Chapter 12

Analysis of $\xi(\,\cdot\,; H, H_0)$ and an application to the Witten index for a class of non-Fredholm operators

Combining Hypotheses 10.16 and 11.1 we next make the following assumptions to describe continuity properties of the spectral shift function for the pair (H, H_0).

Hypothesis 12.1. *Let $n \in \mathbb{N}$ and suppose that $V = \{V_{\ell,\ell'}\}_{1\leq \ell,\ell'\leq N}$ satisfies for some constants $C \in (0,\infty)$ and $\varepsilon > 0$,*

$$\begin{aligned} &V \in [L^\infty(\mathbb{R}^n)]^{N\times N}, \\ &\big|V_{\ell,\ell'}(x)\big| \leq C\langle x\rangle^{-n-1-\varepsilon} \quad \text{for a.e. } x \in \mathbb{R}^n,\ 1 \leq \ell,\ell' \leq N. \end{aligned} \tag{12.1}$$

In addition, assume that $V(x) = \{V_{\ell,\ell'}(x)\}_{1\leq \ell,\ell'\leq N}$ is self-adjoint for a.e. $x \in \mathbb{R}^n$. In accordance with the factorization based on the polar decomposition of V discussed in (10.9) we suppose that $V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}$, where $V_1 = V_1^* = |V|^{1/2}$, $V_2 = U_V |V|^{1/2}$.

Finally, we assume that V satisfies (4.2) and (4.3)[1].

Thus, combining Theorems 9.9, 10.17, and 11.2 yields our principal result:

Theorem 12.2. *Assume Hypothesis 12.1. Then*

$$\xi(\,\cdot\,; H, H_0) \in C\big((-\infty,0) \cup (0,\infty)\big), \tag{12.2}$$

and the left and right limits at zero,

$$\xi(0_\pm; H, H_0) = \lim_{\varepsilon\downarrow 0} \xi(\pm\varepsilon; H, H_0), \tag{12.3}$$

exist. In particular, if 0 is a regular point for H according to Definition 10.6 (iii) and Theorem 10.7 (iii), then

$$\xi(\,\cdot\,; H, H_0) \in C(\mathbb{R}). \tag{12.4}$$

In the remainder of this chapter we describe an application to the Witten index for a class of non-Fredholm operators applicable in the context of multi-dimensional, massless Dirac operators H. We develop some necessary preparations and the basic setup next.

We begin by isolating a bit of notation: Linear operators in the Hilbert space $L^2(\mathbb{R}; dt; \mathcal{H})$, in short, $L^2(\mathbb{R}; \mathcal{H})$, will be denoted by boldface symbols of the type $\boldsymbol{T}$,

[1] The first condition in (4.3) is superseded by assumption (12.1).

to distinguish them from operators T in $\mathcal{H}$. In particular, operators denoted by $\boldsymbol{T}$ in the Hilbert space $L^2(\mathbb{R};\mathcal{H})$ represent operators associated with a family of operators $\{T(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$, defined by

$$\begin{aligned}
&(\boldsymbol{T} f)(t) = T(t)f(t) \quad \text{for a.e. } t\in\mathbb{R},\\
&\quad f\in \operatorname{dom}(\boldsymbol{T}) = \Big\{ g\in L^2(\mathbb{R};\mathcal{H}) \,\Big|\, g(t)\in\operatorname{dom}\big(T(t)\big) \quad \text{for a.e. } t\in\mathbb{R};\\
&\qquad t\mapsto T(t)g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}} dt\, \|T(t)g(t)\|_{\mathcal{H}}^2 < \infty\Big\}.
\end{aligned} \tag{12.5}$$

In the special case, where $\{T(t)\}$ is a family of bounded operators on $\mathcal{H}$ with

$$\sup_{t\in\mathbb{R}} \|T(t)\|_{\mathcal{B}(\mathcal{H})} < \infty,$$

the associated operator $\boldsymbol{T}$ is a bounded operator on $L^2(\mathbb{R};\mathcal{H})$ with $\|\boldsymbol{T}\|_{\mathcal{B}(L^2(\mathbb{R};\mathcal{H}))} = \sup_{t\in\mathbb{R}}\|T(t)\|_{\mathcal{B}(\mathcal{H})}$.

For brevity we will abbreviate $\boldsymbol{I} := I_{L^2(\mathbb{R};\mathcal{H})}$ in the following and note that in the concrete situation of n-dimensional, massless Dirac operators at hand, $\mathcal{H} = [L^2(\mathbb{R}^n)]^N$.

