Appendix B

Asymptotic results for Hankel functions

In this appendix we collect asymptotic results for Hankel functions in the regions of large and small arguments. To set the stage, we recall some details on the analytic behavior of $H_\nu^{(1)}(\,\cdot\,)$ (cf. [1, pp. 358–360]):

$$H_\nu^{(1)}(\zeta) = J_\nu(\zeta) + iY_\nu(\zeta), \quad \nu \in \mathbb{C}, \ \zeta \in \mathbb{C}\backslash\{0\}, \tag{B.1}$$

$$J_\nu(\zeta) = (\zeta/2)^\nu \sum_{k\in\mathbb{N}_0} \big[k!\Gamma(\nu+k+1)\big]^{-1}(-1)^k(\zeta/2)^{2k}, \tag{B.2}$$

$$Y_\nu(\zeta) = \big[\sin(\nu\pi)\big]^{-1}\big[J_\nu(\zeta)\cos(\nu\pi) - J_{-\nu}(\zeta)\big], \tag{B.3}$$

$$\begin{aligned} Y_n(\zeta) = {} & -\pi^{-1}(\zeta/2)^{-n}\sum_{k=0}^{n-1}[k!]^{-1}\big[(n-k-1)!\big](\zeta/2)^{2k} + 2\pi^{-1}J_n(\zeta)\ln(\zeta/2) \\ & - \pi^{-1}(\zeta/2)^n \sum_{k\in\mathbb{N}_0}\big[\psi(k+1)+\psi(n+k+1)\big] \\ & \times \big[k!(n+k)!\big]^{-1}(-1)^k(\zeta/2)^{2k}, \quad n \in \mathbb{N}, \end{aligned} \tag{B.4}$$

$$Y_0(\zeta) = \frac{2}{\pi}\{\ln(\zeta/2) + \gamma_{E-M}\}J_0(\zeta) - \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{k}\frac{1}{\ell}\right)\frac{(-4)^{-k}\zeta^{2k}}{(k!)^2}, \tag{B.5}$$

$$\begin{aligned} & J_{-n}(\zeta) = (-1)^n J_n(\zeta), \quad Y_{-n}(\zeta) = (-1)^n Y_n(\zeta), \quad n \in \mathbb{N}, \\ & H_{-\nu}^{(1)}(\zeta) = e^{i\nu\pi}H_\nu^{(1)}(\zeta), \end{aligned} \tag{B.6}$$

where (cf. [1, p. 256])

$$\psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta), \quad \psi(1) = -\gamma_{E-M}, \quad \psi(\ell) = -\gamma_{E-M} + \sum_{k=1}^{\ell-1}k^{-1}, \tag{B.7}$$

and

$$\gamma_{E-M} := \lim_{m\to\infty}\left(-\ln(m) + \sum_{k=1}^{m}k^{-1}\right) = 0.5772156649\ldots \tag{B.8}$$

denotes the Euler–Mascheroni constant (cf. [1, p. 255]). We also recall the asymptotic behavior (cf. [1, p. 360], [99, pp. 723–724])

$$H_0^{(1)}(\zeta) \underset{\substack{\zeta\to 0 \\ \zeta\in\mathbb{C}\backslash\{0\}}}{=} (2i/\pi)\ln(\zeta) + O\big(|\ln(\zeta)||\zeta|^2\big), \tag{B.9}$$

$$H_\nu^{(1)}(\zeta) \underset{\substack{\zeta\to 0\\ \zeta\in\mathbb{C}\backslash\{0\}}}{=} -(i/\pi)2^\nu\Gamma(\nu)\zeta^{-\nu} + \begin{cases} O\big(|\zeta|^{\min(\nu,-\nu+2)}\big), & \nu\notin\mathbb{N},\\ O\big(\big|\ln(\zeta)\big||\zeta|^\nu\big) + O(\zeta^{-\nu+2}), & \nu\in\mathbb{N}, \end{cases} \quad \operatorname{Re}(\nu)>0, \tag{B.10}$$

$$H_\nu^{(1)}(\zeta) \underset{\zeta\to\infty}{=} (2/\pi)^{1/2}\zeta^{-1/2}e^{i[\zeta-(\nu\pi/2)-(\pi/4)]}, \quad \nu\geq 0,\ \operatorname{Im}(\zeta)\geq 0. \tag{B.11}$$

B.1 Asymptotics of $H_\nu^{(1)}(\zeta)$ as $|\zeta|\to\infty$

Hypothesis B.1. *Let* $\nu\in\mathbb{C}$ *with* $\operatorname{Re}(\nu+(1/2))>0$.

Assuming Hypothesis B.1, the Hankel function $H_\nu^{(1)}(\,\cdot\,)$ permits the following representation (cf., e.g., [88, Equation 8.421.9], [181, Equation 6.12(3)])

$$H_\nu^{(1)}(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{1/2} \frac{e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}}{\Gamma(\nu+(1/2))} \int_0^\infty du\, e^{-u}u^{\nu-(1/2)}\left(1+\frac{iu}{2\zeta}\right)^{\nu-(1/2)}, \tag{B.12}$$

where $\operatorname{Re}(\nu+(1/2))>0$ and $-\pi/2<\arg(\zeta)<3\pi/2$. We will derive the asymptotic behavior of $H_\nu^{(1)}(\zeta)$ as $|\zeta|\to\infty$ closely following the presentation given in [181, Section 7.2].

