Chapter 11

Analysis of G_{H,H_0}

In this chapter, we analyze $G_{H,H_0}(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, and its limiting behavior on $\mathbb{R}$.

One recalls from (9.14) (with $S = H$, $S_0 = H_0$, and $r = n$, cf. Remark 8.4 (iii)), that G_{H,H_0} is of the form

$$\frac{d^n}{dz^n} G_{H,H_0}(z) = \operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}\bigg(\frac{d^{n-1}}{dz^{n-1}} \sum_{j=0}^{n-1} (-1)^{n-j} (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} A(z)^{n-j}\bigg), \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad (11.1)$$

where

$$A(z) = V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

To analyze the trace in (11.1), we use multi-indices (see (9.16) and (9.17)). For each fixed $j \in \mathbb{N}_0$ with $0 \leq j \leq n-1$,

$$\begin{aligned} &\frac{d^{n-1}}{dz^{n-1}} (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1} A(z)^{n-j} \\ &\quad = \sum_{\substack{\underline{k} \in \mathbb{N}_0^{n-j+1} \\ |\underline{k}| = n-1}} c_{j,\underline{k}} (H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_1+1)} \\ &\qquad \times \prod_{\ell=2}^{n-j+1} V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_\ell+1)}, \end{aligned} \quad (11.2)$$

for an appropriate set of z-independent scalars

$$c_{j,\underline{k}} \in \mathbb{R}, \quad \underline{k} \in \mathbb{N}_0^{n-j+1}, \ |\underline{k}| = n-1.$$

Therefore, applying the cyclicity property of the trace, one infers

$$\begin{aligned} \frac{d^n}{dz^n} G_{H,H_0}(z) &= \sum_{j=0}^{n-1} (-1)^{n-j} \sum_{\substack{\underline{k} \in \mathbb{N}_0^{n-j+1} \\ |\underline{k}| = n-1}} c_{j,\underline{k}} \\ &\quad \times \operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}\bigg(\bigg[\prod_{\ell=2}^{n-j} V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_\ell+1)}\bigg] \\ &\qquad \times V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_1+k_{n-j+1}+2)}\bigg), \end{aligned} \quad (11.3)$$

and hence it suffices to analyze the trace

$$
\begin{aligned}
&\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}\Bigg(\Bigg[\prod_{\ell=2}^{n-j} V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_\ell+1)}\Bigg] \\
&\qquad\qquad \times V(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(k_1+k_{n-j+1}+2)}\Bigg) \\
&\quad = \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1)\frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z;x_1,x_2)\bigg] \\
&\qquad \cdots \times V(x_{n-j-1})\frac{1}{k_{n-j}!}\bigg[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z;x_{n-j-1},x_{n-j})\bigg] \\
&\qquad \cdots \times V(x_{n-j})\frac{1}{(k_1+k_{n-j+1}+1)!}\bigg[\frac{\partial^{k_1+k_{n-j+1}+1}}{\partial z^{k_1+k_{n-j+1}+1}} G_0(z;x_{n-j},x_1)\bigg], \\
&\qquad\qquad z \in \mathbb{C}_+,\ \underline{k} \in \mathbb{N}_0^{n-j+1},\ |\underline{k}| = n-1,\ 0 \le j \le n-1, \qquad (11.4)
\end{aligned}
$$

and its properties as $\operatorname{Im}(z) \downarrow 0$.

Here we employed the fact that the integral kernel of

$$
(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-(1+s)} = \frac{1}{s!}\frac{d^s}{dz^s}(H_0 - zI_{[L^2(\mathbb{R}^n)]^N})^{-1}, \quad z \in \mathbb{C}_+,\ s \in \mathbb{N}_0,
$$

is of the form

$$
\frac{1}{s!}\frac{\partial^s}{\partial z^s} G_0(z;x,y), \quad z \in \mathbb{C}_+,\ s \in \mathbb{N}_0,\ x,y \in \mathbb{R}^n,\ x \neq y. \tag{11.5}
$$

Next, we recall the asymptotic relations proved in Appendix C and the estimates (C.30), (C.31). In particular, the estimates (C.30) and (C.31) as $|z||x-y| \ge 1$ necessitate the following strengthening of the estimate (10.77) in Hypothesis 10.16:

Hypothesis 11.1. *Let $n \in \mathbb{N}$ and suppose that $V = \{V_{\ell,\ell'}\}_{1\le \ell,\ell' \le N}$ satisfies for some constant $C \in (0,\infty)$ and $\varepsilon > 0$,*

$$
\begin{aligned}
&V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \\
&|V_{\ell,\ell'}(x)| \le C\langle x\rangle^{-n-1-\varepsilon} \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \le \ell,\ell' \le N. \qquad (11.6)
\end{aligned}
$$

This yields the following result.

Theorem 11.2. *Assume Hypothesis 11.1,*

(i) *Let $n \in \mathbb{N}$ be odd, $n \ge 3$. Then $\frac{d^n}{dz^n} G_{H,H_0}(\cdot)$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$.*

(ii) *Let $n \in \mathbb{N}$ be even. Then $\frac{d^n}{dz^n} G_{H,H_0}(\cdot)$ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}\backslash\{0\}$. If $n \geq 4$, then*

$$\left\| \frac{d^n}{dz^n} G_{H,H_0}(\cdot) \right\|_{\mathcal{B}(\mathbb{C}^N)} \underset{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{-[n-(n/(n-1))]}\big).$$

If $n = 2$, then for any $\delta \in (0, 1)$,

$$\left\| \frac{d^2}{dz^2} G_{H,H_0}(\cdot) \right\|_{\mathcal{B}(\mathbb{C}^2)} \underset{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{-(1+\delta)}\big).$$

Proof. By Lemma 9.5 it suffices to focus on the boundary values of $\frac{d^n}{dz^n} G_{H,H_0}(z)$ as $\operatorname{Im}(z) \downarrow 0$. Utilizing the asymptotic relations (C.6), (C.10), (C.11), (C.12), (C.16), (C.19), (C.20), (C.21), (C.24), (C.28), (C.29), and the fact that $\frac{\partial^k}{\partial z^k} G_0(z; x, y)$, $k \in \mathbb{N}_0$, $0 \leq k \leq n$, is continuous in $z \in \overline{\mathbb{C}_+}$, $x, y \in \mathbb{R}^n$, $x \neq y$, the stated continuity of $\frac{d^n}{dz^n} G_{H,H_0}(\cdot)$ in $\overline{\mathbb{C}_+}$ follows once we derive a z-independent integrable majorant of the integrand in (11.4), appealing to Lebesgue's dominated convergence theorem.

(i) Specializing to $n \in \mathbb{N}$ odd, $n \geq 3$, and employing (11.6) and (C.30), one obtains from (11.4),

$$\begin{aligned}
&\bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) \frac{1}{k_2!} \bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2) \bigg] \\
&\quad \cdots \times V(x_{n-j-1}) \frac{1}{k_{n-j}!} \bigg[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z; x_{n-j-1}, x_{n-j}) \bigg] \\
&\quad \cdots \times V(x_{n-j}) \frac{1}{(k_1 + k_{n-j+1} + 1)!} \bigg[\frac{\partial^{k_1+k_{n-j+1}+1}}{\partial z^{k_1+k_{n-j+1}+1}} G_0(z; x_{n-j}, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq C_n \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2+1-n} \chi_{[0,1]}\big(|z||x_1 - x_2|\big) \\
&\quad + |z|^{(n-1)/2} \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2} \big] \chi_{[1,\infty)}\big(|z||x_1 - x_2|\big) \big\} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-n} \chi_{[0,1]}\big(|z||x_{n-j-1} - x_{n-j}|\big) \\
&\quad + |z|^{(n-1)/2} \big[|x_{n-j-1}|^{(2k_{n-j}+1-n)/2} + |x_{n-j}|^{(2k_{n-j}+1-n)/2} \big] \\
&\quad \times \chi_{[1,\infty)}\big(|z||x_{n-j-1} - x_{n-j}|\big) \big\} \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-n} \chi_{[0,1]}\big(|z||x_{n-j} - x_1|\big) \\
&\quad + |z|^{(n-1)/2} \big[|x_{n-j}|^{(2k_1+2k_{n-j+1}+3-n)/2} + |x_1|^{(2k_1+2k_{n-j+1}+3-n)/2} \big] \\
&\quad \times \chi_{[1,\infty)}\big(|z||x_{n-j} - x_1|\big) \big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \widetilde{C}_n \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2+1-n} \\
&\quad + |z|^{(n-1)/2} \big[1 + |x_1|\big]^{(2k_2+1-n)/2} \big[1 + |x_2|\big]^{(2k_2+1-n)/2} \big\} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-n} \\
&\quad + |z|^{(n-1)/2} \big[1 + |x_{n-j-1}|\big]^{(2k_{n-j}+1-n)/2} \big[1 + |x_{n-j}|\big]^{(2k_{n-j}+1-n)/2} \big\} \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-n} \\
&\quad + |z|^{(n-1)/2} \big[1 + |x_{n-j}|\big]^{(2k_1+2k_{n-j+1}+3-n)/2} \\
&\quad \times \big[1 + |x_1|\big]^{(2k_1+2k_{n-j+1}+3-n)/2} \big\}, \\
&\qquad z \in \mathbb{C}_+, \; \underline{k} \in \mathbb{N}_0^{n-j+1}, \; |\underline{k}| = n-1, \; 0 \leq j \leq n-2,
\end{aligned} \tag{11.7}
$$

where $C_n, \widetilde{C}_n \in (0,\infty)$ are suitable constants and we removed all characteristic functions in the last step (a very crude estimate, but sufficient for our purpose).

We postpone a discussion of the case $j = n-1$ to the end of the proof of part (i).