Denoting

$$A_- = H_0, \quad B_+ = V, \quad A_+ = A_- + B_+ = H,$$

we introduce two families of operators in $[L^2(\mathbb{R}^n)]^N$ by

$$\begin{aligned}
&B(t) = b(t)B_+, \quad t\in\mathbb{R},\\
&b^{(k)}\in C^\infty(\mathbb{R})\cap L^\infty(\mathbb{R};dt),\ k\in\mathbb{N}_0, \quad b'\in L^1(\mathbb{R};dt),\\
&\lim_{t\to\infty} b(t) = 1, \qquad \lim_{t\to-\infty} b(t) = 0,\\
&A(t) = A_- + B(t), \quad t\in\mathbb{R}.
\end{aligned} \tag{12.6}$$

Next, following the general setups described in [38, 41–44, 78, 137] we recall the definitions of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}' = \boldsymbol{B}'$, given in terms of the families $A(t)$, $B(t)$, and $B'(t)$, $t\in\mathbb{R}$, as in (12.5). In addition, $\boldsymbol{A}_-$ in $L^2\big(\mathbb{R};[L^2(\mathbb{R}^n)]^N\big)$ represents the self-adjoint (constant fiber) operator defined by

$$\begin{aligned}
&(\boldsymbol{A}_- f)(t) = A_- f(t) \quad \text{for a.e. } t\in\mathbb{R},\\
&f\in\operatorname{dom}(\boldsymbol{A}_-) = \Big\{ g\in L^2\big(\mathbb{R};[L^2(\mathbb{R}^n)]^N\big) \,\Big|\, g(t)\in\operatorname{dom}(A_-) \quad \text{for a.e. } t\in\mathbb{R},\\
&\quad t\mapsto A_- g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} dt\, \|A_- g(t)\|_{[L^2(\mathbb{R}^n)]^N}^2 < \infty\Big\}.
\end{aligned} \tag{12.7}$$

Next, we introduce the operator $\boldsymbol{D_A}$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ by

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A}) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}(\boldsymbol{A}_-), \tag{12.8}$$

where

$$\boldsymbol{A} = \boldsymbol{A}_- + \boldsymbol{B}, \quad \operatorname{dom}(\boldsymbol{A}) = \operatorname{dom}(\boldsymbol{A}_-),$$

and

$$\|\boldsymbol{B}\|_{\mathcal{B}(L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N))} = \sup_{t\in\mathbb{R}} \big\|B(t)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} < \infty.$$

Here the operator d/dt in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ is defined by

$$\begin{aligned} &\left(\frac{d}{dt} f\right)(t) = f'(t) \quad \text{for a.e. } t \in \mathbb{R}, \\ &f \in \operatorname{dom}(d/dt) = \big\{g \in L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \,\big|\, g \in AC_{\text{loc}}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big), \\ &\qquad\qquad\qquad\qquad\qquad\qquad g' \in L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big\} \\ &\qquad\qquad\qquad = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big). \end{aligned} \tag{12.9}$$

By [78, Lemma 4.4] (which extends to the present setting), $\boldsymbol{D_A}$ is densely defined and closed in $L^2(\mathbb{R}; [L^2(\mathbb{R}^n)]^N)$ and the adjoint operator $\boldsymbol{D_A^*}$ of $\boldsymbol{D_A}$ is given by

$$\boldsymbol{D_A^*} = -\frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A^*}) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}(\boldsymbol{A}_-).$$

This enables one to introduce the nonnegative, self-adjoint operators $\boldsymbol{H}_j$, $j = 1, 2$, in $L^2(\mathbb{R}; [L^2(\mathbb{R}^n)]^N)$ by

$$\boldsymbol{H}_1 = \boldsymbol{D_A^*} \boldsymbol{D_A}, \quad \boldsymbol{H}_2 = \boldsymbol{D_A} \boldsymbol{D_A^*}.$$

In order to effectively describe the domains of $\boldsymbol{H}_j$, $j = 1, 2$, we will decompose the latter as discussed below: To this end, one first observes that

$$\|\boldsymbol{B}'\|_{\mathcal{B}(L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N))} = \sup_{t\in\mathbb{R}} \big\|B'(t)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} < \infty. \tag{12.10}$$

It is convenient to also introduce the operator $\boldsymbol{H}_0$ in $L^2(\mathbb{R}; [L^2(\mathbb{R}^n)]^N)$ by

$$\boldsymbol{H}_0 = -\frac{d^2}{dt^2} + \boldsymbol{A}_-^2, \quad \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \operatorname{dom}(\boldsymbol{A}_-^2). \tag{12.11}$$

Then $\boldsymbol{H}_0$ is self-adjoint by Theorem VIII.33 of [141]. Moreover, since the operator $\boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B}$ is $\boldsymbol{H}_0$-bounded with bound less than one, [108, Theorem VI.4.3] implies the following decomposition of the operators $\boldsymbol{H}_j$, $j = 1, 2$,

$$\begin{aligned} &\boldsymbol{H}_j = -\frac{d^2}{dt^2} + \boldsymbol{A}^2 + (-1)^j \boldsymbol{A}' = \boldsymbol{H}_0 + \boldsymbol{B}\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B} + \boldsymbol{B}^2 + (-1)^j \boldsymbol{B}', \\ &\operatorname{dom}(\boldsymbol{H}_j) = \operatorname{dom}(\boldsymbol{H}_0), \quad j = 1, 2. \end{aligned} \tag{12.12}$$