The factor in parentheses in the integrand in (B.12) may be expanded for any $p\in\mathbb{N}$ according to

$$\left(1+\frac{iu}{2\zeta}\right)^{\nu-(1/2)} = \sum_{m=0}^{p-1}\frac{((1/2)-\nu)_m}{m!}\left(\frac{u}{2i\zeta}\right)^m + \frac{((1/2)-\nu)_p}{(p-1)!}\left(\frac{u}{2i\zeta}\right)^p\int_0^1 dt\,(1-t)^{p-1}\left(1-\frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}, \tag{B.13}$$

where we have employed the Pochhammer symbol,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n\in\mathbb{N},\ a\in\mathbb{C}\backslash\{0,-1,-2,-3,\dots\}.$$

We shall assume for convenience that $p\in\mathbb{N}$ is chosen sufficiently large to guarantee that $\operatorname{Re}(\nu-p-(1/2))\leq 0$, and we will comment on how to remove this restriction later. Next, fix an angle $\delta\in(0,\pi/2)$ which satisfies

$$\big|\arg(\zeta)-(\pi/2)\big|\leq\pi-\delta.$$

With δ so chosen, one infers

$$\left|1-\frac{ut}{2i\zeta}\right|\geq\sin(\delta), \quad \left|\arg\left(1-\frac{ut}{2i\zeta}\right)\right|<\pi,$$

for all $t \in [0, 1]$ and all $u \in (0, \infty)$. In particular,

$$\left|\left(1 - \frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}\right| \le e^{\pi|\operatorname{Im}(\nu)|}\big[\sin(\delta)\big]^{\operatorname{Re}(\nu-p-(1/2))} =: C_{\nu,p,\delta}, \tag{B.14}$$

where $C_{\nu,p,\delta}$ is independent of ζ. Using the expansion (B.13) in (B.12), one obtains

$$\begin{aligned} H_\nu^{(1)}(\zeta) &= \left(\frac{2}{\pi\zeta}\right)^{1/2} \frac{e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}}{\Gamma\big(\nu+(1/2)\big)} \\ &\quad \times \int_0^\infty du\, e^{-u} u^{\nu-(1/2)} \Bigg[\sum_{m=0}^{p-1} \frac{\big((1/2)-\nu\big)_m}{m!} \left(\frac{u}{2i\zeta}\right)^m \\ &\quad + \frac{\big((1/2)-\nu\big)_p}{(p-1)!}\left(\frac{u}{2i\zeta}\right)^p \int_0^1 dt\,(1-t)^{p-1}\left(1-\frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}\Bigg] \\ &= \left(\frac{2}{\pi\zeta}\right)^{1/2} \frac{e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}}{\Gamma\big(\nu+(1/2)\big)} \Bigg[\sum_{m=0}^{p-1} \frac{\big((1/2)-\nu\big)_m \Gamma\big(\nu+m+(1/2)\big)}{m!(2i\zeta)^m} \\ &\quad + \frac{\big((1/2)-\nu\big)_p}{(p-1)!} \int_0^\infty du\, e^{-u} u^{\nu-(1/2)} \left(\frac{u}{2i\zeta}\right)^p \\ &\quad \times \int_0^1 dt\,(1-t)^{p-1}\left(1-\frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}\Bigg] \\ &= \left(\frac{2}{\pi\zeta}\right)^{1/2} e^{i[\zeta-(\pi/2)\nu-(\pi/4)]} \Bigg[\sum_{m=0}^{p-1} \frac{\big((1/2)-\nu\big)_m \big(\nu+(1/2)\big)_m}{m!(2i\zeta)^m} + R_{\nu,p}^{(1)}(\zeta)\Bigg], \end{aligned} \tag{B.15}$$

where

$$\begin{aligned} R_{\nu,p}^{(1)}(\zeta) &:= \frac{\big((1/2)-\nu\big)_p}{(p-1)!\Gamma\big(\nu+(1/2)\big)} \\ &\quad \times \int_0^\infty du\, e^{-u} u^{\nu-(1/2)} \left(\frac{u}{2i\zeta}\right)^p \int_0^1 dt\,(1-t)^{p-1}\left(1-\frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}. \end{aligned} \tag{B.16}$$

One observes that

$$\begin{aligned} &\big|R_{\nu,p}^{(1)}(\zeta)\big| \\ &\le \frac{C_{\nu,p,\delta}}{(p-1)!} \left|\frac{\big((1/2)-\nu\big)_p}{\Gamma\big(\nu+(1/2)\big)(2i\zeta)^p}\right| \left[\int_0^1 dt\,(1-t)^{p-1}\right]\left[\int_0^\infty e^{-u}\big|u^{\nu+p-(1/2)}\big|\,du\right] \\ &= \widetilde{C}_{\nu,p,\delta}|\zeta|^{-p}. \end{aligned} \tag{B.17}$$

As a consequence of (B.15), (B.16), and (B.17), one infers that for any fixed $\delta \in (0, \pi/2)$,

$$H_\nu^{(1)}(\zeta) \underset{|\zeta|\to\infty}{=} \left(\frac{2}{\pi\zeta}\right)^{1/2} e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}\left[\sum_{m=0}^{p-1}\frac{\big((1/2)-\nu\big)_m\big(\nu+(1/2)\big)_m}{m!(2i\zeta)^m}+O\big(|\zeta|^{-p}\big)\right],$$
$$\big|\arg(\zeta)-(\pi/2)\big| \leq \pi-\delta. \quad \text{(B.18)}$$