Next, one notes that all terms originally multiplied by an "exterior" characteristic function $\chi_{[1,\infty)}(|z||\cdot|)$, that is, all terms of the type

$$
\begin{aligned}
&|z|^{(n-1)/2} \big[1 + |x_{n-j-1}|\big]^{(2k_{n-j}+1-n)/2} \big[1 + |x_{n-j}|\big]^{(2k_{n-j}+1-n)/2}, \dots, \\
&|z|^{(n-1)/2} \big[1+|x_{n-j}|\big]^{(2k_1+2k_{n-j+1}+3-n)/2} \big[1+|x_1|\big]^{(2k_1+2k_{n-j+1}+3-n)/2},
\end{aligned} \tag{11.8}
$$

can be grouped together with

$$
\langle x_{n-j-1} \rangle^{-n-1-\varepsilon}, \langle x_{n-j} \rangle^{-n-1-\varepsilon}, \dots, \langle x_{n-j} \rangle^{-n-1-\varepsilon}, \langle x_1 \rangle^{-n-1-\varepsilon},
$$

due to the decay assumptions imposed in (11.6), and hence we can simply disregard all these contributions in the following as they lead to finite integrals. To illustrate this fact we look at the extreme case where only these terms are considered. Indeed, ignoring all numerical constants and the factors $|z|^{(n-1)/2}$ for simplicity, this leads to the integral,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big[1 + |x_1|\big]^{(2k_2+1-n)/2} \big[1 + |x_2|\big]^{(2k_2+1-n)/2} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big[1 + |x_{n-j-1}|\big]^{(2k_{n-j}+1-n)/2} \big[1 + |x_{n-j}|\big]^{(2k_{n-j}+1-n)/2} \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |x_{n-j}|\big]^{(2k_1+2k_{n-j+1}+3-n)/2} \\
&\quad \times \big[1 + |x_1|\big]^{(2k_1+2k_{n-j+1}+3-n)/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^n} d^n x_1 \, \langle x_1 \rangle^{-n-1-\varepsilon} \big[1 + |x_1|\big]^{k_1+k_2+k_{n-j+1}+2-n} \\
&\quad \times \int_{\mathbb{R}^n} d^n x_2 \, \langle x_2 \rangle^{-n-1-\varepsilon} \big[1 + |x_2|\big]^{k_2+k_3+1-n} \\
&\quad \vdots \\
&\quad \times \int_{\mathbb{R}^n} d^n x_{n-j-1} \, \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big[1 + |x_{n-j-1}|\big]^{k_{n-j-1}+k_{n-j}+1-n} \\
&\quad \times \int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |x_{n-j}|\big]^{k_1+k_{n-j}+k_{n-j+1}+2-n} \\
&\leq \int_{\mathbb{R}^n} d^n x_1 \, \langle x_1 \rangle^{-n-1-\varepsilon} \big[1 + |x_1|\big] \bigg[\int_{\mathbb{R}^n} d^n y \langle y \rangle^{-n-1-\varepsilon} \bigg]^{n-j-2} \\
&\quad \times \int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |x_{n-j}|\big] \\
&< \infty, \quad \underline{k} \in \mathbb{N}_0^{n-j+1}, \ |\underline{k}| = n-1, \ 0 \leq j \leq n-2,
\end{aligned} \tag{11.9}
$$

employing (11.6).

Thus, without loss of generality, we now focus on the terms originally multiplied by an “interior” characteristic function $\chi_{[0,1]}(|z|| \cdot |)$ and hence arrive at the need to estimate the integral

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_1 \rangle^{-n-1-\varepsilon} |x_1 - x_2|^{k_2+1-n} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-n} \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-n}, \\
&\quad \underline{k} \in \mathbb{N}_0^{n-j+1}, \ |\underline{k}| = n-1, \ 0 \leq j \leq n-2.
\end{aligned} \tag{11.10}
$$

For this purpose we recall the following special case of Lemma 6.4,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n y \, |y_1 - y|^{\alpha-n} \langle y \rangle^{-\gamma-\varepsilon} |y - y_2|^{\beta-n} \\
&\leq \widetilde{C}_{n,\alpha,\beta,\gamma,\varepsilon} \begin{cases} |y_1 - y_2|^{\min(n,\alpha+\beta)-n}, & |y_1 - y_2| \leq 1, \\ |y_1 - y_2|^{\max(\alpha,\beta)-n}, & |y_1 - y_2| \geq 1, \end{cases} \\
&\leq \widetilde{C}_{n,\alpha,\beta,\gamma,\varepsilon} \begin{cases} |y_1 - y_2|^{\min(n,\alpha+\beta)-n}, & |y_1 - y_2| \leq 1, \\ 1, & |y_1 - y_2| \geq 1, \end{cases} \\
&\leq C_{n,\alpha,\beta,\gamma,\varepsilon} \big[|y_1 - y_2|^{\min(n,\alpha+\beta)-n} + 1 \big], \\
&\qquad \alpha, \beta \in (0, n], \ \gamma > (\alpha + \beta) - n, \ \varepsilon > 0,
\end{aligned} \tag{11.11}
$$

for appropriate constants $\widetilde{C}_{n,\alpha,\beta,\gamma,\varepsilon}, C_{n,\alpha,\beta,\gamma,\varepsilon} \in (0, \infty)$.

Hence,

$$\int_{\mathbb{R}^n} d^n x_2\, |x_1 - x_2|^{k_2+1-n} \langle x_2\rangle^{-n-1-\varepsilon} |x_2 - x_3|^{k_3+1-n}$$
$$\leq c_{2,n}\big[|x_1 - x_3|^{\min(n,k_2+k_3+2)-n} + 1\big], \qquad (11.12)$$

for some $c_{2,n} \in (0,\infty)$. For precisely the same reason as in the context of (11.9), we will simply disregard the additive term $+1$ on the right-hand side of (11.12) as the latter is bounded and we want to focus on the possibly most singular contribution to the integral in (11.10) when probing whether or not this integral is finite.

Thus, with these simplifications of ignoring 1's and at the same time focusing on the possibly most singular contribution, the next integral over x_3 becomes

$$\int_{\mathbb{R}^n} d^n x_3\, |x_1 - x_3|^{k_2+k_3+2-n} \langle x_3\rangle^{-n-1-\varepsilon} |x_3 - x_4|^{k_4+1-n}$$
$$\leq c_{3,n}\big[|x_1 - x_4|^{\min(n,k_2+k_3+k_4+3)-n} + 1\big], \qquad (11.13)$$

for some $c_{3,n} \in (0,\infty)$. Repeating this process (again disregarding 1's at each step and focusing on the possibly most singular contributions only) leads to

$$\int_{\mathbb{R}^n} d^n x_{n-j-1}\, |x_1 - x_{n-j-1}|^{k_2+k_3+\cdots+k_{n-j-1}+(n-j-2)-n} \langle x_{n-j-1}\rangle^{-n-1-\varepsilon}$$
$$\times |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-n}$$
$$\leq c_{n-j-1,n}\big[|x_1 - x_{n-j}|^{\min(n,k_2+k_3+\cdots+k_{n-j}+(n-j-1))-n} + 1\big], \qquad (11.14)$$

for some $c_{n-j-1,n} \in (0,\infty)$. Thus, disregarding once more the additive constant $+1$ in (11.14) results in the following integral over x_{n-j}, $0 \leq j \leq n-2$,

$$\int_{\mathbb{R}^n} d^n x_{n-j} \left[\begin{cases} |x_1 - x_{n-j}|^{k_2+k_3+\cdots+k_{n-j}-j-1}, & \big(\sum_{q=2}^{n-j} k_q\big) - j - 1 \leq 0, \\ 1, & \big(\sum_{q=2}^{n-j} k_q\big) - j - 1 \geq 0 \end{cases}\right]$$
$$\times \langle x_{n-j}\rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-n}$$
$$\leq c \int_{\mathbb{R}^n} d^n x_{n-j}\, \langle x_{n-j}\rangle^{-n-1-\varepsilon} |x_1 - x_{n-j}|^{k_1+k_2+\cdots+k_{n-j+1}-j+1-n}$$
$$+ d \int_{\mathbb{R}^n} d^n x_{n-j}\, \langle x_{n-j}\rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-n}$$
$$\leq C \int_{\mathbb{R}^n} d^n x_{n-j}\, \langle x_{n-j}\rangle^{-n-1-\varepsilon}\big[|x_1 - x_{n-j}|^{-j} + |x_1 - x_{n-j}|^{-m}\big]$$
$$\text{for some } -1 \leq m \leq n-2, \text{ where } m = k_1 + k_{n-j+1} + 2 - n, \qquad (11.15)$$

for appropriate $c, d, C \in (0,\infty)$, employing again that

$$\underline{k} = (k_1, \dots, k_{n-j+1}) \in \mathbb{N}_0^{n-j+1}, \quad |\underline{k}| = k_1 + k_2 + \cdots k_{n-j+1} = n - 1.$$

At this point we invoke the special case $\alpha = n$ in (11.11), resulting in

$$\int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[|x_1 - x_{n-j}|^{-j} + |x_1 - x_{n-j}|^{-m}\big] \leq C_{j,m},$$
$$0 \leq j \leq n-2, \; 0 \leq m \leq n-2, \quad (11.16)$$

for some $C_{j,m} \in (0, \infty)$. The remaining case $m = -1$ in (11.15) leads to

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_{n-j} \rangle^{-n-1-\varepsilon} |x_1 - x_{n-j}| \\ &\quad \leq \int_{\mathbb{R}^n} d^n x_{n-j} \, \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |x_1| + |x_{n-j}|\big] \\ &\quad \leq C_n + D_n\big[1 + |x_1|\big], \end{aligned} \quad (11.17)$$

for some $C_n, D_n \in (0, \infty)$, since

$$\int_{\mathbb{R}^n} d^n y \, \langle y \rangle^{-n-1-\varepsilon} \big[1 + |y|\big] < \infty. \quad (11.18)$$

Thus, altogether, (11.15)–(11.17) finally yield

$$(11.15) \leq C_0\big[1 + |x_1|\big], \quad 0 \leq j \leq n-2, \quad (11.19)$$

for appropriate $C_0 \in (0, \infty)$. Hence, applying (11.18) once more, the integral (11.10) is finite.