Next, we introduce an approximation procedure as follows: Consider the characteristic function for the interval $[-\ell, \ell] \subset \mathbb{R}$,

$$\chi_\ell(\nu) = \chi_{[-\ell,\ell]}(\nu), \quad \nu \in \mathbb{R},\ \ell \in \mathbb{N}, \tag{12.13}$$

and hence

$$\operatorname*{s-lim}_{\ell\to\infty} \chi_\ell(A_-) = I_{[L^2(\mathbb{R}^n)]^N}. \tag{12.14}$$

Introducing

$$\begin{aligned} &A_\ell(t) = A_- + \chi_\ell(A_-)B(t)\chi_\ell(A_-) = A_- + B_\ell(t), \\ &\operatorname{dom}\big(A_\ell(t)\big) = \operatorname{dom}(A_-), \quad \ell \in \mathbb{N},\ t \in \mathbb{R}, \end{aligned} \tag{12.15}$$

$$A_{+,\ell} = A_- + \chi_\ell(A_-)B_+\chi_\ell(A_-), \quad \operatorname{dom}(A_{+,\ell}) = \operatorname{dom}(A_-),\ \ell \in \mathbb{N}, \tag{12.16}$$

where

$$B_\ell(t) = \chi_\ell(A_-)B(t)\chi_\ell(A_-), \quad \operatorname{dom}\big(B_\ell(t)\big) = [L^2(\mathbb{R}^n)]^N,\ \ell \in \mathbb{N},\ t \in \mathbb{R},$$

one concludes that

$$A_{+,\ell} - A_- = \chi_\ell(A_-)B_+\chi_\ell(A_-) \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad \ell \in \mathbb{N}, \tag{12.17}$$

$$A_\ell'(t) = B_\ell'(t) = \chi_\ell(A_-)B'(t)\chi_\ell(A_-) \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad \ell \in \mathbb{N},\ t \in \mathbb{R}. \tag{12.18}$$

As a consequence of (12.17), which follows from

$$\begin{aligned} &\big\|\chi_\ell(A_-)B_+\chi_\ell(A_-)\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \\ &\quad \le \big\|\chi_\ell(A_-)B_+(A_- - iI_{[L^2(\mathbb{R}^n)]^N})^{-n-1}\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \\ &\qquad \times \big\|(A_- - iI_{[L^2(\mathbb{R}^n)]^N})^{n+1}\chi_\ell(A_-)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)} < \infty \end{aligned} \tag{12.19}$$

(cf. (7.2)), the spectral shift functions $\xi(\,\cdot\,; A_{+,\ell}, A_-)$, $\ell \in \mathbb{N}$, exist and are uniquely determined by

$$\xi(\,\cdot\,; A_{+,\ell}, A_-) \in L^1(\mathbb{R}; d\nu), \quad \ell \in \mathbb{N}, \tag{12.20}$$

implying

$$\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N}\big(f(A_{+,\ell}) - f(A_-)\big) = \int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-)\,d\nu\, f'(\nu), \quad f \in C_0^\infty(\mathbb{R}).$$

We also note the analogous decompositions,

$$\begin{aligned} &\boldsymbol{H}_{j,\ell} = -\frac{d^2}{dt^2} + \boldsymbol{A}_\ell^2 + (-1)^j \boldsymbol{A}_\ell' = \boldsymbol{H}_0 + \boldsymbol{B}_\ell\boldsymbol{A}_- + \boldsymbol{A}_-\boldsymbol{B}_\ell + \boldsymbol{B}_\ell^2 + (-1)^j \boldsymbol{B}_\ell', \\ &\operatorname{dom}(\boldsymbol{H}_{j,\ell}) = \operatorname{dom}(\boldsymbol{H}_0) = W^{2,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big), \quad \ell \in \mathbb{N},\ j = 1, 2, \end{aligned}$$

with

$$\boldsymbol{B}_\ell = \chi_\ell(\boldsymbol{A}_-)\boldsymbol{B}\chi_\ell(\boldsymbol{A}_-), \quad \boldsymbol{B}'_\ell = \chi_\ell(\boldsymbol{A}_-)\boldsymbol{B}'\chi_\ell(\boldsymbol{A}_-), \quad \ell \in \mathbb{N},$$

implying

$$\boldsymbol{H}_2 - \boldsymbol{H}_1 = 2\boldsymbol{B}', \tag{12.21}$$

$$\boldsymbol{H}_{2,\ell} - \boldsymbol{H}_{1,\ell} = 2\boldsymbol{B}'_\ell = 2\chi_\ell(\boldsymbol{A}_-)\boldsymbol{B}'\chi_\ell(\boldsymbol{A}_-), \quad \ell \in \mathbb{N}. \tag{12.22}$$