To obtain similar expansions when $\mathrm{Re}(\nu - p - (1/2)) > 0$, one chooses $q \in \mathbb{N}$ so large that $\mathrm{Re}(\nu - q - (1/2)) < 0$, which requires $p < q$. Then (B.15), (B.16), (B.17), and (B.18) hold with p replaced by q. In particular, by (B.15), for any fixed $\delta \in (0, \pi/2)$,

$$\begin{aligned} H_\nu^{(1)}(\zeta) &= \left(\frac{2}{\pi\zeta}\right)^{1/2} e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}\left[\sum_{m=0}^{q-1}\frac{((1/2)-\nu)_m(\nu+(1/2))_m}{m!(2i\zeta)^m}+R_{\nu,q}^{(1)}(\zeta)\right] \\ &= \left(\frac{2}{\pi\zeta}\right)^{1/2} e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}\left[\sum_{m=0}^{p-1}\frac{((1/2)-\nu)_m(\nu+(1/2))_m}{m!(2i\zeta)^m}+\widetilde{R}_{\nu,q}^{(1)}(\zeta)\right], \end{aligned}$$
$$\big|\arg(\zeta)-(\pi/2)\big| \leq \pi-\delta, \quad \text{(B.19)}$$

where

$$\widetilde{R}_{\nu,p,q}^{(1)}(\zeta) = \sum_{m=p}^{q-1}\frac{\big((1/2)-\nu\big)_m\big(\nu+(1/2)\big)_m}{m!(2i\zeta)^m}+R_{\nu,q}^{(1)}(\zeta). \quad \text{(B.20)}$$

The following lemma provides sufficient conditions for the differentiability of an integral depending on a complex parameter.

Lemma B.2 ([119]). *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $G \subset \mathbb{C}$ be an open set, and let $f : G \times X \to \mathbb{C}$ be a function which satisfies the following conditions:*

(i) *$f(\zeta, \cdot)$ is $\mathcal{M}$-measurable for every $\zeta \in G$,*

(ii) *$f(\cdot, x)$ is holomorphic in G for every $x \in X$, and*

(iii) *$\int_X d\mu\, |f(\cdot, x)|$ is locally bounded; that is, for every $\zeta_0 \in G$, there exists $\varepsilon(\zeta_0) > 0$ such that*

$$\sup_{\substack{\zeta\in G\\ |\zeta-\zeta_0|\leq\varepsilon(\zeta_0)}} \int_X d\mu\,\big|f(\zeta, x)\big| < \infty.$$

Then $\int_X d\mu\, f(\cdot, x)$ is holomorphic in G and

$$\frac{d^n}{d\zeta^n}\int_X d\mu\, f(\zeta, x) = \int_X d\mu\, \frac{\partial^n}{\partial\zeta^n} f(\zeta, x) \quad \text{in } G \text{ for every } n \in \mathbb{N}.$$

Proposition B.3. *Assume Hypothesis B.1. Let $p \in \mathbb{N}$, $u \in (0, \infty)$, and suppose* $\mathrm{Re}(\nu - p - (1/2)) \leq 0$. *If*

$$\Omega_0 := \{\zeta \in \mathbb{C} \mid |\arg(\zeta) - (\pi/2)| < \pi\}, \tag{B.21}$$

then the function $a_{u,p,\nu} : \Omega_0 \to \mathbb{C}$ defined by

$$a_{u,p,\nu}(\zeta) = \int_0^1 dt\, (1-t)^p \left(1 - \frac{ut}{2i\zeta}\right)^{\nu - p - (1/2)}, \quad \zeta \in \Omega_0, \tag{B.22}$$

is analytic in Ω_0 and

$$\frac{d^n}{d\zeta^n} a_{u,p,\nu}(\zeta) = \int_0^1 dt\, (1-t)^p \frac{\partial^n}{\partial^n \zeta}\left(1 - \frac{ut}{2i\zeta}\right)^{\nu - p - (1/2)}, \quad \zeta \in \Omega_0. \tag{B.23}$$

Proof. Let $p \in \mathbb{N}$, $u \in (0, \infty)$, $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu - p - (1/2)) \leq 0$. It suffices to apply Lemma B.2 to the function

$$f_{u,p,\nu}(\zeta, t) = (1-t)^p \left(1 - \frac{ut}{2i\zeta}\right)^{\nu - p - (1/2)}, \quad \zeta \in \Omega_0,\ t \in (0,1). \tag{B.24}$$

Of course, (B.24) defines a function which is Lebesgue measurable for each $\zeta \in \Omega_0$ and analytic in Ω_0 for every $t \in (0,1)$. Therefore, it remains to verify condition (iii) in Lemma B.2. To this end, let $\zeta_0 \in \Omega_0$. Choose $\delta \in (0, \pi/2)$ such that

$$\zeta_0 \in \Omega_\delta := \{\zeta \in \mathbb{C} \mid |\arg(\zeta) - (\pi/2)| < \pi - \delta\}.$$

By (B.14), one then infers

$$|a_{u,p,\nu}(\zeta)| = \left|\int_0^1 dt\, f_{u,p,\nu}(\zeta,t)\right| \leq \int_0^1 dt\, |f_{u,p,\nu}(\zeta,t)| \leq \frac{C_{\nu,p,\delta}}{p}, \quad \zeta \in \Omega_\delta. \tag{B.25}$$

In particular, choosing $\varepsilon(\zeta_0) \in (0,1)$ so small that

$$\{\zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < \varepsilon(\zeta_0)\} \subset \Omega_\delta,$$

one concludes

$$\sup_{\substack{\zeta \in \Omega_0 \\ |\zeta - \zeta_0| < \varepsilon(\zeta_0)}} \left|\int_0^1 dt\, f_{u,p,\nu}(\zeta,t)\right| < \infty.$$