If $j = n-1$ in (11.2), (11.3) one is left to consider $\underline{k} = (k_1, k_2)$, $|\underline{k}| = k_1 + k_2 = n-1$, and hence obtains

$$\begin{aligned} &\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N} \bigg(\frac{d^{n-1}}{dz^{n-1}} (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} A(z) \bigg) \\ &\quad = \operatorname{tr}_{[L^2(\mathbb{R}^n)]^N} \bigg(\frac{d^{n-1}}{dz^{n-1}} (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} V (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-1} \bigg) \\ &\quad = \operatorname{tr}_{[L^2(\mathbb{R}^n)]^N} \Big(V (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-(k_1+k_2+2)} \Big) \\ &\quad = \operatorname{tr}_{[L^2(\mathbb{R}^n)]^N} \Big(V (H_0 - z I_{[L^2(\mathbb{R}^n)]^N})^{-(n+1)} \Big), \quad z \in \mathbb{C}_+. \end{aligned} \quad (11.20)$$

Since by (C.7)–(C.9)

$$\frac{d^n}{dz^n} G_0(z; x, y) \underset{|x-y| \to 0}{=} O\big(|x-y|\big), \quad (11.21)$$

the trace in (11.17) vanishes and hence extends continuously to $z \in \overline{\mathbb{C}_+}$.

(ii) Next, we specialize to $n \in \mathbb{N}$ even. We investigate each term in (11.3) separately. To this end, let $0 \leq j \leq n-1$ and $\underline{k} \in \mathbb{N}_0^{n-j+1}$ with $|\underline{k}| = n-1$ be fixed. We distinguish the following cases:

Case 1. $n \geq 4$ with $0 \leq j \leq n-3$ and $k_1 + k_{n-j+1} \neq n-1$.

Case 2. $n \geq 4$ with $0 \leq j \leq n-3$ and $k_1 + k_{n-j+1} = n-1$.

Case 3. $n \geq 2$ with $j = n-2$ and $k_1 + k_3 \neq n-1$.

Case 4. $n \geq 2$ with $j = n-2$ and $k_1 + k_3 = n-1$.

Case 5. $n \geq 2$ with $j = n-1$.

We begin with *Case 1*. The assumptions in *Case 1* imply

$$0 \leq k_\ell \leq n-1 \text{ for all } 2 \leq \ell \leq n-j \text{ and } k_1 + k_{n-j+1} + 1 < n. \tag{11.22}$$

Define the quantity $\delta = \delta(n, j)$ by

$$\delta := \frac{n-j-2}{n-j-1}, \tag{11.23}$$

so that $\delta \in (0, 1)$ and

$$\delta \geq \frac{n-j-\ell}{n-j-\ell+1}, \quad 2 \leq \ell \leq n-j-1. \tag{11.24}$$

Invoking the final estimate in (C.31), one obtains

$$\begin{aligned}
&\bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) \frac{1}{k_2!}\left[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2)\right] \\
&\quad \cdots \times V(x_{n-j-1}) \frac{1}{k_{n-j}!}\left[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z; x_{n-j-1}, x_{n-j})\right] \\
&\quad \times V(x_{n-j}) \frac{1}{(k_1 + k_{n-j+1} + 1)!}\left[\frac{\partial^{k_1 + k_{n-j+1}+1}}{\partial z^{k_1 + k_{n-j+1}+1}} G_0(z; x_{n-j}, x_1)\right] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq C_{n,j,\delta} |z|^{-(n-j)\delta} \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2 + 1 - \delta - n} \chi_{[0,1]}\big(|z||x_1 - x_2|\big) \\
&\quad + |z|^{(n-1+2\delta)/2} \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2} \big] \chi_{[1,\infty)}\big(|z||x_1 - x_2|\big) \big\} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \chi_{[0,1]}\big(|z||x_{n-j-1} - x_{n-j}|\big) \\
&\quad + |z|^{(n-1+2\delta)/2} \big[|x_{n-j-1}|^{(2k_{n-j}+1-n)/2} + |x_{n-j}|^{(2k_{n-j}+1-n)/2} \big] \\
&\quad \times \chi_{[1,\infty)}\big(|z||x_{n-j-1} - x_{n-j}|\big) \big\} \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big\{ |x_{n-j} - x_1|^{k_1 + k_{n-j+1} + 2 - \delta - n} \chi_{[0,1]}\big(|z||x_{n-j} - x_1|\big) \\
&\quad + |z|^{(n-1+2\delta)/2} \big[|x_{n-j}|^{(2k_1 + 2k_{n-j+1}+3-n)/2} + |x_1|^{(2k_1+2k_{n-j+1}+3-n)/2} \big] \\
&\quad \times \chi_{[1,\infty)}\big(|z||x_{n-j} - x_1|\big) \big\}
\end{aligned}$$

$$\leq \widetilde{C}_{n,j,\delta}|z|^{-(n-j)\delta}\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}$$
$$\times \langle x_1\rangle^{-n-1-\varepsilon}\big\{|x_1-x_2|^{k_2+1-\delta-n}$$
$$+ |z|^{(n-1+2\delta)/2}\big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2}\big]\big\}$$
$$\vdots$$
$$\times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon}\big\{|x_{n-j-1}-x_{n-j}|^{k_{n-j}+1-\delta-n}$$
$$+ |z|^{(n-1+2\delta)/2}\big[|x_{n-j-1}|^{(2k_{n-j}+1-n)/2} + |x_{n-j}|^{(2k_{n-j}+1-n)/2}\big]\big\}$$
$$\times \langle x_{n-j}\rangle^{-n-1-\varepsilon}\big\{|x_{n-j}-x_1|^{k_1+k_{n-j+1}+2-\delta-n}$$
$$+ |z|^{(n-1+2\delta)/2}\big[|x_{n-j}|^{(2k_1+2k_{n-j+1}+3-n)/2} + |x_1|^{(2k_1+2k_{n-j+1}+3-n)/2}\big]\big\},$$
$$z \in \mathbb{C}_+, \quad (11.25)$$

where $C_{n,j,\delta}, \widetilde{C}_{n,j,\delta} \in (0,\infty)$ are suitable constants and we removed all characteristic functions in the last step (again, a very crude estimate, but sufficient for our purpose).

We claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.25) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables $x_1,\dots,x_{n-j}$. Since $|z|^{(n-1+2\delta)/2}$ is locally bounded, to justify the claim, it suffices to establish convergence of the following integral:

$$\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}$$
$$\times \langle x_1\rangle^{-n-1-\varepsilon}\big\{|x_1-x_2|^{k_2+1-\delta-n}$$
$$+ \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2}\big]\big\}$$
$$\vdots$$
$$\times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon}\big\{|x_{n-j-1}-x_{n-j}|^{k_{n-j}+1-\delta-n}$$
$$+ \big[|x_{n-j-1}|^{(2k_{n-j}+1-n)/2} + |x_{n-j}|^{(2k_{n-j}+1-n)/2}\big]\big\}$$
$$\times \langle x_{n-j}\rangle^{-n-1-\varepsilon}\big\{|x_{n-j}-x_1|^{k_1+k_{n-j+1}+2-\delta-n}$$
$$+ \big[|x_{n-j}|^{(2k_1+2k_{n-j+1}+3-n)/2} + |x_1|^{(2k_1+2k_{n-j+1}+3-n)/2}\big]\big\}. \quad (11.26)$$

In turn, as in the argument for the proof of part (i), it suffices to focus on the most singular term in (11.26) and thus disregard the terms originally multiplied by the factor $|z|^{(n-1+2\delta)/2}$ in (11.25) (following the same line of reasoning used throughout (11.8)–(11.9)). With this simplification, the claim reduces to establishing convergence of the integral

$$\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}$$
$$\times \langle x_1\rangle^{-n-1-\varepsilon}|x_1-x_2|^{k_2+1-\delta-n}$$
$$\vdots$$

$$\times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon}|x_{n-j-1}-x_{n-j}|^{k_{n-j}+1-\delta-n}$$
$$\times \langle x_{n-j}\rangle^{-n-1-\varepsilon}|x_{n-j}-x_1|^{k_1+k_{n-j+1}+2-\delta-n}. \tag{11.27}$$

The integrals over the inner variables $x_2,\ldots,x_{n-j-1}$ in (11.27) can be estimated successively as follows. Beginning with the integral with respect to x_2, an application of (11.11) with the choices

$$\alpha = k_2+1-\delta, \quad \beta = k_3+1-\delta, \quad \gamma = n+1, \tag{11.28}$$

implies

$$\begin{aligned}&\int d^n x_2\,|x_1-x_2|^{k_2+1-\delta-n}\langle x_2\rangle^{-n-1-\varepsilon}|x_2-x_3|^{k_3+1-\delta-n}\\ &\le c_{2,n,\delta}\big[|x_1-x_3|^{k_2+k_3+2(1-\delta)-n}+1\big],\end{aligned} \tag{11.29}$$

for some $c_{2,n,\delta}\in(0,\infty)$. The conditions on α and β in (11.11) are satisfied by the choices in (11.28) since (11.22) implies

$$0\le k_2+1-\delta\le n \quad \text{and} \quad 0<k_3+1-\delta\le n,$$

together with

$$(\alpha+\beta)-n = k_2+k_3+2(1-\delta)-n \le n-2\delta < n+1=\gamma. \tag{11.30}$$

