Appendix A

Some remarks on block matrix operators

In this appendix, we collect some useful (and well-known) material on linear operators in connection with pointwise domination, boundedness, compactness, and the Hilbert-Schmidt property.

Definition A.1. Let $(M; \mathcal{M}; \mu)$ be a σ-finite, separable measure space, μ a non-negative measure with $0 < \mu(M) \leq \infty$, and consider the linear operators $A, B \in \mathcal{B}\big(L^2(M; d\mu)\big)$. Then B pointwise dominates A

$$\text{if for all } f \in L^2(M; d\mu),\ \big|(Af)(\cdot)\big| \leq \big(B|f|\big)(\cdot)\ \mu-\text{a.e. on } M. \tag{A.1}$$

For a linear block operator matrix $T = \{T_{j,k}\}_{1 \leq j,k \leq N}$, $N \in \mathbb{N}$, in the Hilbert space $[L^2(M; d\mu)]^N$ (where $[L^2(M; d\mu)]^N = L^2(M; d\mu; \mathbb{C}^N)$), we recall that $T \in \mathcal{B}_2([L^2(M; d\mu)]^N)$ if and only if $T_{j,k} \in \mathcal{B}_2(L^2(M; d\mu))$, $1 \leq j, k \leq N$. Moreover, we recall that (cf. e.g., [27, Theorem 11.3.6])

$$\begin{aligned}
\|T\|^2_{\mathcal{B}_2(L^2(M;d\mu)^N)} &= \int_{M\times M} d\mu(x)\, d\mu(y)\, \big\|T(x,y)\big\|^2_{\mathcal{B}_2(\mathbb{C}^N)} \\
&= \int_{M\times M} d\mu(x)\, d\mu(y) \sum_{j,k=1}^{N} \big|T_{j,k}(x,y)\big|^2 \\
&= \sum_{j,k=1}^{N} \int_{M\times M} d\mu(x)\, d\mu(y)\, \big|T_{j,k}(x,y)\big|^2 \\
&= \sum_{j,k=1}^{N} \|T_{j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))},
\end{aligned} \tag{A.2}$$

where, in obvious notation, $T(\cdot\,,\,\cdot)$ denotes the $N \times N$ matrix-valued integral kernel of T in $[L^2(M; d\mu)]^N$, and $T_{j,k}(\cdot\,,\,\cdot)$ represents the integral kernel of $T_{j,k}$ in $L^2(M; d\mu)$, $1 \leq j, k \leq N$.

In addition, employing the fact that for any $N \times N$ matrix $D \in \mathbb{C}^{N\times N}$,

$$\|D\|_{\mathcal{B}(\mathbb{C}^N)} \leq \|D\|_{\mathcal{B}_2(\mathbb{C}^N)} \leq N^{1/2} \|D\|_{\mathcal{B}(\mathbb{C}^N)}, \tag{A.3}$$

one also obtains

$$\|T\|^2_{\mathcal{B}_2(L^2(M;d\mu)^N)} \leq N \int_{M\times M} d\mu(x)\, d\mu(y)\, \big\|T(x,y)\big\|^2_{\mathcal{B}(\mathbb{C}^N)}. \tag{A.4}$$

More generally, for $\mathcal{H}$ a complex separable Hilbert space and $T = \{T_{j,k}\}_{1\le j,k\le N}$, $N \in \mathbb{N}$, a block operator matrix in $\mathcal{H}^N$, one confirms that

$$\begin{aligned} &T \in \mathcal{B}(\mathcal{H}^N) \ \big(\text{resp., } T \in \mathcal{B}_p\big(\mathcal{H}^N\big),\ p \in [1,\infty) \cup \{\infty\}\big) \\ &\quad \text{if and only if for each } 1 \le j,k \le N,\ T_{j,k} \in \mathcal{B}(\mathcal{H}) \\ &\quad \big(\text{resp., } T_{j,k} \in \mathcal{B}_p\big(\mathcal{H}^N\big),\ p \in [1,\infty) \cup \{\infty\}\big). \end{aligned} \tag{A.5}$$

In other words, for membership of T in $\mathcal{B}\big(\mathcal{H}^N\big)$ or $\mathcal{B}_p\big(\mathcal{H}^N\big)$, $p \in [1,\infty) \cup \{\infty\}$, it suffices to focus on each of its matrix elements $T_{j,k}$, $1 \le j,k \le N$. (For necessity of the last line in (A.5) it suffices to multiply T from the left and right by $N \times N$ diagonal matrices with $I_{\mathcal{H}}$ on the jth and kth position, resp., to isolate $T_{j,k}$ and appeal to the ideal property. For sufficiency, it suffices to write T as a sum of N^2 terms with $T_{j,k}$ at the j,kth position and zeros otherwise.)

The next result is useful in connection with Chapters 5 and 6.