Therefore, condition (iii) in Lemma B.2 holds, and it follows that $a_{u,p,\nu}$ is analytic in Ω_0. ■

Proposition B.4. *Assume Hypothesis B.1, let $p \in \mathbb{N}$, and let Ω_0 be defined as in (B.21). The following statements hold:*

(i) *If* $\mathrm{Re}(\nu - p - (1/2)) \leq 0$, *then the function $R_{\nu,p}^{(1)} : \Omega_0 \to \mathbb{C}$ defined by (B.16) is analytic in Ω_0.*

(ii) *If* $\mathrm{Re}(\nu - p - (1/2)) > 0$, *then the function $\widetilde{R}_{\nu,p,q}^{(1)} : \Omega_0 \to \mathbb{C}$ defined by (B.20) is analytic in Ω_0 for every $q \in \mathbb{N}$ such that* $\mathrm{Re}(\nu - q - (1/2)) < 0$.

Proof. Let $p \in \mathbb{N}$ and suppose $\mathrm{Re}(\nu - p - (1/2)) \leq 0$. We begin with the proof of (i). It suffices to show that the function $b_{p,\nu} : \Omega_0 \to \mathbb{C}$ defined by (cf. (B.22))

$$\begin{aligned} b_{p,\nu}(\zeta) &= \int_0^\infty du\, e^{-u} u^{p+\nu-(1/2)} \int_0^1 dt\, (1-t)^{p-1} \left(1 - \frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)} \\ &= \int_0^\infty du\, e^{-u} u^{p+\nu-(1/2)} a_{u,p,\nu}(\zeta), \quad \zeta \in \Omega_0, \end{aligned} \tag{B.26}$$

is analytic in Ω_0. The function $e^{-u} u^{p+\nu-(1/2)} a_{u,p,\nu}(\zeta)$ is a measurable function of $u \in (0,\infty)$ for each $\zeta \in \Omega_0$ and is, by Proposition B.3, an analytic function of $\zeta \in \Omega_0$ for each $u \in (0,\infty)$. Therefore, by Lemma B.2, it suffices to prove that for each $\zeta_0 \in \Omega_0$, there exists $\varepsilon(\zeta_0) \in (0,\infty)$ such that

$$\sup_{\substack{\zeta \in \Omega_0 \\ |\zeta - \zeta_0| < \varepsilon(\zeta_0)}} \left| \int_0^\infty du\, e^{-u} u^{p+\nu-(1/2)} a_{u,p,\nu}(\zeta) \right| < \infty. \tag{B.27}$$

To this end, let $\zeta_0 \in \Omega_0$. Choose $\delta \in (0, \pi/2)$ such that

$$\zeta_0 \in \Omega_\delta := \big\{\zeta \in \mathbb{C} \,\big|\, \big|\arg(\zeta) - (\pi/2)\big| < \pi - \delta\big\}.$$

An application of (B.25) yields the following estimate:

$$\begin{aligned} \left| \int_0^\infty du\, e^{-u} u^{p+\nu-(1/2)} a_{u,p,\nu}(\zeta) \right| &\leq \frac{C_{\nu,p,\delta}}{p} \int_0^\infty du\, e^{-u} u^{\mathrm{Re}(p+\nu-(1/2))} \\ &= \frac{C_{\nu,p,\delta}}{p} \Gamma\big(\mathrm{Re}\,(p + \nu + (1/2))\big), \ \zeta \in \Omega_\delta. \end{aligned}$$

Thus, one obtains (B.27) by choosing $\varepsilon(\zeta_0) \in (0,1)$ so small that

$$\{\zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < \varepsilon(\zeta_0)\} \subset \Omega_\delta.$$

Finally, to prove item (ii), suppose that $\mathrm{Re}(\nu - p - (1/2)) > 0$ and $q \in \mathbb{N}$ with $\mathrm{Re}(\nu - q - (1/2)) < 0$. The first term on the right-hand side in (B.20) is analytic in $\mathbb{C}\backslash\{0\}$, while the second term on the right-hand side in (B.20) is analytic in Ω_0 by the statement in (i). Hence, the statement in (ii) follows from the subspace property of analytic functions. ∎

Remark B.5. Of course, analyticity of $R^{(1)}_{\nu,p}$ (resp., $\widetilde{R}^{(1)}_{\nu,p,q}$) follows immediately from (B.15) (resp., (B.19)). However, the proof of Proposition B.4 shows that the ζ-derivatives of $R^{(1)}_{\nu,p}$ may be computed by differentiating under the integrals in (B.16). In fact, as a consequence of (B.23) and the proof of Proposition B.4, one infers that under the assumptions of Proposition B.4,

$$\frac{\partial^n}{\partial^n \zeta} b_{p,\nu}(\zeta) = \int_0^\infty du\, e^{-u} u^{p+\nu-(1/2)} \int_0^1 dt\, (1-t)^{p-1} \frac{\partial^n}{\partial^n \zeta} \left(1 - \frac{ut}{2i\zeta}\right)^{\nu-p-(1/2)}, \quad \zeta \in \Omega_0,\ n \in \mathbb{N}. \tag{B.28}$$

◇

In order to state the next result, we introduce $\widetilde{O}$-notation. Recall that if $\Omega \subseteq \mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$, then one writes

$$f(\zeta) = O\big(g(\zeta)\big), \quad \zeta \in \Omega,$$

if and only if there exists a constant $C \in (0, \infty)$ (independent of $\zeta \in \Omega$) such that

$$\big|f(\zeta)\big| \leq C\big|g(\zeta)\big|, \quad \zeta \in \Omega.$$

One writes

$$f(\zeta) = \widetilde{O}\big(g(\zeta)\big), \quad \zeta \in \Omega,$$

if and only if for each $n \in \mathbb{N}_0$,

$$\frac{d^n f}{d\zeta^n} = O\bigg(\frac{d^n g}{d\zeta^n}\bigg), \quad \zeta \in \Omega. \tag{B.29}$$

It is understood that the constant corresponding to (B.29) will, in general, depend on $n \in \mathbb{N}_0$.