The inequality in (11.24) with $\ell = n-j-1$ implies $1-2\delta\le 0$, so that

$$k_2+k_3+2(1-\delta)\le n-1+2(1-\delta) = n+1-2\delta\le n,$$

which yields

$$\min(n,\alpha+\beta) = \min\big(n,k_2+k_3+2(1-\delta)\big) = k_2+k_3+2(1-\delta),$$

and the estimate in (11.29) follows. If $j=n-3$, then x_2 is the only inner variable, and the integration over the inner variables is complete with (11.29). For $j\le n-4$ the process continues and there are $n-j-3$ remaining inner integrals to estimate. Applying (11.29) in (11.27), the next inner integral is with respect to x_3:

$$\begin{aligned}&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3\,|x_1-x_2|^{k_2+1-\delta-n}\langle x_2\rangle^{-n-1-\varepsilon}|x_2-x_3|^{k_3+1-\delta-n}\\ &\quad\times\langle x_3\rangle^{-n-1-\varepsilon}|x_3-x_4|^{k_4+1-\delta-n}\\ &\le c_{2,n,\delta}\int_{\mathbb{R}^n} d^n x_3\,\big[|x_1-x_3|^{k_2+k_3+2(1-\delta)-n}+1\big]\langle x_3\rangle^{-n-1-\varepsilon}|x_3-x_4|^{k_4+1-\delta-n}\\ &= c_{2,n,\delta}\bigg[\int_{\mathbb{R}^n} d^n x_3\,|x_1-x_3|^{k_2+k_3+2(1-\delta)-n}\langle x_3\rangle^{-n-1-\varepsilon}|x_3-x_4|^{k_4+1-\delta-n}\\ &\quad+\int_{\mathbb{R}^n} d^n x_3\,\langle x_3\rangle^{-n-1-\varepsilon}|x_3-x_4|^{k_4+1-\delta-n}\bigg]\\ &=: c_{2,n,\delta}\big[\mathcal{I}_1(x_1,x_4)+\mathcal{I}_2(x_4)\big].\end{aligned} \tag{11.31}$$

An application of (11.11) with the choices

$$\alpha = n, \quad \beta = k_4 + 1 - \delta, \quad \gamma = n + 1,$$

immediately yields (note that in this case $\min(n, \alpha + \beta) = n$)

$$\mathcal{I}_2(x_4) \le c''_{3,n,\delta}, \quad x_4 \in \mathbb{R}^n, \tag{11.32}$$

for some $c''_{3,n,\delta} \in (0, \infty)$. Another application of (11.11), this time with the choices

$$\alpha = k_2 + k_3 + 2(1 - \delta), \quad \beta = k_4 + 1 - \delta, \quad \gamma = n + 1, \tag{11.33}$$

implies

$$\mathcal{I}_1(x_1, x_4) \le c'_{3,n,\delta}\big[|x_1 - x_4|^{k_2+k_3+k_4+3(1-\delta)-n} + 1\big] \tag{11.34}$$

for some $c'_{3,n,\delta} \in (0, \infty)$. The conditions on α and β in (11.11) are satisfied by the choices in (11.33) since (cf. (11.30))

$$0 < k_2 + k_3 + 2(1 - \delta) \le n - 1 + 2 - 2\delta = n + 1 - 2\delta \le n$$

and

$$0 < k_4 + 1 - \delta \le n - \delta \le n$$

together with

$$\begin{aligned}(\alpha + \beta) - n &= \underbrace{k_2 + k_3}_{\le n-1} + \underbrace{k_4}_{\le n-1} + 3(1 - \delta) - n \le 2(n - 1) + 3(1 - \delta) - n \\ &= n + 1 - 3\delta < n + 1 = \gamma.\end{aligned}$$

The inequality in (11.24) with $\ell = n - j - 2$ implies $2 - 3\delta \le 0$, so that

$$k_2 + k_3 + k_4 + 3(1 - \delta) \le n - 1 + 3(1 - \delta) = n + 2 - 3\delta \le n,$$

which yields

$$\begin{aligned}\min(n, \alpha + \beta) &= \min\big(n, k_2 + k_3 + k_4 + 3(1 - \delta)\big) \\ &= k_2 + k_3 + k_4 + 3(1 - \delta),\end{aligned}$$

and the estimate in (11.34) follows. Finally, combining (11.31), (11.32), and (11.34), one obtains

$$\begin{aligned}&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3\, |x_1 - x_2|^{k_2+1-\delta-n} \langle x_2\rangle^{-n-1-\varepsilon} |x_2 - x_3|^{k_3+1-\delta-n} \\ &\quad \times \langle x_3\rangle^{-n-1-\varepsilon} |x_3 - x_4|^{k_4+1-\delta-n} \\ &\quad \le c_{3,n,\delta}\big[|x_1 - x_4|^{k_2+k_3+k_4+3(1-\delta)-n} + 1\big]\end{aligned} \tag{11.35}$$

for some $c_{3,n,\delta} \in (0,\infty)$. Continuing systematically in this way, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \\
&\quad \times |x_1 - x_2|^{k_2+1-\delta-n} \langle x_2\rangle^{-n-1-\varepsilon} |x_2 - x_3|^{k_3+1-\delta-n} \\
&\quad \times \langle x_3\rangle^{-n-1-\varepsilon} |x_3 - x_4|^{k_4+1-\delta-n} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \\
&\leq c_{n-j-2,n,\delta} \int_{\mathbb{R}^n} d^n x_{n-j-1} \big[|x_1 - x_{n-j-1}|^{k_2+\cdots+k_{n-j-1}+1(n-j-2)(1-\delta)-n} + 1\big] \\
&\quad \times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \\
&= c_{n-j-2,n,\delta} \bigg[\int_{\mathbb{R}^n} d^n x_{n-j-1}\, |x_1 - x_{n-j-1}|^{k_2+\cdots+k_{n-j-1}+1(n-j-2)(1-\delta)-n} \\
&\quad \times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \\
&\quad + \int_{\mathbb{R}^n} d^n x_{n-j-1}\, \langle x_{n-j-1}\rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \bigg] \\
&=: c_{n-j-2,n,\delta} \big[\mathcal{I}_1(x_1, x_{n-j}) + \mathcal{I}_2(x_{n-j})\big]
\end{aligned} \tag{11.36}$$

for some $c_{n-j-2,n,\delta} \in (0,\infty)$. An application of (11.11) with the choices

$$\alpha = n, \quad \beta = k_{n-j} + 1 - \delta, \quad \gamma = n + 1$$

immediately yields (note that in this case $\min(n, \alpha + \beta) = n$)

$$\mathcal{I}_2(x_{n-j}) \leq c''_{n-j-1,n,\delta}, \quad x_{n-j} \in \mathbb{R}^n, \tag{11.37}$$

for some $c''_{n-j-1,n,\delta} \in (0,\infty)$. Another application of (11.11), this time with the choices

$$\alpha = k_2 + \cdots + k_{n-j-1} + (n-j-2)(1-\delta), \quad \beta = k_{n-j} + 1 - \delta, \quad \gamma = n + 1, \tag{11.38}$$

implies

$$\mathcal{I}_1(x_1, x_{n-j}) \leq c'_{n-j-1,n,\delta} \big[|x_1 - x_{n-j}|^{k_2+\cdots+k_{n-j}+(n-j-1)(1-\delta)-n} + 1\big] \tag{11.39}$$

for some $c'_{n-j-1,n,\delta} \in (0,\infty)$. The conditions on α and β in (11.11) are satisfied by the choices in (11.38) since

$$\begin{aligned}
0 &< k_2 + \cdots + k_{n-j-1} + (n-j-2)(1-\delta) \\
&\leq n - 1 + (n-j-2)(1-\delta) \\
&= n + (n-j-3) - (n-j-2)\delta \\
&\leq n,
\end{aligned} \tag{11.40}$$

and

$$0 < k_{n-j} + 1 - \delta \leq n - \delta \leq n.$$

The final inequality in (11.40) follows by choosing $\ell = 3$ in (11.24). In addition,

$$\begin{aligned}
(\alpha + \beta) - n &= k_2 + \cdots + k_{n-j} + (n - j - 1)(1 - \delta) - n \\
&= \underbrace{k_2 + \cdots + k_{n-j}}_{\leq n-1} -(j + 1) - (n - j - 1)(1 - \delta) \\
&\leq n - 2 - j - (n - j - 2)(1 - \delta) \\
&< n + 1 \\
&= \gamma.
\end{aligned}$$

The inequality in (11.25) with $\ell = 2$ implies $(n - j - 2) - (n - j - 1)\delta \leq 0$, so that

$$\begin{aligned}
k_2 + \cdots + k_{n-j} + (n - j - 1)\delta &= k_2 + \cdots + k_{n-j} + (n - j - 1) + (n - j - 1)\delta \\
&\leq n + (n - j - 2) - (n - j - 1)\delta \\
&\leq n,
\end{aligned}$$

which yields

$$\min(n, \alpha + \beta) = k_2 + \cdots + k_{n-j} + (n - j - 1)\delta,$$

and the estimate in (11.39) follows. Finally, combining (11.36), (11.37), and (11.39), one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \\
&\quad \times |x_1 - x_2|^{k_2+1-\delta-n} \langle x_2 \rangle^{-n-1-\varepsilon} |x_2 - x_3|^{k_3+1-\delta-n} \\
&\quad \times \langle x_3 \rangle^{-n-1-\varepsilon} |x_3 - x_4|^{k_4+1-\delta-n} \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{k_{n-j}+1-\delta-n} \\
&\leq c_{n-j-1,n,\delta} \big[|x_1 - x_{n-j}|^{k_2+\cdots+k_{n-j}+(n-j-1)(1-\delta)-n} + 1 \big]
\end{aligned} \tag{11.41}$$

for some $c_{n-j-1,n,\delta} \in (0, \infty)$.