Next, we recall the fact that for $\varepsilon > 0$,

$$L^2\big(\mathbb{R}^n; (1+|x|)^{(n/2)+\varepsilon} d^n x\big) \subset \ell^1(L^2)(\mathbb{R}^n)$$

(see, e.g., [159, p. 38] for the definition of the Birman–Solomyak space $\ell^1(L^2)(\mathbb{R}^n)$) and, given $\alpha > n$,

$$\big(1 + |\cdot|\big)^{-\alpha} \in L^2\big(\mathbb{R}^n; (1+|x|)^{(n/2)+\varepsilon} d^n x\big)$$

for $0 < \varepsilon$ sufficiently small (depending on a). This is of relevance here so that [44, Section 8] becomes applicable in our context.

We continue with the following basic result in [44, Theorems 5.2 and 8.4]:

Theorem 12.3. *In addition to Hypothesis 12.1 suppose that*

$$V_{\ell,\ell'} \in W^{4n,\infty}(\mathbb{R}^n), \quad 1 \leq \ell, \ell' \leq N.$$

Then, abbreviating

$$q = \lceil n/2 \rceil = \begin{cases} (n+1)/2, & n \text{ odd}, \\ n/2, & n \text{ even}, \end{cases}$$

one obtains

$$\big[(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-q} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-q}\big], \big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-q} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-q}\big] \\ \in \mathcal{B}_1\big(L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big), \quad \ell \in \mathbb{N}, \ z \in \mathbb{C}\backslash[0,\infty), \tag{12.23}$$

and

$$\lim_{\ell\to\infty} \big\|\big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-q} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-q}\big] \\ - \big[(\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-q} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-q}\big]\big\|_{\mathcal{B}_1(L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N))} = 0, \\ z \in \mathbb{C}\backslash[0,\infty). \tag{12.24}$$

For the fact that $q = \lceil n/2 \rceil$ in (12.23) can be replaced by any $r \geq q$, $r \in \mathbb{N}$, see, for instance, [184, p. 210]; similarly, (12.24) extends to $r \geq q$, $r \in \mathbb{N}$, by [40].

Relations (12.23) together with the fact that $\boldsymbol{H}_j \geq 0$, $\boldsymbol{H}_{j,\ell} \geq 0$, $\ell \in \mathbb{N}$, $j = 1, 2$, implies the existence and uniqueness of spectral shift functions $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ and $\xi(\,\cdot\,; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})$ for the pair of operators $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ and $(\boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})$, $\ell \in \mathbb{N}$, respectively, employing the normalization

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = 0, \quad \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) = 0, \quad \lambda < 0,\ \ell \in \mathbb{N} \tag{12.25}$$

(cf. [184, Section 8.9]). Moreover,

$$\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1) \in L^1\big(\mathbb{R}; \big(1 + |\lambda|\big)^{-q-1} d\lambda\big). \tag{12.26}$$

Since in analogy to (12.19),

$$\begin{aligned}
&\big\|A_\ell'(\cdot)\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} = \big\|B_\ell'(\cdot)\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \\
&= \big\|\chi_\ell(A_-)B'(\cdot)\chi_\ell(A_-)\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \\
&\leq \big\|\chi_\ell(A_-)B_+(A_- - iI_{[L^2(\mathbb{R}^n)]^N})^{-n-1}\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \\
&\quad \times \big\|(A_- - iI_{[L^2(\mathbb{R}^n)]^N})^{n+1}\chi_\ell(A_-)\big\|_{\mathcal{B}([L^2(\mathbb{R}^n)]^N)}\, b'(\cdot) \in L^1(\mathbb{R}; dt), \\
&\hspace{20em} \ell \in \mathbb{N},
\end{aligned} \tag{12.27}$$

employing $b'(\cdot) \in L^1(\mathbb{R}; dt)$ (cf. (12.6)), one obtains

$$\int_{\mathbb{R}} dt\, \big\|A_\ell'(t)\big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} < \infty, \quad \ell \in \mathbb{N}. \tag{12.28}$$

Given (12.28), the results in [137] (see also [78]) actually imply that

$$\big[(\boldsymbol{H}_{2,\ell} - z\,\boldsymbol{I})^{-1} - (\boldsymbol{H}_{1,\ell} - z\,\boldsymbol{I})^{-1}\big] \in \mathcal{B}_1\big(L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)\big), \quad \ell \in \mathbb{N},$$

and

$$\xi(\,\cdot\,; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) \in L^1\big(\mathbb{R}; (1 + |\lambda|)^{-2} d\lambda\big), \quad \ell \in \mathbb{N}.$$