The principal asymptotic result for $H_\nu^{(1)}(\zeta)$ as $|\zeta| \to \infty$ can be summarized as follows:

Lemma B.6. *Assume Hypothesis B.1 holds. If $\delta \in (0, \pi/2)$, then*

$$H_\nu^{(1)}(\zeta) = e^{i\zeta}\omega_\nu(\zeta), \quad \zeta \in \Omega_\delta,$$

where

$$\omega_\nu(\zeta) \underset{|\zeta|\to\infty}{=} \widetilde{O}\Big(\big(1 + |\zeta|\big)^{-1/2}\Big), \quad \zeta \in \Omega_\delta \cap \big\{z \in \mathbb{C} \mid |z| \geq 1\big\}. \tag{B.30}$$

Proof. Assume Hypothesis B.1 holds. We distinguish two cases: $\mathrm{Re}(\nu - (3/2)) \leq 0$ and $\mathrm{Re}(\nu - (3/2)) > 0$. If $\mathrm{Re}(\nu - (3/2)) \leq 0$, then one may take $p = 1$ in (B.15) to obtain

$$H_\nu^{(1)}(\zeta) = \bigg(\frac{2}{\pi\zeta}\bigg)^{1/2} e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}\big[1 + R_{\nu,1}^{(1)}(\zeta)\big] = e^{i\zeta}\omega_\nu(\zeta), \quad \zeta \in \Omega_\delta,$$

where

$$\omega_\nu(\zeta) = \bigg(\frac{2}{\pi\zeta}\bigg)^{1/2} e^{i[-(\pi/2)\nu-(\pi/4)]}\big[1 + R_{\nu,1}^{(1)}(\zeta)\big], \quad \zeta \in \Omega_\delta. \tag{B.31}$$

It remains to prove $\omega_\nu(\,\cdot\,)$ defined by (B.31) satisfies (B.30). To prove this, it suffices to show that

$$\zeta^{-1/2}\big[1 + R_{\nu,1}^{(1)}(\zeta)\big] \underset{|\zeta|\to\infty}{=} \widetilde{O}\Big(\big(1 + |\zeta|\big)^{-1/2}\Big), \quad \zeta \in \Omega_\delta \cap \big\{z \in \mathbb{C} \mid |z| \geq 1\big\}; \tag{B.32}$$

that is,

$$\frac{d^n}{d\zeta^n}\zeta^{-1/2}\big[1+R^{(1)}_{\nu,1}(\zeta)\big] \underset{|\zeta|\to\infty}{=} O\bigg(\frac{d^n}{d\zeta^n}(1+|\zeta|)^{-1/2}\bigg),$$
$$\zeta\in\Omega_\delta\cap\big\{z\in\mathbb{C}\mid |z|\geq 1\big\},\ n\in\mathbb{N}_0. \quad \text{(B.33)}$$

For $n=0$, the relation in (B.33) follows immediately from (B.17). To treat the derivatives in (B.33), one differentiates under the integrals in (B.16). For simplicity, we only treat the case $n=1$ and omit the details for $n\geq 2$. One computes

$$\begin{aligned}
\frac{d}{d\zeta}\zeta^{-1/2}\big[1+R^{(1)}_{\nu,1}(\zeta)\big] &= -\frac{1}{2\zeta^{3/2}}\big[1+R^{(1)}_{\nu,1}(\zeta)\big]+\zeta^{-1/2}\frac{d}{d\zeta}R^{(1)}_{\nu,1}(\zeta)\\
&\underset{|\zeta|\to\infty}{=} O\big((1+|\zeta|)^{-3/2}\big)\\
&\quad -\zeta^{-5/2}\frac{((1/2)-\nu)_1}{2i\Gamma(\nu+(1/2))}\int_0^\infty du\, e^{-u}u^{\nu+(1/2)}\int_0^1 dt\left(1-\frac{ut}{2i\zeta}\right)^{\nu-(3/2)}\\
&\quad -\zeta^{-7/2}\frac{((1/2)-\nu)_1(\nu-(3/2))}{4\Gamma(\nu+(1/2))}\int_0^\infty du\, e^{-u}u^{\nu+(3/2)}\int_0^1 dt\, t\left(1-\frac{ut}{2i\zeta}\right)^{\nu-(5/2)}\\
&\underset{|\zeta|\to\infty}{=} O\big((1+|\zeta|)^{-3/2}\big),\quad \zeta\in\Omega_\delta\cap\big\{z\in\mathbb{C}\mid |z|\geq 1\big\}.
\end{aligned} \quad \text{(B.34)}$$

To obtain the final equality in (B.34), one applies (B.14) to bound the two (ζ-dependent) integrals with respect to $t\in(0,1)$. This settles the case when $\mathrm{Re}(\nu-(3/2))\leq 0$.