The estimate in (11.41) implies

$$\begin{aligned}
(11.27) \leq c_{n-j-1,n,\delta} &\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\times \langle x_1 \rangle^{-n-1-\varepsilon} \big[|x_1 - x_{n-j}|^{k_2+\cdots+k_{n-j}+(n-j-1)(1-\delta)-n} + 1 \big] \\
&\qquad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-\delta-n}.
\end{aligned} \tag{11.42}$$

Focusing on the integral over x_{n-j} in (11.42),

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_{n-j} \left[|x_1 - x_{n-j}|^{k_2+\cdots+k_{n-j}+(n-j-1)(1-\delta)-n} + 1\right] \\
&\qquad \times \langle x_{n-j}\rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-\delta-n} \\
&\quad = \int_{\mathbb{R}^n} d^n x_{n-j}\, |x_1 - x_{n-j}|^{(n-j)(1-\delta)-n} \langle x_{n-j}\rangle^{-n-1-\varepsilon} \\
&\qquad + \int_{\mathbb{R}^n} d^n x_{n-j}\, \langle x_{n-j}\rangle^{-n-1-\varepsilon} |x_{n-j} - x_1|^{k_1+k_{n-j+1}+2-\delta-n} \\
&\quad = \mathcal{I}_1(x_1) + \mathcal{I}_2(x_1), \qquad (11.43)
\end{aligned}$$

an application of (11.11) with the choices

$$\alpha = (n-j)(1-\delta), \quad \beta = n, \quad \gamma = n+1, \qquad (11.44)$$

yields

$$\mathcal{I}_1(x_1) \le c'_{n-j,n,\delta}, \quad x_1 \in \mathbb{R}^n, \qquad (11.45)$$

for some $c'_{n-j,n,\delta} \in (0,\infty)$. The conditions on α and β in (11.11) are satisfied by the choices in (11.44) since $\alpha = (n-j)(1-\delta), \beta = n \in (0,n]$ and

$$(\alpha+\beta) - n = \alpha = (n-j)(1-\delta) \le n < n+1 = \gamma.$$

Since $\min(n, \alpha+\beta) = n$, the estimate in (11.11) results in (11.45). To estimate $\mathcal{I}_2(\cdot)$, one applies (11.11) with the choices

$$\alpha = n, \quad \beta = k_1 + k_{n-j+1} + 2 - \delta, \quad \gamma = n+1, \qquad (11.46)$$

to obtain

$$\mathcal{I}_2(x_1) \le c''_{n-j,n,\delta}, \quad x_1 \in \mathbb{R}^n, \qquad (11.47)$$

for some $c''_{n-j,n,\delta} \in (0,\infty)$. The conditions on α and β in (11.11) are satisfied by the choices in (11.46) since $\alpha = n \in (0,n]$,

$$0 < \beta = \underbrace{k_1 + k_{n-j+1}}_{\le n-2} + 2 - \delta \le n - \delta \le n,$$

and $(\alpha+\beta) - n = \beta \le n < n+1$. In this case, $\min(n, \alpha+\beta) = n$, and (11.11) results in (11.47). Combining (11.43), (11.45), and (11.47), one obtains

$$(11.43) \le c_{n-j,n,\delta}, \quad x_1 \in \mathbb{R}^n,$$

for some $c_{n-j,n,\delta} \in (0,\infty)$. As a consequence,

$$(11.42) \le c_{n-j-1,n,\delta}\, c_{n-j,n,\delta} \int_{\mathbb{R}^n} d^n x_1\, \langle x_1\rangle^{-n-1-\varepsilon} < \infty.$$

Hence, this establishes the claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.25) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables $x_1, ..., x_{n-j}$. As a consequence of this claim, (11.25) implies that for each bounded subset $\Omega \subset \mathbb{C}_+$, the following estimate holds:

$$(11.25) \leq C_{n,j,\delta,\Omega}|z|^{-(n-j)\delta}, \quad z \in \Omega,$$

for some $C_{n,j,\delta,\Omega} \in (0, \infty)$. In summary, for *Case 1*, one has that for any bounded subset $\Omega \subset \mathbb{C}_+$,

$$\begin{aligned} &\bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) \frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2)\bigg] \\ &\quad \cdots \times V(x_{n-j-1}) \frac{1}{k_{n-j}!}\bigg[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z; x_{n-j-1}, x_{n-j})\bigg] \\ &\quad \times V(x_{n-j}) \frac{1}{(k_1 + k_{n-j+1} + 1)!}\bigg[\frac{\partial^{k_1+k_{n-j+1}+1}}{\partial z^{k_1+k_{n-j+1}+1}} G_0(z; x_{n-j}, x_1)\bigg]\bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\leq C_{n,j,\delta,\Omega}|z|^{-(n-j)\delta}, \quad z \in \Omega, \end{aligned} \tag{11.48}$$

where δ is defined by (11.23). In addition, since $|z|^{-(n-j)\delta}$ is bounded in $\overline{\Omega}$ if $0 \notin \overline{\Omega}$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) \frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2)\bigg] \\ &\quad \cdots \times V(x_{n-j-1}) \frac{1}{k_{n-j}!}\bigg[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z; x_{n-j-1}, x_{n-j})\bigg] \\ &\quad \times V(x_{n-j}) \frac{1}{(k_1 + k_{n-j+1} + 1)!}\bigg[\frac{\partial^{k_1+k_{n-j+1}+1}}{\partial z^{k_1+k_{n-j+1}+1}} G_0(z; x_{n-j}, x_1)\bigg], \\ &\qquad z \in \Omega, \end{aligned}$$

is analytic in Ω and extends continuously to $\overline{\Omega}$ if $0 \notin \overline{\Omega}$. This settles *Case 1*.

Next, we treat *Case 2*. The assumptions in *Case 2* imply

$$k_\ell = 0 \text{ for all } 2 \leq \ell \leq n-j \text{ and } k_1 + k_{n-j+1} + 1 = n. \tag{11.49}$$

Let $\delta \in (0, 1)$ be fixed. Applying the final estimate in (C.31), one obtains:

$$\begin{aligned} &\bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) \frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2)\bigg] \\ &\quad \cdots \times V(x_{n-j-1}) \frac{1}{k_{n-j}!}\bigg[\frac{\partial^{k_{n-j}}}{\partial z^{k_{n-j}}} G_0(z; x_{n-j-1}, x_{n-j})\bigg] \\ &\quad \times V(x_{n-j}) \frac{1}{(k_1 + k_{n-j+1} + 1)!}\bigg[\frac{\partial^{k_1+k_{n-j+1}+1}}{\partial z^{k_1+k_{n-j+1}+1}} G_0(z; x_{n-j}, x_1)\bigg]\bigg\|_{\mathcal{B}(\mathbb{C}^N)} \end{aligned}$$

$$
\begin{aligned}
&= \bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) G_0(z; x_1, x_2) \\
&\quad \cdots \times V(x_{n-j-1}) G_0(z; x_{n-j-1}, x_{n-j}) \\
&\quad \times V(x_{n-j}) \frac{1}{n!} \bigg[\frac{\partial^n}{\partial z^n} G_0(z; x_{n-j}, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq C_{n,j,\delta} |z|^{-(n-j-1)\delta - 1} \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big[|x_1 - x_2|^{1-\delta-n} + |z|^{(n-1+2\delta)/2} |x_1 - x_2|^{(1-n)/2} \big] \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big[|x_{n-j-1} - x_{n-j}|^{1-\delta-n} \\
&\quad + |z|^{(n-1+2\delta)/2} |x_{n-j-1} - x_{n-j}|^{(1-n)/2} \big] \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |z|^{(n+1)/2} |x_{n-j} - x_1|^{(n+1)/2} \big] \\
&\leq \widetilde{C}_{n,j,\delta} |z|^{-(n-j-1)\delta - 1} \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big[|x_1 - x_2|^{1-\delta-n} \\
&\quad + |z|^{(n-1+2\delta)/2} \big[1 + |x_1|\big]^{(1-n)/2} \big[1 + |x_2|\big]^{(1-n)/2} \big] \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big[|x_{n-j-1} - x_{n-j}|^{1-\delta-n} \\
&\quad + |z|^{(n-1+2\delta)/2} \big[1 + |x_{n-j-1}|\big]^{(1-n)/2} \big[1 + |x_{n-j}|\big]^{(1-n)/2} \big] \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + |z|^{(n+1)/2} \big[1 + |x_{n-j}|\big]^{(n+1)/2} \big[1 + |x_1|\big]^{(n+1)/2} \big], \\
&\qquad\qquad z \in \mathbb{C}_+, \quad (11.50)
\end{aligned}
$$

where $C_{n,j,\delta}, \widetilde{C}_{n,j,\delta} \in (0, \infty)$ are suitable constants. We claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.50) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables $x_1, \dots, x_{n-j}$. Since $|z|^{(n-1+2\delta)/2}$ and $|z|^{(n+1)/2}$ are locally bounded, to justify the claim, it suffices to establish convergence of the following integral:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon} \big[|x_1 - x_2|^{1-\delta-n} + \big[1 + |x_1|\big]^{(1-n)/2} \big[1 + |x_2|\big]^{(1-n)/2} \big] \\
&\quad \vdots \\
&\quad \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} \big[|x_{n-j-1} - x_{n-j}|^{1-\delta-n} \\
&\quad + \big[1 + |x_{n-j-1}|\big]^{(1-n)/2} \big[1 + |x_{n-j}|\big]^{(1-n)/2} \big] \\
&\quad \times \langle x_{n-j} \rangle^{-n-1-\varepsilon} \big[1 + \big[1 + |x_{n-j}|\big]^{(n+1)/2} \big[1 + |x_1|\big]^{(n+1)/2} \big]. \qquad (11.51)
\end{aligned}
$$