Theorem A.2. *Let $N \in \mathbb{N}$ and suppose that T_1, T_2 are linear $N \times N$ block operator matrices defined on $[L^2(M;d\mu)]^N$, such that for each $1 \le j,k \le N$, $T_{2,j,k}$ pointwise dominates $T_{1,j,k}$. Then the following items* (i)–(iii) *hold:*

(i) *If $T_2 \in \mathcal{B}([L^2(M;d\mu)]^N)$ then $T_1 \in \mathcal{B}([L^2(M;d\mu)]^N)$ and*

$$\|T_1\|_{\mathcal{B}([L^2(M;d\mu)]^N)} \le \|T_2\|_{\mathcal{B}([L^2(M;d\mu)]^N)}. \tag{A.6}$$

(ii) *If $T_2 \in \mathcal{B}_\infty([L^2(M;d\mu)]^N)$ then $T_1 \in \mathcal{B}_\infty([L^2(M;d\mu)]^N)$ and*

$$\|T_1\|_{\mathcal{B}([L^2(M;d\mu)]^N)} \le \|T_2\|_{\mathcal{B}([L^2(M;d\mu)]^N)}. \tag{A.7}$$

(iii) *If $T_2 \in \mathcal{B}_2([L^2(M;d\mu)]^N)$ then $T_1 \in \mathcal{B}_2([L^2(M;d\mu)]^N)$ and*

$$\|T_1\|_{\mathcal{B}_2([L^2(M;d\mu)]^N)} \le \|T_2\|_{\mathcal{B}_2([L^2(M;d\mu)]^N)}. \tag{A.8}$$

Proof. For item (ii) we refer to [51, 130] (see also [113]) combined with (A.5) as we will not use it in this manuscript. While the proofs of items (i) and (iii) are obviously well known, we briefly recall them here as we will be using these facts in Chapters 5 and 6. Starting with item (i), we introduce the notation $f = (f_1, \dots, f_N) \in [L^2(M;d\mu)]^N$ and $|f| = (|f_1|, \dots, |f_N|) \in [L^2(M;d\mu)]^N$ and compute,

$$\begin{aligned} \|T_1 f\|^2_{[L^2(M;d\mu)]^N} &= \sum_{j=1}^N \|(T_1 f)_j\|^2_{L^2(M;d\mu)} \\ &= \sum_{j=1}^N \big((T_1 f)_j, (T_1 f)_j\big)_{L^2(M;d\mu)} \\ &= \sum_{j=1}^N \left| \sum_{k,\ell=1}^N (T_{1,j,k} f_k, T_{1,j,\ell} f_\ell)_{L^2(M;d\mu)} \right| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{N} \sum_{k,\ell=1}^{N} \big|(T_{1,j,k} f_k, T_{1,j,\ell} f_\ell)_{L^2(M;d\mu)}\big| \\
&\le \sum_{j=1}^{N} \sum_{k,\ell=1}^{N} \big(|T_{1,j,k} f_k|, |T_{1,j,\ell} f_\ell|\big)_{L^2(M;d\mu)} \\
&\le \sum_{j=1}^{N} \sum_{k,\ell=1}^{N} \big(T_{2,j,k}|f_k|, T_{2,j,\ell}|f_\ell|\big)_{L^2(M;d\mu)} \\
&= \sum_{j=1}^{N} \big((T_2|f|)_j, (T_2|f|)_j\big)_{L^2(M;d\mu)} \\
&= \big\|T_2|f|\big\|^2_{[L^2(M;d\mu)]^N} \\
&\le \|T_2\|^2_{\mathcal{B}(L^2(M;d\mu)^N)} \big\||f|\big\|^2_{[L^2(M;d\mu)]^N} \\
&= \|T_2\|^2_{\mathcal{B}(L^2(M;d\mu)^N)} \|f\|^2_{[L^2(M;d\mu)]^N},
\end{aligned} \tag{A.9}
$$

implying item (i). For item (iii) we recall from [159, Theorem 2.13] that $T_{1,j,k} \in \mathcal{B}_2(L^2(M;d\mu))$, $1 \le j,k \le N$, and $\|T_{1,j,k}\|_{\mathcal{B}_2(L^2(M;d\mu))} \le \|T_{2,j,k}\|_{\mathcal{B}_2(L^2(M;d\mu))}$, $1 \le j,k \le N$, and hence by (A.2),

$$
\begin{aligned}
\|T_1\|^2_{\mathcal{B}_2([L^2(M;d\mu)]^N)} &= \sum_{j,k=1}^{N} \|T_{1,j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))} \\
&\le \sum_{j,k=1}^{N} \|T_{2,j,k}\|^2_{\mathcal{B}_2(L^2(M;d\mu))} \\
&= \|T_2\|^2_{\mathcal{B}_2([L^2(M;d\mu)]^N)}.
\end{aligned} \tag{A.10}
$$

■

Remark A.3. We complete this appendix with the observation that the subordination assumption $|(Af)(\cdot)| \le (B|f|)(\cdot)$ μ-a.e. on M, if A and B are integral operators in $L^2(M;d\mu)$ with integral kernels $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$, respectively, is implied by the condition $|A(\cdot,\cdot)| \le B(\cdot,\cdot)$ $\mu \otimes \mu$-a.e. on $M \times M$ since

$$
\begin{aligned}
\big|(Af)(x)\big| &= \bigg|\int_M d\mu(y)\, A(x,y) f(y)\bigg| \le \int_M d\mu(y)\, \big|A(x,y)\big|\big|f(y)\big| \\
&\le \int_M d\mu(y)\, B(x,y)\big|f(y)\big| = \big(B|f|\big)(x) \quad \text{for a.e. } x \in M.
\end{aligned} \tag{A.11}
$$