If $\mathrm{Re}(\nu-(3/2))>0$, one chooses $q\in\mathbb{N}$ such that $\mathrm{Re}(\nu-q-(1/2))<0$. Then

$$H^{(1)}_\nu(\zeta)=\left(\frac{2}{\pi\zeta}\right)^{1/2}e^{i[\zeta-(\pi/2)\nu-(\pi/4)]}\big[1+\widetilde{R}^{(1)}_{\nu,1,q}(\zeta)\big]=e^{i\zeta}\omega_\nu(\zeta),\quad \zeta\in\Omega_\delta,$$

where

$$\omega_\nu(\zeta)=\left(\frac{2}{\pi\zeta}\right)^{1/2}e^{i[-(\pi/2)\nu-(\pi/4)]}\big[1+\widetilde{R}^{(1)}_{\nu,1,q}(\zeta)\big],\quad \zeta\in\Omega_\delta.$$

Then, as a consequence of (B.20) and (B.32), one obtains

$$\zeta^{-1/2}\big[1+\widetilde{R}^{(1)}_{\nu,1,q}(\zeta)\big] \underset{|\zeta|\to\infty}{=} \widetilde{O}\big((1+|\zeta|)^{-1/2}\big),\quad \zeta\in\Omega_\delta\cap\big\{z\in\mathbb{C}\mid |z|\geq 1\big\}. \quad \blacksquare$$

B.2 Asymptotics of $H^{(1)}_\nu(\zeta)$ as $|\zeta|\to 0$

Since the asymptotics derived here will be applied to Dirac operators, we only consider $\nu\in[0,\infty)$ from this point on. We distinguish two cases:

(i) $\nu\in(0,\infty)\backslash\mathbb{N}$,

and

(ii) $\nu\in\mathbb{N}_0$.

Case (i): $\nu \in (0,\infty)\backslash\mathbb{N}$. In this case, one has the following representation for $H_\nu^{(1)}$ in terms of Bessel functions:

$$H_\nu^{(1)}(\zeta) = \big[1 + i\cot(\nu\pi)\big]J_\nu(\zeta) - i\big[\sin(\nu\pi)\big]^{-1}J_{-\nu}(\zeta), \quad \zeta \in \mathbb{C}.$$

Repeated term-by-term differentiation of the series representations for $J_{\pm\nu}$ reveals

$$\begin{aligned}
&\frac{d^k}{d\zeta^k}J_{\pm\nu}(\zeta)\\
&\underset{|\zeta|\to 0}{=} \left(\frac{d^k}{d\zeta^k}\left[\frac{1}{\Gamma(1\pm\nu)}(\zeta/2)^{\pm\nu}\right]\right)\big[1+O\big(|\zeta|^2\big)\big]\\
&\underset{|\zeta|\to 0}{=} \frac{(\pm\nu)_k}{2^k\Gamma(1\pm\nu)}(\zeta/2)^{\pm\nu-k}\big[1+O\big(|\zeta|^2\big)\big], \quad |\zeta|\le 1,\ k\in\mathbb{N}_0.
\end{aligned}$$

As a result,

$$\begin{aligned}
\frac{d^k}{d\zeta^k}H_\nu^{(1)}(\zeta) &\underset{|\zeta|\to 0}{=} \big[1+i\cot(\nu\pi)\big]\frac{(\nu)_k}{2^k\Gamma(\nu+1)}(\zeta/2)^{\nu-k}\big[1+O\big(|\zeta|^2\big)\big]\\
&- i\big[\sin(\nu\pi)\big]^{-1}\frac{(-\nu)_k}{2^k\Gamma(1-\nu)}(\zeta/2)^{-\nu-k}\big[1+O\big(|\zeta|^2\big)\big], \quad |\zeta|\le 1,\ k\in\mathbb{N}_0,
\end{aligned}$$

which settles Case (i).

Case (ii): $\nu \in \mathbb{N}_0$. Since ν is a nonnegative integer, we write

$$\tilde{n} := \nu \in \mathbb{N}_0.$$

First, we treat the case $\tilde{n} \in \mathbb{N}$. Then,

$$J_{\tilde{n}}(\zeta) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!(\tilde{n}+m)!}(\zeta/2)^{2m+\tilde{n}}, \quad \zeta\in\mathbb{C}, \tag{B.35}$$

and

$$\begin{aligned}
Y_{\tilde{n}}(\zeta) &= -\frac{1}{\pi}(\zeta/2)^{-\tilde{n}}\sum_{m=0}^{\tilde{n}-1}\frac{(\tilde{n}-m-1)!}{m!}(\zeta/2)^{2m} + \frac{2}{\pi}\ln(\zeta/2)J_{\tilde{n}}(\zeta)\\
&\quad - \frac{1}{\pi}(\zeta/2)^{\tilde{n}}\sum_{m=0}^{\infty}\big[\psi(m+1)+\psi(\tilde{n}+m+1)\big]\frac{(-1)^m}{m!(\tilde{n}+m)!}(\zeta/2)^{2m}\\
&=: Y_{1,\tilde{n}}(\zeta) + Y_{2,\tilde{n}}(\zeta) + Y_{3,\tilde{n}}(\zeta), \quad \zeta\in\mathbb{C}\backslash\{0\}.
\end{aligned}$$