As with *Case 1*, it suffices to focus on the most singular term in (11.51) and thus disregard the terms originally multiplied by $|z|^{(n-1+2\delta)/2}$ or $|z|^{(n+1)/2}$ in (11.50) (following the same line of reasoning used throughout (11.8)–(11.9)). With this simplification, the claim reduces to establishing convergence of the integral

$$\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j} \langle x_1 \rangle^{-n-1-\varepsilon} |x_1 - x_2|^{1-\delta-n} \\ \times \cdots \times \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{1-\delta-n} \langle x_{n-j} \rangle^{-n-1-\varepsilon}. \quad (11.52)$$

In analogy to *Case 1*, one successively estimates the integrals over the inner variables $x_2, \dots, x_{n-j-1}$ in (11.52) as follows. Beginning with the integral with respect to x_2, an application of (11.11) with the choices

$$\alpha = \beta = 1 - \delta, \quad \gamma = n + 1, \quad (11.53)$$

yields

$$\int_{\mathbb{R}^n} d^n x_2 \, |x_1 - x_2|^{1-\delta-n} \langle x_2 \rangle^{-n-1-\varepsilon} |x_2 - x_3|^{1-\delta-n} \\ \leq c_{2,n} \big[|x_1 - x_3|^{2(1-\delta)-n} + 1 \big]. \quad (11.54)$$

The assumptions on α and β in (11.11) are satisfied by the choices in (11.53). In fact,

$$\alpha = \beta = 1 - \delta \in (0, n] \quad \text{and} \quad (\alpha + \beta) - n = 2(1 - \delta) - n < n + 1 = \gamma.$$

Finally, $\min(n, 2(1-\delta)) = 2(1-\delta)$ and (11.11) results in (11.54). If $j = n - 3$, then x_2 is the only inner variable, and the integration over the inner variables is complete with (11.54). For $j \leq n - 4$ the process continues and there are $n - j - 3$ remaining inner integrals to estimate. Applying (11.54) in (11.52), the next inner integral is with respect to x_3:

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3 \, |x_1 - x_2|^{1-\delta-n} \langle x_2 \rangle^{-n-1-\varepsilon} |x_2 - x_3|^{1-\delta-n} \\
&\quad \times \langle x_3 \rangle^{-n-1-\varepsilon} |x_3 - x_4|^{1-\delta-n} \\
&\leq c_{2,n,\delta} \int_{\mathbb{R}^n} d^n x_3 \Big[|x_1 - x_3|^{2(1-\delta)-n} + 1 \Big] \langle x_3 \rangle^{-n-1-\varepsilon} |x_3 - x_4|^{1-\delta-n} \\
&= c_{2,n,\delta} \bigg[\int_{\mathbb{R}^n} d^n x_3 \, |x_1 - x_3|^{2(1-\delta)-n} \langle x_3 \rangle^{-n-1-\varepsilon} |x_3 - x_4|^{1-\delta-n} \\
&\quad + \int_{\mathbb{R}^n} d^n x_3 \, \langle x_3 \rangle^{-n-1-\varepsilon} |x_3 - x_4|^{1-\delta-n} \bigg] \\
&=: c_{2,n,\delta} [\mathcal{I}_1(x_1, x_4) + \mathcal{I}_2(x_4)] \qquad (11.55)
\end{aligned}$$

for some $c_{2,n,\delta} \in (0,\infty)$. An application of (11.11) with the choices

$$\alpha = 2(1-\delta), \quad \beta = 1-\delta, \quad \gamma = n+1, \tag{11.56}$$

yields

$$\mathcal{I}_1(x_1,x_4) \le c'_{3,n,\delta}\big[|x_1-x_4|^{3(1-\delta)-n}+1\big] \tag{11.57}$$

for some $c'_{3,n,\delta} \in (0,\infty)$. With the choices in (11.56), it is clear that $\alpha,\beta \in (0,n]$ and $(\alpha+\beta)-n = 3(1-\delta)-n < n+1$ since

$$n+1-\big[3(1-\delta)-n\big] = 2(n-1)+3\delta > 0,$$

so the assumptions on α and β in (11.11) are satisfied. Finally, $\min(n, 3(1-\delta)) = 3(1-\delta)$, and (11.11) results in (11.57). A second application of (11.11), this time with the choices

$$\alpha = n, \quad \beta = 1-\delta, \quad \gamma = n+1$$

yields

$$\mathcal{I}_2(x_4) \le c''_{3,n,\delta}, \quad x_4 \in \mathbb{R}^n, \tag{11.58}$$

for some $c''_{3,n,\delta} \in (0,\infty)$. As a result, (11.55), (11.57), and (11.58) imply

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \int_{\mathbb{R}^n} d^n x_3\, |x_1-x_2|^{1-\delta-n}\langle x_2\rangle^{-n-1-\varepsilon}|x_2-x_3|^{1-\delta-n} \\
&\quad \times \langle x_3\rangle^{-n-1-\varepsilon}|x_3-x_4|^{1-\delta-n} \\
&\le c_{3,n,\delta}\big[|x_1-x_4|^{3(1-\delta)-n}+1\big]
\end{aligned} \tag{11.59}$$

for some $c_{3,n,\delta} \in (0,\infty)$. Continuing systematically in this way, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} |x_1-x_2|^{1-\delta-n}\langle x_2\rangle^{-n-1-\varepsilon} \\
&\quad \times \cdots \times |x_{n-j-2}-x_{n-j-1}|^{1-\delta-n}\langle x_{n-j-1}\rangle^{-n-1-\varepsilon}|x_{n-j-1}-x_{n-j}|^{1-\delta-n} \\
&\le c_{n-j-2,n,\delta}\int_{\mathbb{R}^n} d^n x_{n-j-1}\,\big[|x_1-x_{n-j-1}|^{(n-j-2)(1-\delta)-n}+1\big] \\
&\quad \times \langle x_{n-j-1}\rangle^{-n-1-\varepsilon}|x_{n-j-1}-x_{n-j}|^{1-\delta-n} \\
&= c_{n-j-2,n,\delta}\bigg[\int_{\mathbb{R}^n} d^n x_{n-j-1}\,|x_1-x_{n-j-1}|^{(n-j-2)(1-\delta)-n} \\
&\quad + \int_{\mathbb{R}^n} d^n x_{n-j-1}\,\langle x_{n-j-1}\rangle^{-n-1-\varepsilon}|x_{n-j-1}-x_{n-j}|^{1-\delta-n}\bigg] \\
&=: c_{n-j-2,n,\delta}\big[\mathcal{I}_1(x_1,x_{n-j})+\mathcal{I}_2(x_{n-j})\big]
\end{aligned} \tag{11.60}$$

for some $c_{n-j-2,n,\delta} \in (0,\infty)$. Applying (11.11) with the choices

$$\alpha = (n-j-2)(1-\delta), \quad \beta = 1-\delta, \quad \gamma = n+1, \tag{11.61}$$

yields

$$\mathcal{I}_1(x_1, x_{n-j}) \le c'_{n-j-1,n,\delta}\big[|x_1 - x_{n-j}|^{(n-j-1)(1-\delta)-n} + 1\big]. \tag{11.62}$$

The assumptions on α and β in (11.11) are satisfied by the choices in (11.61). In fact,

$$0 < \alpha = (n-j-2)(1-\delta) \le n-2 \le n \quad \text{and} \quad 0 < \beta = 1-\delta \le n,$$

while

$$(\alpha + \beta) - n = (n-j-1)(1-\delta) - n \le 0 < n+1 = \gamma.$$

Finally, $\min(n, (n-j-1)(1-\delta)) = (n-j-1)(1-\delta)$ and (11.11) results in (11.62). A second application of (11.11), this time with the choices

$$\alpha = n, \quad \beta = 1-\delta, \quad \gamma = n+1, \tag{11.63}$$

yields

$$\mathcal{I}_2(x_{n-j}) \le c''_{n-j-1,n,\delta}, \quad x_{n-j} \in \mathbb{R}^n, \tag{11.64}$$

for some $c''_{n-j-1,n,\delta} \in (0,\infty)$. Combining (11.60), (11.62), and (11.64), one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_2 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} |x_1 - x_2|^{1-\delta-n} \langle x_2 \rangle^{-n-1-\varepsilon} \\
&\quad \times \cdots \times |x_{n-j-2} - x_{n-j-1}|^{1-\delta-n} \langle x_{n-j-1} \rangle^{-n-1-\varepsilon} |x_{n-j-1} - x_{n-j}|^{1-\delta-n} \\
&\le c_{n-j-1,n,\delta}\big[|x_1 - x_{n-j}|^{(n-j-1)(1-\delta)-n} + 1\big]
\end{aligned} \tag{11.65}$$

for some $c_{n-j-1,n,\delta} \in (0,\infty)$.