In particular,

$$\begin{aligned}
\operatorname{tr}_{L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N)}\big(f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)\big) &= \int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) d\lambda\, f'(\lambda), \\
\operatorname{tr}_{L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N)}\big(f(\boldsymbol{H}_{2,\ell}) - f(\boldsymbol{H}_{1,\ell})\big) &= \int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda), \\
&\qquad \ell \in \mathbb{N},\ f \in C_0^\infty(\mathbb{R}).
\end{aligned}$$

In addition, as derived in [137] (see also, [78]), (12.20), (12.25), and (12.28) imply the approximate trace formula,

$$\int_{[0,\infty)} \frac{\xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell})\, d\lambda}{(\lambda - z)^2} = \frac{1}{2} \int_{\mathbb{R}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\nu^2 - z)^{3/2}}, \quad \ell \in \mathbb{N},\ z \in \mathbb{C}\backslash[0,\infty), \tag{12.29}$$

which in turn implies Pushnitski's formula [137],

$$\xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) = \begin{cases} \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\lambda-\nu^2)^{1/2}}, & \text{for a.e. } \lambda > 0, \\ 0, & \lambda < 0, \end{cases} \quad \ell \in \mathbb{N}, \quad (12.30)$$

via a Stieltjes inversion argument (cf. [78, Section 8]).

As shown in [40], (12.24) implies for $f \in C_0^\infty(\mathbb{R})$,

$$\lim_{\ell\to\infty} \big\| \big[f(\boldsymbol{H}_{2,\ell}) - f(\boldsymbol{H}_{1,\ell})\big] - \big[f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)\big]\big\|_{\mathcal{B}_1(L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N))} = 0, \quad (12.31)$$

and hence

$$\begin{aligned} &\lim_{\ell\to\infty} \int_{[0,\infty)} d\lambda\, \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) f'(\lambda) \\ &\quad = \lim_{\ell\to\infty} \operatorname{tr}_{L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N)}\big(f(\boldsymbol{H}_{2,\ell}) - f(\boldsymbol{H}_{1,\ell})\big) \\ &\quad = \operatorname{tr}_{L^2(\mathbb{R};[L^2(\mathbb{R}^n)]^N)}\big(f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)\big) \\ &\quad = \int_{[0,\infty)} d\lambda\, \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) f'(\lambda). \end{aligned} \quad (12.32)$$

Abbreviating

$$q_0 = 2\lfloor n/2 \rfloor + 1 = \begin{cases} n, & n \text{ odd}, \\ n+1, & n \text{ even}, \end{cases}$$

and assuming Hypothesis 7.1, one recalls that Theorem 7.4 implies

$$\big[(A_+ - zI_{[L^2(\mathbb{R}^n)]^N})^{-r_0} - (A_- - zI_{[L^2(\mathbb{R}^n)]^N})^{-r_0}\big] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad r_0 \in \mathbb{N},\ r_0 \geq q_0,\ z \in \mathbb{C}\backslash\mathbb{R}. \quad (12.33)$$

Since q_0 is always odd, [185, Theorem 2.2] yields the existence of a spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ for the pair (A_+, A_-) satisfying

$$\xi(\,\cdot\,; A_+, A_-) \in L^1\big(\mathbb{R}; \big(1 + |\nu|\big)^{-q_0-1} d\nu\big) \quad (12.34)$$

and hence

$$\operatorname{tr}_{[L^2(\mathbb{R}^n)]^N} \big(f(A_+) - f(A_-)\big) = \int_{\mathbb{R}} \xi(\nu; A_+, A_-) d\nu\, f'(\nu), \quad f \in C_0^\infty(\mathbb{R}). \quad (12.35)$$

While $\xi(\,\cdot\,; A_+, A_-)$ in (12.34), (12.35) is not unique, we will select a unique candidate using Theorem 12.4 below.

The next result is essentially [40, Theorem 4.7]; due to its importance we reproduce the proof here. To prepare the stage, we temporarily go beyond the approximation $A_{+,\ell}$ of A_+ and now introduce the following path of self-adjoint operators

$\{A_+(s)\}_{s\in[0,1]}$, in $[L^2(\mathbb{R}^n)]^N$, where

$$A_+(s) = A_- + P_s B_+ P_s, \quad \operatorname{dom}\big(A_+(s)\big) = \operatorname{dom}(A_-),\ s \in [0,1], \tag{12.36}$$
$$P_s = \chi_{[-(1-s)^{-1},(1-s)^{-1}]}(A_-),\ s \in [0,1), \quad P_1 = I_{[L^2(\mathbb{R}^n)]^N}, \tag{12.37}$$

in particular,

$$A_+(0) = A_{+,1} \text{ (cf. (12.16) with } \ell = 1) \quad \text{and} \quad A_+(1) = A_+. \tag{12.38}$$