Repeated differentiation of the series representation for $J_{\tilde{n}}$ yields

$$\frac{d^k}{d\zeta^k}J_{\tilde{n}}(\zeta) \underset{|\zeta|\to 0}{=} \frac{(\tilde{n})_k}{2^k\tilde{n}!}(\zeta/2)^{\tilde{n}-k}\big[1+O\big(|\zeta|^2\big)\big], \quad |\zeta|\le 1,\ k\in\mathbb{N}\cap[0,\tilde{n}], \tag{B.36}$$

and

$$\frac{d^k}{d\zeta^k} J_{\tilde{n}}(\zeta) \underset{|\zeta|\to 0}{=} \begin{cases} \frac{(-1)^{(k-\tilde{n})/2} k!}{2^k ((k-\tilde{n})/2)!((k+\tilde{n})/2)!} \big[1+O\big(|\zeta|^2\big)\big], & \tilde{n}+k \text{ even}, \\ \frac{(-1)^{(k-\tilde{n}+1)/2}(k+1)!}{2^{k+1}((k-\tilde{n}+1)/2)!((k+\tilde{n}+1)/2)!} \zeta\big[1+O\big(|\zeta|^2\big)\big], & \tilde{n}+k \text{ odd}, \end{cases}$$
$$|\zeta| \leq 1,\ k \in \mathbb{N} \cap (\tilde{n}, \infty). \quad \text{(B.37)}$$

Repeated differentiation of $Y_{1,\tilde{n}}$ yields

$$\begin{aligned} \frac{d^k}{d\zeta^k} Y_{1,\tilde{n}}(\zeta) &\underset{|\zeta|\to 0}{=} -\pi^{-1}\bigg(\frac{d^k}{d\zeta^k}\big[(\zeta/2)^{-\tilde{n}}\big]\bigg)\big[1 + O\big(|\zeta|^2\big)\big] \\ &\underset{|\zeta|\to 0}{=} -\frac{(-\tilde{n})_k}{2^k \pi}(\zeta/2)^{-\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big], \quad |\zeta| \leq 1,\ k \in \mathbb{N}_0. \end{aligned} \quad \text{(B.38)}$$

In view of (B.35),

$$Y_{2,\tilde{n}}(\zeta) \underset{|\zeta|\to 0}{=} \frac{2}{\pi \tilde{n}!} \ln(\zeta/2)(\zeta/2)^{\tilde{n}} + \widetilde{O}\big(\zeta^{\tilde{n}+2}\ln(\zeta)\big), \quad |\zeta| \leq 1. \quad \text{(B.39)}$$

Differentiation of the series representation of $Y_{3,\tilde{n}}$ yields

$$\frac{d^k}{d\zeta^k} Y_{3,\tilde{n}}(\zeta) \underset{|\zeta|\to 0}{=} -\frac{1}{\pi}\frac{\big[\psi(\tilde{n}+1)-\gamma\big](\tilde{n})_k}{2^k \tilde{n}!}(\zeta/2)^{\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big],$$
$$|\zeta| \leq 1,\ k \in \mathbb{N} \cap [0, \tilde{n}], \quad \text{(B.40)}$$

and

$$\frac{d^k}{d\zeta^k} Y_{3,\tilde{n}}(\zeta)$$
$$\underset{|\zeta|\to 0}{=} \begin{cases} -\frac{1}{\pi}\frac{[\psi([(k-\tilde{n})/2]+1)+\psi([(k+\tilde{n})/2]+1)](-1)^{(k-\tilde{n})/2}k!}{2^k((k-\tilde{n})/2)!((k+\tilde{n})/2)!}\big[1+O\big(|\zeta|^2\big)\big], \\ \qquad \tilde{n}+k \text{ even}, \\ -\frac{1}{\pi}\frac{[\psi([(k-\tilde{n}+1)/2]+1)+\psi([(k+\tilde{n}+1)/2]+1)](-1)^{(k-\tilde{n}+1)/2}k!}{2^k((k-\tilde{n}+1)/2)!((k+\tilde{n}+1)/2)!}\zeta\big[1+O\big(|\zeta|^2\big)\big], \\ \qquad \tilde{n}+k \text{ odd}, \end{cases}$$
$$|\zeta| \leq 1,\ k \in \mathbb{N} \cap (\tilde{n}, \infty). \quad \text{(B.41)}$$

In the remaining case $\tilde{n} = 0$, one obtains (cf., e.g., [66, (11) and (12)])

$$J_0(\zeta) \underset{|\zeta|\to 0}{=} 1 + O(\zeta^2), \quad |\zeta| \leq 1, \quad \text{(B.42)}$$

with

$$\frac{d^k}{d\zeta^k} J_0(\zeta) \underset{|\zeta|\to 0}{=} \begin{cases} \frac{(-1)^{k/2}k!}{2^k[(k/2)!]^2}\big[1+O\big(|\zeta|^2\big)\big], & k \text{ even}, \\ \frac{(-1)^{(k+1)/2}(k+1)!}{2^{k+1}[((k+1)/2)!]^2}\zeta\big[1+O\big(|\zeta|^2\big)\big], & k \text{ odd}, \end{cases} \quad |\zeta| \leq 1,\ k \in \mathbb{N}, \quad \text{(B.43)}$$

and

$$Y_0(\zeta) \underset{|\zeta|\to 0}{=} \frac{2}{\pi}\ln(\zeta/2) + \frac{2\gamma}{\pi} + \widetilde{O}\big(\zeta^2 \ln(\zeta)\big), \quad |\zeta| \le 1. \tag{B.44}$$

Finally, to obtain expressions for the derivatives of $H_{\tilde{n}}^{(1)}$, one applies the representation

$$H_{\tilde{n}}^{(1)}(\zeta) = J_{\tilde{n}}(\zeta) + iY_{\tilde{n}}(\zeta), \quad \zeta \in \mathbb{C}\backslash\{0\},\ \tilde{n} \in \mathbb{N}_0.$$