The estimate in (11.65) implies

$$\begin{aligned}
(11.52) &\le c_{n-j-1,n,\delta} \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_{n-j} \\
&\quad \times \langle x_1 \rangle^{-n-1-\varepsilon}\big[|x_1 - x_{n-j}|^{(n-j-1)(1-\delta)-n} + 1\big] \langle x_{n-j} \rangle^{-n-1-\varepsilon}.
\end{aligned} \tag{11.66}$$

Focusing on the integral over x_{n-j} in (11.66),

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_{n-j} \big[|x_1 - x_{n-j}|^{(n-j-1)(1-\delta)-n} + 1\big] \langle x_{n-j} \rangle^{-n-1-\varepsilon} \\
&\quad = \int_{\mathbb{R}^n} d^n x_{n-j}\, |x_1 - x_{n-j}|^{(n-j-1)(1-\delta)-n} \langle x_{n-j} \rangle^{-n-1-\varepsilon} \\
&\qquad + \int_{\mathbb{R}^n} d^n x_{n-j}\, \langle x_{n-j} \rangle^{-n-1-\varepsilon} \\
&\quad =: \mathcal{I}_1(x_1) + \mathcal{I}_2,
\end{aligned} \tag{11.67}$$

one infers that

$$\mathcal{I}_2 = c_n < \infty. \tag{11.68}$$

An application of (11.11) with the choices

$$\alpha = (n-j-1)(1-\delta), \quad \beta = n, \quad \gamma = n+1$$

yields

$$\mathcal{I}_1(x_1) \le c'_{n-j,n,\delta}, \quad x_1 \in \mathbb{R}^n. \tag{11.69}$$

Thus, (11.67), (11.68), and (11.69) imply

$$(11.67) \le c_{n-j,n,\delta}, \quad x_1 \in \mathbb{R}^n, \tag{11.70}$$

for some $c_{n-j,n,\delta} \in (0,\infty)$. As a result, (11.66) and (11.70) imply

$$(11.52) \le c_{n-j-1,n,\delta} c_{n-j,n,\delta} \int_{\mathbb{R}^n} d^n x_1 \langle x_1 \rangle^{-n-1-\varepsilon} < \infty.$$

Hence, this establishes the claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.50) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables $x_1, \dots, x_{n-j}$. As a consequence of this claim, (11.50) implies that for each $\delta \in (0,1)$ and each bounded subset $\Omega \subset \mathbb{C}_+$, the following estimate holds:

$$(11.50) \le C_{n,j,\delta,\Omega} |z|^{-(n-j-1)\delta-1}, \quad z \in \Omega, \tag{11.71}$$

for some $C_{n,j,\delta,\Omega} \in (0,\infty)$. In summary, for *Case 2*, one has that for any $\delta \in (0,1)$ and any bounded subset $\Omega \subset \mathbb{C}_+$,

$$\begin{aligned} &\bigg\| \int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) G_0(z; x_1, x_2) \\ &\quad \cdots \times V(x_{n-j-1}) G_0(z; x_{n-j-1}, x_{n-j}) V(x_{n-j}) \frac{1}{n!} \bigg[\frac{\partial^n}{\partial z^n} G_0(z; x_{n-j}, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\le C_{n,j,\delta,\Omega} |z|^{-(n-j-1)\delta-1}, \quad z \in \Omega. \end{aligned} \tag{11.72}$$

In addition, since $|z|^{-(n-j-1)(1-\delta)-1}$ is bounded in $\overline{\Omega}$ if $0 \notin \overline{\Omega}$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x_1 \cdots \int_{\mathbb{R}^n} d^n x_{n-j-1} \int_{\mathbb{R}^n} d^n x_{n-j}\, V(x_1) G_0(z; x_1, x_2) \\ &\quad \cdots \times V(x_{n-j-1}) G_0(z; x_{n-j-1}, x_{n-j}) V(x_{n-j}) \frac{1}{n!} \bigg[\frac{\partial^n}{\partial z^n} G_0(z; x_{n-j}, x_1) \bigg] \end{aligned}$$

is analytic in Ω and extends continuously to $\overline{\Omega}$ if $0 \notin \overline{\Omega}$. This settles *Case 2*.

Turning to *Case 3*, we assume that $j = n-2$. In this case, $\underline{k} = (k_1, k_2, k_3)$ with $k_1 + k_3 \ne n-1$. Let $\delta \in (0,1)$ be fixed. Invoking the final estimate in (11.27), one obtains

$$\begin{aligned}
&\bigg\| \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1) \frac{1}{k_2!} \bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2) \bigg] \\
&\times V(x_2) \frac{1}{(k_1+k_3+1)!} \bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}} G_0(z; x_2, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq C_{n,j,\delta} |z|^{-2\delta} \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2+1-\delta-n} \chi_{[0,1]}\big(|z||x_1-x_2|\big) \\
&+ |z|^{(n-1+2\delta)/2} \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2} \big] \chi_{[1,\infty)}\big(|z||x_1-x_2|\big) \big\} \\
&\times \langle x_2 \rangle^{-n-1-\varepsilon} \big\{ |x_2 - x_1|^{k_1+k_3+2-\delta-n} \chi_{[0,1]}\big(|z||x_2-x_1|\big) \\
&+ |z|^{(n-1+2\delta)/2} \big[|x_2|^{(2k_1+2k_3+3-n)/2} + |x_1|^{(2k_1+2k_3+3-n)/2} \big] \chi_{[1,\infty)}\big(|z||x_2-x_1|\big) \big\} \\
&\leq \widetilde{C}_{n,j,\delta} |z|^{-2\delta} \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2+1-\delta-n} \\
&+ |z|^{(n-1+2\delta)/2} \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2} \big] \big\} \\
&\times \langle x_2 \rangle^{-n-1-\varepsilon} \big\{ |x_2 - x_1|^{k_1+k_3+2-\delta-n} \\
&+ |z|^{(n-1+2\delta)/2} \big[|x_2|^{(2k_1+2k_3+3-n)/2} + |x_1|^{(2k_1+2k_3+3-n)/2} \big] \big\}, \quad z \in \mathbb{C}_+, \qquad (11.73)
\end{aligned}$$

where $C_{n,n-2,\delta}, \widetilde{C}_{n,n-2,\delta} \in (0,\infty)$ are suitable constants. We claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.73) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables x_1, x_2. Since $|z|^{(n-1+2\delta)/2}$ is locally bounded, to justify the claim, it suffices to establish convergence of the following integral:

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, \langle x_1 \rangle^{-n-1-\varepsilon} \big\{ |x_1 - x_2|^{k_2+1-\delta-n} \\
&\quad + \big[|x_1|^{(2k_2+1-n)/2} + |x_2|^{(2k_2+1-n)/2} \big] \big\} \langle x_2 \rangle^{-n-1-\varepsilon} \big\{ |x_2 - x_1|^{k_1+k_3+2-\delta-n} \\
&\quad + \big[|x_2|^{(2k_1+2k_3+3-n)/2} + |x_1|^{(2k_1+2k_3+3-n)/2} \big] \big\}. \qquad (11.74)
\end{aligned}$$

In turn, as in the argument for the proof of part (i) and Cases 1 and 2, it suffices to focus on the most singular term in (11.74) and thus disregard the terms originally multiplied by the factor $|z|^{(n-1+2\delta)/2}$ in (11.73) (following the same line of reasoning used throughout (11.8)–(11.9)). With this simplification, the claim reduces to establishing convergence of the integral:

$$\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, \langle x_1 \rangle^{-n-1-\varepsilon} |x_1 - x_2|^{k_2+1-\delta-n} \langle x_2 \rangle^{-n-1-\varepsilon} |x_2 - x_1|^{k_1+k_3+2-\delta-n} \\
&\quad = \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \langle x_1 \rangle^{-n-1-\varepsilon} |x_1 - x_2|^{2(1-\delta)-n} \langle x_2 \rangle^{-n-1-\varepsilon}. \qquad (11.75)
\end{aligned}$$

Applying (11.11) with $\alpha = 2(1-\delta)$, $\beta = n$, and $\gamma = n+1$, one infers that

$$\int_{\mathbb{R}^n} d^n x_2 |x_1 - x_2|^{2(1-\delta)-n} \langle x_2 \rangle^{-n-1-\varepsilon} \leq c_{2,n,\delta}, \quad x_1 \in \mathbb{R}^n, \tag{11.76}$$

for some $c_{2,n,\delta} \in (0,\infty)$. In turn, (11.76) implies

$$(11.75) \leq c_{2,n,\delta} \int_{\mathbb{R}^n} d^n x_1 \, \langle x_1 \rangle^{-n-1-\varepsilon} < \infty. \tag{11.77}$$

Hence, this establishes the claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.73) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables $x_1, \dots, x_2$. As a consequence of this claim, (11.73) implies that for each $\delta \in (0,1)$ and each bounded subset $\Omega \subset \mathbb{C}_+$, the following estimate holds:

$$(11.73) \leq C_{n,n-2,\delta,\Omega} |z|^{-2\delta}, \quad z \in \Omega,$$

for some $C_{n,n-2,\delta,\Omega} \in (0,\infty)$. In summary, for *Case 3*, one has that for any $\delta \in (0,1)$ and any bounded subset $\Omega \subset \mathbb{C}_+$,

$$\begin{aligned} &\bigg\| \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1) \frac{1}{k_2!} \bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2) \bigg] \\ &\qquad \times V(x_2) \frac{1}{(k_1+k_3+1)!} \bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}} G_0(z; x_2, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\ &\quad \leq C_{n,n-2,\delta,\Omega} |z|^{-2\delta}, \quad z \in \Omega. \end{aligned} \tag{11.78}$$

In addition, since $|z|^{-2\delta}$ is bounded in $\overline{\Omega}$ if $0 \notin \overline{\Omega}$, Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1) \frac{1}{k_2!} \bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2) \bigg] \\ &\qquad\qquad \times V(x_2) \frac{1}{(k_1+k_3+1)!} \bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}} G_0(z; x_2, x_1) \bigg] \end{aligned}$$

is analytic in Ω and extends continuously to $\overline{\Omega}$ if $0 \notin \overline{\Omega}$. This settles *Case 3*.