Theorem 12.4. *Assume Hypothesis 12.1 and suppose that*

$$V_{\ell,\ell'} \in W^{4n,\infty}(\mathbb{R}^n), \quad 1 \leq \ell, \ell' \leq N.$$

Then there exists a unique spectral shift function $\xi(\,\cdot\,; A_+, A_-)$ such that

$$\begin{aligned}\xi(\,\cdot\,; A_+, A_-) &= \xi\big(\,\cdot\,; A_+(1), A_-\big) \\ &= \lim_{\ell\to\infty} \xi(\,\cdot\,; A_{+,\ell}, A_-) \quad \text{in } L^1\big(\mathbb{R}; (1+|\nu|)^{-q_0-1} d\nu\big).\end{aligned} \tag{12.39}$$

Moreover, assume that $g \in L^\infty(\mathbb{R}; d\nu)$. Then

$$\lim_{\ell\to\infty} \big\| \xi(\,\cdot\,; A_{+,\ell}, A_-) g - \xi(\,\cdot\,; A_+, A_-) g \big\|_{L^1(\mathbb{R};(1+|\nu|)^{-q_0-1}d\nu)} = 0, \tag{12.40}$$

and hence,

$$\lim_{\ell\to\infty} \int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-) d\nu\, h(\nu) = \int_{\mathbb{R}} \xi(\nu; A_+, A_-) d\nu\, h(\nu) \tag{12.41}$$

for all $h \in L^\infty(\mathbb{R}; d\nu)$ such that $\operatorname{ess.\,sup}_{\nu\in\mathbb{R}} |h(\nu)|(1+|\nu|)^{q_0+1} < \infty$.

Proof. Since by (12.17), $\chi_\ell(A-)B_+\chi_\ell(A-) \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big)$, also

$$A_+(s) - A_- = P_s B_+ P_s \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big), \quad s \in [0,1),$$

and hence there exists a uniques spectral shift function $\xi(\,\cdot\,; A_+(s), A_-)$ for the pair $(A_+(s), A_-)$ satisfying

$$\xi(\,\cdot\,; A_+(s), A_-) \in L^1(\mathbb{R}; d\nu).$$

Moreover, in complete analogy to (12.33), the family $A_+(s)$ depends continuously on $s \in [0,1]$ with respect to the pseudometric

$$\begin{aligned} &d_{q_0,z}(A, A') \\ &\quad = \big\| (A - zI_{[L^2(\mathbb{R}^n)]^N})^{-q_0} - (A' - zI_{[L^2(\mathbb{R}^n)]^N})^{-q_0} \big\|_{\mathcal{B}_1([L^2(\mathbb{R}^n)]^N)} \end{aligned} \tag{12.42}$$

for A, A' in the set of self-adjoint operators which satisfy for all $\zeta \in i\mathbb{R}\backslash\{0\}$,

$$\begin{aligned}&\big[(A - \zeta I_{[L^2(\mathbb{R}^n)]^N})^{-q_0} - (A_- - \zeta I_{[L^2(\mathbb{R}^n)]^N})^{-q_0}\big],\\ &\big[(A' - \zeta I_{[L^2(\mathbb{R}^n)]^N})^{-q_0} - (A_- - \zeta I_{[L^2(\mathbb{R}^n)]^N})^{-q_0}\big] \in \mathcal{B}_1\big([L^2(\mathbb{R}^n)]^N\big).\end{aligned}$$

Thus, the hypotheses of [40, Theorem 4.7] are satisfied and one concludes the existence of a unique spectral shift function $\xi(\cdot\,; A_+(s), A_-)$ for the pair $(A_+(s), A_-)$ depending continuously on $s \in [0,1]$ in the space $L^1\big(\mathbb{R}; (1+|\nu|)^{-q_0-1}d\nu\big)$, satisfying $\xi(\cdot\,; A_+(0), A_-) = \xi(\cdot\,; A_{+,1}, A_-)$. Taking $s = (\ell-1)/\ell$, $\ell \in \mathbb{N}$, yields

$$\begin{aligned}\xi(\cdot\,; A_+, A_-) &= \xi(\cdot\,; A_+(1), A_-) = \lim_{s\uparrow 1} \xi(\cdot\,; A_+(s), A_-)\\ &= \lim_{\ell\to\infty} \xi(\cdot\,; A_{+,\ell}, A_-) \quad \text{in } L^2\big(\mathbb{R}; (1+|\nu|)^{-q_0-1}d\nu\big).\end{aligned}$$

Hence an appropriate subsequence, again denoted by $\{\xi(\cdot\,; A_{+,\ell}, A_-)\}_{\ell\in\mathbb{N}}$, converges pointwise a.e. to $\xi(\cdot\,; A_+, A_-)$ as $\ell \to \infty$. Since each $\xi(\cdot\,; A_{+,\ell}, A_-) \in L^1(\mathbb{R}; d\nu)$, $\ell \in \mathbb{N}$, is uniquely defined one obtains a unique spectral shift function satisfying (12.42).