In fact, the converse is true as well as shown next in a concrete situation. ◇

Lemma A.4. *Let $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, suppose $(\Omega, \Sigma, d\sigma)$ represents the standard Lebesgue measure on Ω (i.e., $d\sigma = d^n x$), and denote the Lebesgue measure of a set*

$S \in \Sigma$ by $|S|$. Consider bounded, linear integral operators $A, B \in \mathcal{B}(L^2(\Omega; d^n x))$, with integral kernels $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$, respectively. Then

B pointwise dominates A, that is,

$$\big|(Af)(\cdot)\big| \leq \big(B|f|\big)(\cdot) \quad \sigma\text{-a.e. on } \Omega \text{ for all } f \in L^2(\Omega; d^n x), \tag{A.12}$$

if and only if

$$\big|A(\cdot\,,\cdot)\big| \leq B(\cdot\,,\cdot) \quad \sigma \otimes \sigma\text{-a.e. on } \Omega \times \Omega. \tag{A.13}$$

Proof. That (A.13) implies (A.12) has just been shown in (A.11).

To prove the converse, suppose (A.12) holds. Since the operators A and B are bounded operators on $L^2(\Omega; d^n x)$, it follows from Tonelli's theorem that the integral kernels $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ are locally integrable functions on $\Omega \times \Omega$ (with respect to $\sigma \otimes \sigma$).

Let $x, y \in \Omega$ be arbitrary and let $\alpha \in \mathbb{Q}$. For $\varepsilon > 0$ consider the open ball $B_\varepsilon(x) \subset \Omega$ of radius ε with the center at $x \in \Omega$. The fact $f_{\varepsilon,x} = \chi_{B_\varepsilon(x)}(\cdot) \in L^2(\Omega; d^n x)$ and assumption (A.12) imply that

$$\begin{aligned} \left| \int_{B_\varepsilon(x)} d^n x'\, A(x', y) \right| &= \big|(Af_{\varepsilon,x})(y)\big| \leq \big(B|f_{\varepsilon,x}|\big)(y) \\ &\leq \int_{B_\varepsilon(x)} d^n x'\, B(x', y), \end{aligned}$$

for σ-a.e. $y \in \Omega$. Since $\operatorname{Re}\big(e^{i\alpha} z\big) \leq |z|$ for all $z \in \mathbb{C}$, it follows that

$$\int_{B_\varepsilon(x)} d^n x'\, \operatorname{Re}\big(e^{i\alpha} A(x', y)\big) \leq \int_{B_\varepsilon(x)} d^n x'\, B(x', y), \tag{A.14}$$

for σ-a.e. $y \in \Omega$. Integrating inequality (A.14) over the ball $B_\varepsilon(y)$ implies

$$\begin{aligned} &\int_{B_\varepsilon(y)} d^n y' \int_{B_\varepsilon(x)} d^n x'\, \operatorname{Re}\big(e^{i\alpha} A(x', y')\big) \\ &\quad \leq \int_{B_\varepsilon(y)} d^n y' \int_{B_\varepsilon(x)} d^n x'\, B(x', y'). \end{aligned}$$

Since $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ are locally integrable functions, an application of Fubini's theorem yields

$$\begin{aligned} &\frac{1}{|B_\varepsilon(\cdot)|^2} \int_{B_\varepsilon(x) \times B_\varepsilon(y)} d(\sigma \otimes \sigma)(x', y')\, \operatorname{Re}\big(e^{i\alpha} A(x', y')\big) \\ &\quad \leq \frac{1}{|B_\varepsilon(\cdot)|^2} \int_{B_\varepsilon(x) \times B_\varepsilon(y)} d(\sigma \otimes \sigma)(x', y')\, B(x', y'). \end{aligned} \tag{A.15}$$

Moreover, since both $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ are locally integrable functions, it follows that almost every point $(x, y) \in \Omega \times \Omega$ is a Lebesgue point for $B(\cdot\,,\cdot)$ and for

$\mathrm{Re}\left(e^{i\alpha}A(\cdot\,,\,\cdot)\right)$. Hence, letting $\varepsilon \to 0$ in inequality (A.15), one infers that for any $\alpha \in \mathbb{Q}$

$$\mathrm{Re}\left(e^{i\alpha}A(x,y)\right) \leq B(x,y),$$

for $\sigma \otimes \sigma$-a.e. $(x,y) \in \Omega \times \Omega$. Taking the supremum over all $\alpha \in \mathbb{Q}$, one obtains

$$\left|A(x,y)\right| \leq B(x,y),$$

for $\sigma \otimes \sigma$-a.e. $(x,y) \in \Omega \times \Omega$, proving (A.13). ■