If $\tilde{n} \in \mathbb{N}$, then

$$\begin{aligned}
\frac{d^k}{d\zeta^k} H_{\tilde{n}}^{(1)}(\zeta) &\underset{|\zeta|\to 0}{=} \frac{(\tilde{n})_k}{2^k \tilde{n}!}(\zeta/2)^{\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad - i\,\frac{(-\tilde{n})_k}{2^k \pi}(\zeta/2)^{-\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad + i\,\frac{2}{\pi \tilde{n}!}\frac{d^k}{d\zeta^k}\big[\ln(\zeta/2)(\zeta/2)^{\tilde{n}}\big] + O\bigg(\frac{d^k}{d\zeta^k}\big[\zeta^{\tilde{n}+2}\ln(\zeta)\big]\bigg) \\
&\quad - \frac{i}{\pi}\frac{\big[\psi(\tilde{n}+1) - \gamma\big](\tilde{n})_k}{2^k \tilde{n}!}(\zeta/2)^{\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big], \quad |\zeta| \le 1,\ k \in \mathbb{N} \cap [0, \tilde{n}],
\end{aligned}$$

while

$$\begin{aligned}
\frac{d^k}{d\zeta^k} H_{\tilde{n}}^{(1)}(\zeta) &\underset{|\zeta|\to 0}{=} \frac{(-1)^{(k-\tilde{n})/2} k!}{2^k \big((k-\tilde{n})/2\big)!\big((k+\tilde{n})/2\big)!}\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad - i\,\frac{(-\tilde{n})_k}{2^k \pi}(\zeta/2)^{-\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad + i\,\frac{2}{\pi \tilde{n}!}\frac{d^k}{d\zeta^k}\big[(\zeta/2)^{\tilde{n}}\ln(\zeta/2)\big] + O\bigg(\frac{d^k}{d\zeta^k}\big[\zeta^{\tilde{n}+2}\ln(\zeta)\big]\bigg) \\
&\quad - \frac{i}{\pi}\frac{\big[\psi\big(\big[(k-\tilde{n})/2\big]+1\big) + \psi\big(\big[(k+\tilde{n})/2\big]+1\big)\big](-1)^{(k-\tilde{n})/2} k!}{2^k \big((k-\tilde{n})/2\big)!\big((k+\tilde{n})/2\big)!}\big[1 + O\big(|\zeta|^2\big)\big], \\
&\qquad |\zeta| \le 1,\ k \in \mathbb{N} \cap (\tilde{n}, \infty),\ \tilde{n} + k \text{ even},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^k}{d\zeta^k} H_{\tilde{n}}^{(1)}(\zeta) &\underset{|\zeta|\to 0}{=} \frac{(-1)^{(k-\tilde{n}+1)/2}(k+1)!}{2^{k+1}\big((k-\tilde{n}+1)/2\big)!\big((k+\tilde{n}+1)/2\big)!}\zeta\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad - i\,\frac{(-\tilde{n})_k}{2^k \pi}(\zeta/2)^{-\tilde{n}-k}\big[1 + O\big(|\zeta|^2\big)\big] \\
&\quad + i\,\frac{2}{\pi \tilde{n}!}\frac{d^k}{d\zeta^k}\big[(\zeta/2)^{\tilde{n}}\ln(\zeta/2)\big] + O\bigg(\frac{d^k}{d\zeta^k}\big[\zeta^{\tilde{n}+2}\ln(\zeta)\big]\bigg) \\
&\quad - \frac{i}{\pi}\frac{\big[\psi\big(\big[(k-\tilde{n}+1)/2\big]+1\big) + \psi\big(\big[(k+\tilde{n}+1)/2\big]+1\big)\big](-1)^{(k-\tilde{n}+1)/2} k!}{2^k \big((k-\tilde{n}+1)/2\big)!\big((k+\tilde{n}+1)/2\big)!}\zeta \\
&\quad \times \big[1 + O\big(|\zeta|^2\big)\big], \quad |\zeta| \le 1,\ k \in \mathbb{N} \cap (\tilde{n}, \infty),\ \tilde{n} + k \text{ odd}.
\end{aligned}$$

In the case $\tilde{n} = 0$,

$$\frac{d^k}{d\zeta^k} H_0^{(1)}(\zeta) \underset{|\zeta|\to 0}{=} \frac{(-1)^{k/2} k!}{2^k \big[(k/2)!\big]^2} \big[1 + O\big(|\zeta|^2\big)\big] + i \frac{d^k}{d\zeta^k} \left[\frac{2}{\pi} \ln(\zeta/2) + \frac{2\gamma}{\pi}\right] + O\left(\frac{d^k}{d\zeta^k}\big[\zeta^2 \ln(\zeta)\big]\right), \quad |\zeta| \le 1,\ k \in \mathbb{N}_0,\ k \text{ even},$$

and

$$\frac{d^k}{d\zeta^k} H_0^{(1)}(\zeta) \underset{|\zeta|\to 0}{=} \frac{(-1)^{(k+1)/2} (k+1)!}{2^{k+1} \big[\big((k+1)/2\big)!\big]^2} \zeta \big[1 + O\big(|\zeta|^2\big)\big] + i \frac{d^k}{d\zeta^k} \left[\frac{2}{\pi} \ln(\zeta/2) + \frac{2\gamma}{\pi}\right] + O\left(\frac{d^k}{d\zeta^k}\big[\zeta^2 \ln(\zeta)\big]\right), \quad |\zeta| \le 1,\ k \in \mathbb{N},\ k \text{ odd}.$$