The facts (12.40) and (12.41) are now evident. ∎

In the following we will always employ $\xi(\cdot\,; A_+, A_-)$ as determined by the limiting relation (12.39) as the spectral shift function for the pair (A_+, A_-).

The next result is fundamental, it establishes (12.30) in the limit $\ell \to \infty$.

Theorem 12.5. *Assume Hypothesis 12.1 and suppose that*

$$V_{\ell,\ell'} \in W^{4n,\infty}(\mathbb{R}^n), \quad 1 \le \ell, \ell' \le N.$$

Then,

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda-\nu^2)^{1/2}} \quad \textit{for a.e. } \lambda > 0. \tag{12.43}$$

Proof. We start by multiplying the approximate relation (12.30) by the derivative f' of a test function $f \in C_0^\infty(\mathbb{R})$, and integrate to get,

$$\begin{aligned}\int_{\mathbb{R}} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda) &= \int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda)\\ &= \frac{1}{\pi}\int_{[0,\infty)} d\lambda\, f'(\lambda) \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_{+,\ell}, A_-)\, d\nu}{(\lambda-\nu^2)^{1/2}}\\ &= \frac{1}{\pi}\int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-) d\nu\, F'(\nu), \quad \ell \in \mathbb{N},\end{aligned} \tag{12.44}$$

where F' is defined by

$$F'(\nu) = \int_{\nu^2}^{\infty} d\lambda\, f'(\lambda)(\lambda - \nu^2)^{-1/2}, \quad \nu \in \mathbb{R}. \tag{12.45}$$

We claim that

$$F' \in C_0^{\infty}(\mathbb{R}),$$

rendering the manipulations leading to (12.44) well defined. Clearly, $F' \in C_0(\mathbb{R})$ since $f' \in C_0^{\infty}(\mathbb{R})$. To show that $F' \in C_0^{\infty}(\mathbb{R})$, it suffices to repeatedly integrate by parts and allude to the following representations of F',

$$\begin{aligned} F'(\nu) &= \int_{\nu^2}^{\infty} d\lambda\, f'(\lambda)(\lambda - \nu^2)^{-1/2} \\ &= -2 \int_{\nu^2}^{\infty} d\lambda\, f''(\lambda)(\lambda - \nu^2)^{1/2} \\ &= 2\frac{2}{3} \int_{\nu^2}^{\infty} d\lambda\, f'''(\lambda)(\lambda - \nu^2)^{3/2} \\ &\;\;\vdots \\ &= c_k \int_{\nu^2}^{\infty} d\lambda\, f^{(k)}(\lambda)(\lambda - \nu^2)^{k-(3/2)} \quad \nu \in \mathbb{R},\ k \in \mathbb{N}, \end{aligned} \tag{12.46}$$

for appropriate constants c_k, $k \in \mathbb{N}$. Thus, (12.44) yields the following,

$$\int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-) d\nu\, F'(\nu), \quad \ell \in \mathbb{N}, \tag{12.47}$$

where $f \in C_0^{\infty}(\mathbb{R})$ was arbitrary, and $F' \in C_0^{\infty}(\mathbb{R})$ (depending on f') is given by (12.45) or equivalently, by any of the expressions in (12.46).

It remains to control the limits $\ell \to \infty$ on either side of (12.47): By (12.32), the left-hand side of (12.47) converges as $\ell \to \infty$,

$$\lim_{\ell \to \infty} \int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_{2,\ell}, \boldsymbol{H}_{1,\ell}) d\lambda\, f'(\lambda) = \int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) d\lambda\, f'(\lambda). \tag{12.48}$$

For the right-hand side of (12.47) one applies Theorem 12.4, especially, (12.41), and concludes that

$$\lim_{\ell \to \infty} \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_{+,\ell}, A_-) d\nu\, F'(\nu) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_+, A_-) d\nu\, F'(\nu), \tag{12.49}$$

since by (12.42)

$$\lim_{\ell \to \infty} \big\| \xi(\cdot\,; A_{+,\ell}, A_-) - \xi(\cdot\,; A_+, A_-) \big\|_{L^1(\mathbb{R}; (1+|\nu|)^{-q_0-1} d\nu)} = 0.$$

Combining (12.47)–(12.49) finally yields

$$\int_{[0,\infty)} \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) d\lambda\, f'(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(\nu; A_+, A_-) d\nu\, F'(\nu)$$
$$= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda\, f'(\lambda) \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \chi_{[0,\infty)}(\lambda), \quad f \in C_0^\infty(\mathbb{R}).$$

An application of the Du Bois–Raymond Lemma (see, e.g., [114, Theorem 6.11]), thus implies for some constant $c \in \mathbb{R}$,

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \chi_{[0,\infty)}(\lambda) + c \quad \text{for a.e. } \lambda \in \mathbb{R}.$$

Due to our normalization (12.25), $c = 0$, proving (12.43). ■

Having established (12.43), we turn to the resolvent regularized Witten index of the densely defined and closed operator $\boldsymbol{D}_{\boldsymbol{A}}$. We refer to [31, 38, 41–44, 78, 84, 137] and the references therein for a bit of history on this subject.