Turning to *Case 4*, we assume that $j = n-2$. In this case, $\underline{k} = (k_1, 0, k_3)$ with $k_1 + k_3 = n-1$. Let $\delta \in (0,1)$ be fixed. Invoking the final estimate in (11.27), one obtains

$$\begin{aligned} &\bigg\| \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1) \frac{1}{k_2!} \bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z; x_1, x_2) \bigg] \\ &\quad \times V(x_2) \frac{1}{(k_1+k_3+1)!} \bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}} G_0(z; x_2, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\ &= \bigg\| \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1) G_0(z; x_1, x_2) V(x_2) \frac{1}{n!} \bigg[\frac{\partial^n}{\partial z^n} G_0(z; x_2, x_1) \bigg] \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \end{aligned}$$

$$
\begin{aligned}
&\le C_{n,j,\delta}|z|^{-\delta-1}\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \\
&\quad \times \langle x_1\rangle^{-n-1-\varepsilon}\big\{|x_1-x_2|^{1-\delta-n}\chi_{[0,1]}\big(|z||x_1-x_2|\big) \\
&\quad + |z|^{(n-1+2\delta)/2}\big[|x_1|^{(1-n)/2}+|x_2|^{(1-n)/2}\big]\chi_{[1,\infty)}\big(|z||x_1-x_2|\big)\big\} \\
&\quad \times \langle x_2\rangle^{-n-1-\varepsilon}\big\{\chi_{[0,1]}\big(|z||x_2-x_1|\big) \\
&\quad + |z|^{(n+1)/2}\big[|x_2|^{(n+1)/2}+|x_1|^{(n+1)/2}\big]\chi_{[1,\infty)}\big(|z||x_2-x_1|\big)\big\} \\
&\le \widetilde{C}_{n,j,\delta}|z|^{-\delta-1}\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \\
&\quad \times \langle x_1\rangle^{-n-1-\varepsilon}\big\{|x_1-x_2|^{1-\delta-n} + |z|^{(n-1+2\delta)/2}\big[|x_1|^{(1-n)/2}+|x_2|^{(1-n)/2}\big]\big\} \\
&\quad \times \langle x_2\rangle^{-n-1-\varepsilon}\big\{1 + |z|^{(n+1)/2}\big[|x_2|^{(n+1)/2}+|x_1|^{(n+1)/2}\big]\big\}, \quad z\in\mathbb{C}_+, \qquad (11.79)
\end{aligned}
$$

where $C_{n,n-2,\delta}, \widetilde{C}_{n,n-2,\delta} \in (0,\infty)$ are suitable constants. We claim that for each bounded subset $\Omega \subset \mathbb{C}_+$, the integrand under the iterated integral on the right-hand side in (11.79) is uniformly bounded with respect to $z \in \Omega$ by an integrable function of the variables x_1, x_2. Since $|z|^{(n-1+2\delta)/2}$ and $|z|^{(n+1)/2}$ are locally bounded, to justify the claim, it suffices to establish convergence of the following integral:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \\
&\quad \times \langle x_1\rangle^{-n-1-\varepsilon}\big\{|x_1-x_2|^{1-\delta-n} + |z|^{(n-1+2\delta)/2}\big[|x_1|^{(1-n)/2}+|x_2|^{(1-n)/2}\big]\big\} \\
&\quad \times \langle x_2\rangle^{-n-1-\varepsilon}\big\{1 + |z|^{(n+1)/2}\big[|x_2|^{(n+1)/2}+|x_1|^{(n+1)/2}\big]\big\}. \qquad (11.80)
\end{aligned}
$$

In turn, as in the argument for the proof of part (i) and *Cases 1*, *2*, and *3*, it suffices to focus on the most singular term in (11.80) and thus disregard the terms originally multiplied by the factor $|z|^{(n-1+2\delta)/2}$ or $|z|^{(n+1)/2}$ in (11.79) (following the same line of reasoning used throughout (11.8)–(11.9)). With this simplification, the claim reduces to establishing convergence of the integral:

$$
\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \,\langle x_1\rangle^{-n-1-\varepsilon}|x_1-x_2|^{1-\delta-n}\langle x_2\rangle^{-n-1-\varepsilon}. \qquad (11.81)
$$

The integral in (11.81) is similar to the integral in (11.75). An argument entirely analogous to that used throughout (11.75)–(11.77) to show the integral in (11.75) is finite yields that the integral in (11.81) is finite. We omit further details at this point. In summary, for *Case 4*, one has that for any $\delta \in (0,1)$ and any bounded subset $\Omega \subset \mathbb{C}_+$,

$$
\begin{aligned}
&\bigg\| \int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2 \, V(x_1)\frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}}G_0(z;x_1,x_2)\bigg] \\
&\quad \times V(x_2)\frac{1}{(k_1+k_3+1)!}\bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}}G_0(z;x_2,x_1)\bigg]\bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\le C_{n,n-2,\delta,\Omega}|z|^{-\delta-1}, \quad z\in\Omega, \qquad (11.82)
\end{aligned}
$$

for some $C_{n,n-2,\delta,\Omega} \in (0,\infty)$. In addition, since $|z|^{-\delta-1}$ is bounded in $\overline{\Omega}$ if $0 \notin \overline{\Omega}$, Lebesgue's dominated convergence theorem implies that

$$\int_{\mathbb{R}^n} d^n x_1 \int_{\mathbb{R}^n} d^n x_2\, V(x_1) \frac{1}{k_2!}\bigg[\frac{\partial^{k_2}}{\partial z^{k_2}} G_0(z;x_1,x_2)\bigg] \\ \times V(x_2) \frac{1}{(k_1+k_3+1)!}\bigg[\frac{\partial^{k_1+k_3+1}}{\partial z^{k_1+k_3+1}} G_0(z;x_2,x_1)\bigg]$$

is analytic in Ω and extends continuously to $\overline{\Omega}$ if $0 \notin \overline{\Omega}$. This settles *Case 4*.

In *Case 5*, we assume that $j = n-1$. In this case, $\underline{k} = (k_1,k_2)$ with $k_1 + k_2 = n-1$. Invoking the final estimate in (C.31), one obtains

$$\begin{aligned} \bigg\| \int_{\mathbb{R}^n} d^n x_1\, V(x_1)\bigg[\frac{\partial^n}{\partial z^n} G_0(z;x_1,x_1)\bigg]\bigg\|_{\mathcal{B}(\mathbb{C}^N)} &\leq C_n |z|^{-1} \int_{\mathbb{R}^n} d^n x_1\, \langle x_1\rangle^{-n-1-\varepsilon} \\ &= \widetilde{C}_n |z|^{-1}, \quad z \in \mathbb{C}_+, \end{aligned} \tag{11.83}$$

for some $C_n, \widetilde{C}_n \in (0,\infty)$. In addition, since $|z|^{-1}$ is bounded outside any neighborhood of 0, Lebesgue's dominated convergence theorem implies that

$$\int_{\mathbb{R}^n} d^n x_1\, V(x_1)\bigg[\frac{\partial^n}{\partial z^n} G_0(z;x_1,x_1)\bigg] \tag{11.84}$$

is analytic in Ω and extends continuously to $\overline{\mathbb{C}_+}\backslash\{0\}$.

Now, looking at the bounds (11.48), (11.72), (11.78), (11.82), and (11.83), we identify the bound which is the most singular as $z \to 0$ in $\mathbb{C}_+$. The bounds from (11.48) are (up to z-independent constant multiples)

$$|z|^{-(n-j)\delta}, \quad \delta = \frac{n-j-2}{n-j-1},\ 0 \leq j \leq n-3. \tag{11.85}$$

The singularity in (11.85) is strongest when $(n-j)\delta$, $0 \leq j \leq n-3$, is largest. Since the expression

$$(n-j)\delta = (n-j) - \frac{n-j}{n-j-1}$$

is decreasing with respect to the parameter j, its maximum value is attained for $j = 0$:

$$\max_{0\leq j\leq n-3} (n-j)\delta = n - \frac{n}{n-1}.$$

Thus, the strongest singularity in (11.85) corresponds to $j = 0$ and is

$$|z|^{-[n-(n/(n-1))]}. \tag{11.86}$$

The bounds from (11.72) are (up to z-independent constant multiples)

$$|z|^{-(n-j-1)\delta-1}, \quad \delta \in (0,1),\ 0 \leq j \leq n-3. \tag{11.87}$$

Choosing $\delta = \eta/(n-j-1)$, $\eta \in (0,1)$, the bound in (11.87) may be recast as

$$|z|^{-(1+\eta)}, \quad \eta \in (0,1),\ 0 \leq j \leq n-3. \tag{11.88}$$

The bound from (11.78) is (up to a z-independent constant multiple)

$$|z|^{-2\delta}, \quad \delta \in (0,1),\ j = n-2. \tag{11.89}$$

Choosing $\delta = \eta/2$, $\eta \in (0,1)$, the bound in (11.89) may be recast as

$$|z|^{-\eta}, \quad \eta \in (0,1),\ j = n-2. \tag{11.90}$$

The bound from (11.82) is (up to a z-independent constant multiple)

$$|z|^{-(1+\delta)}, \quad \delta \in (0,1),\ j = n-2. \tag{11.91}$$

The bound from (11.83) is (up to a z-independent constant multiple)

$$|z|^{-1}, \quad j = n-1. \tag{11.92}$$

If $n \geq 4$, then the strongest singularity from (11.85), (11.88), (11.90), (11.91), and (11.92) is given by (11.86). Therefore, combining the results of *Case 1*–*Case 5* above with (11.3) and (11.4), one concludes that $\frac{d^n}{dz^n}G_{H,H_0}(\cdot)$ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}\backslash\{0\}$ and

$$\left\| \frac{d^n}{dz^n} G_{H,H_0}(\cdot) \right\|_{\mathcal{B}(\mathbb{C}^N)} \underset{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{-[n-(n/(n-1))]}\big). \tag{11.93}$$

If $n = 2$, then the strongest singularity in (11.90) and (11.92) is $|z|^{-(1+\delta)}$. Therefore, combining the results of *Case 4* and *Case 5* above with (11.3) and (11.4), one concludes that $\frac{d^2}{dz^2}G_{H,H_0}(\cdot)$ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}\backslash\{0\}$ and for any $\delta \in (0,1)$,

$$\left\| \frac{d^2}{dz^2} G_{H,H_0}(\cdot) \right\|_{\mathcal{B}(\mathbb{C}^2)} \underset{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}}{=} O\big(|z|^{-(1+\delta)}\big). \tag{11.94}$$

■