Fredholm determinant

$$\det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\Big(I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*}\Big)$$

as $z \to 0$, $z \in \overline{\mathbb{C}_+}\backslash\{0\}$, is identical to the structure of the zero of the modified Fredholm determinant (see, e.g., [159, Theorem 9.2(d)])

$$\begin{aligned}
&\det{}_{[L^2(\mathbb{R}^n)]^N,n+1}\Big(I_{[L^2(\mathbb{R}^n)]^N} \\
&\quad - P_{0,+}\big[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\Big) \\
&= \det{}_{[P_{0,+}L^2(\mathbb{R}^n)]^N}\Big(P_{0,+}I_{[L^2(\mathbb{R}^n)]^N} \\
&\quad - P_{0,+}\big[I_{[P_{0,+}L^2(\mathbb{R}^n)]^N} + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\Big) \\
&\quad \times \exp\Bigg(\sum_{j=1}^{n} j^{-1}\mathrm{tr}_{[L^2(\mathbb{R}^n)]^N}\Big(\big[P_{0,+}\big[I_{[P_{0,+}L^2(\mathbb{R}^n)]^N} \\
&\qquad + \overline{V_2\big(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N}\big)^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\big]^j\Big)\Bigg),
\end{aligned} \tag{10.85}$$

which now reduces to a finite-dimensional determinant. The behavior of the latter as $z \to 0$, $z \in \overline{\mathbb{C}_+}\backslash\{0\}$,

$$\begin{aligned}
&\det{}_{[P_{0,+}L^2(\mathbb{R}^n)]^N}\Big(P_{0,+}I_{[L^2(\mathbb{R}^n)]^N} - P_{0,+}\big[I_{[P_{0,+}L^2(\mathbb{R}^n)]^N} \\
&\qquad\qquad + \overline{V_2(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*} + P_{0,+}\big]^{-1}P_{0,+}\Big)
\end{aligned}$$

in turn, is governed by Lemma 10.11 and hence in leading order is a polynomial $\mathcal{P}(\cdot\,,\cdot)$ in the two variables $z\ln(z)$ and z (the $z\ln(z)$ part being absent in odd space dimensions). By (10.83), $\mathcal{P}(\cdot\,,\cdot)$ has no constant term and hence its leading order is of the form

$$\begin{aligned}
&\mathcal{P}\big(z\ln(z), z\big) \\
&\underset{\substack{z\to 0\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} c z^{M_1}\big[\ln(z)\big]^{M_2}\big[1 + o(1)\big], \quad M_1 \in \mathbb{N},\ M_2 \in \mathbb{N}_0,\ c \in \mathbb{C}.
\end{aligned} \tag{10.86}$$

Setting $z = \varepsilon e^{i\varphi}$, $\varphi \in [0, \pi]$, and letting $\varepsilon \downarrow 0$ in (10.86) then readily yields

$$\mathrm{Im}\Big(\ln\Big(\mathcal{P}\big(z\ln(z), z\big)\Big)\Big) \underset{\substack{z\to 0\\ z\in\mathbb{R}\backslash\{0\}}}{=} \mathrm{Im}\big(\ln(c)\big) + \begin{cases} 0, & \varphi = 0,\\ M_1\pi, & \varphi = \pi,\end{cases}$$

and hence proves the claim (10.80). ∎