Since $\sigma(A_\pm) = \mathbb{R}$, in particular, $0 \notin \rho(A_+) \cap \rho(A_-)$,

$$\boldsymbol{D}_{\boldsymbol{A}} \text{ is a non-Fredholm operator.}$$

This follows from the criterion for Fredholm operators established in [43, Theorem 2.6] (which extends to the current setting by replacing the resolvent of $A_\pm$ by appropriate powers of the resolvent in the proof).

In the following we will show that even though $\boldsymbol{D}_{\boldsymbol{A}}$ is a non-Fredholm operator, its Witten index is well defined and expressible in terms of the spectral shift functions for the pair of operators $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ and (A_+, A_-).

To introduce an appropriately (resolvent regularized) Witten index of $\boldsymbol{D}_{\boldsymbol{A}}$, we consider a densely defined, closed operator T in the complex, separable Hilbert space $\mathcal{K}$ and assume that for some $k \in \mathbb{N}$, and all $\lambda < 0$

$$\big[(T^*T - \lambda I_{\mathcal{K}})^{-k} - (TT^* - \lambda I_{\mathcal{K}})^{-k}\big] \in \mathcal{B}_1(\mathcal{K}).$$

Then the kth resolvent regularized Witten index of T is defined by

$$W_{k,r}(T) = \lim_{\lambda \uparrow 0} (-\lambda)^k \operatorname{tr}_{\mathcal{K}} \big((T^*T - \lambda I_{\mathcal{K}})^{-k} - (TT^* - \lambda I_{\mathcal{K}})^{-k}\big),$$

whenever the limit exists. The case $k = 1$ as well as the approach where resolvents are replaced by semigroups has been studied in great detail in [43], the extension to $k \geq 2$ was discussed in [44].

It is well known that the (regularized) Witten index is generally not an integer, in fact, it can take on any real value (cf. [31, 84]). The intrinsic value of $W_{k,r}(T)$ lies in its stability properties with respect to additive perturbations, analogous to stability properties of the Fredholm index. Indeed, as long as one replaces the familiar relative compactness assumption on an additive perturbation in connection with the Fredholm index, by appropriate relative trace class conditions in connection with the resolvent regularized Witten index, stability of the Witten index was proved in [31] (for $k = 1$, see also [45]) and, in connection with the analogous semigroup regularized Witten index, in [84] (the semigroup approach then yielding stability for $W_{k,r}(\,\cdot\,)$, $k \in \mathbb{N}$).

The following result, the first of this kind applicable to non-Fredholm operators in a partial differential operator setting involving multi-dimensional massless Dirac operators, then characterizes the Witten index of $\boldsymbol{D}_{\boldsymbol{A}}$ in terms of spectral shift functions:

Theorem 12.6. *Assume Hypothesis 12.1 and suppose that*

$$V_{\ell,\ell'} \in W^{4n,\infty}(\mathbb{R}^n), \quad 1 \leq \ell, \ell' \leq N.$$

Then 0 *is a right Lebesgue point of* $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$, *denoted by* $\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1)$, *and*

$$\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \big[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)\big]/2.$$

In addition, the resolvent regularized Witten index $W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}})$ *of* $\boldsymbol{D}_{\boldsymbol{A}}$ *exists for all* $k \in \mathbb{N}$, $k \geq q$ *and equals*

$$\begin{aligned} W_{k,r}(\boldsymbol{D}_{\boldsymbol{A}}) = \xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) &= \big[\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)\big]/2 \\ &= \big[\xi(0_+; H, H_0) + \xi(0_-; H, H_0)\big]/2. \end{aligned} \tag{12.50}$$

Proof. The key new input for the proof is the existence of 0 as a left and right Lebesgue point of $\xi(\,\cdot\,; A_+, A_-) = \xi(\,\cdot\,; H, H_0)$. This is established in Theorem 12.2, in fact, more is proved since left and right limits of $\xi(\,\cdot\,; H, H_0)$ at 0 are shown to exist. For $q = 1$, the remaining assertions are proved in [43, Theorem 4.3], the extension to $q \geq 2$ is discussed in [44, Section 7]. ■

The actual computation of the right-hand side of (12.50) in terms of the potential V is left for a future investigation.