Appendix C

Expansions and estimates of the free Dirac Green's matrix

In this chapter, we investigate the behavior of the Green's function (5.9) of the massless Dirac operator and certain of its partial derivatives with respect to the energy parameter $z \in \mathbb{C}\backslash\mathbb{R}$. Throughout, we assume that $n \in \mathbb{N}\backslash\{1\}$.

By (5.9),

$$\begin{aligned} G_0(z;x,y) &= i 2^{-1-(n/2)} \pi^{1-(n/2)} z^{n/2} |x-y|^{1-(n/2)} H^{(1)}_{(n/2)-1}\big(z|x-y|\big) I_N \\ &\quad - 2^{-1-(n/2)} \pi^{1-(n/2)} z^{n/2} |x-y|^{1-(n/2)} H^{(1)}_{n/2}\big(z|x-y|\big) \alpha \cdot \frac{(x-y)}{|x-y|}, \\ &\qquad z \in \mathbb{C}_+,\ x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N},\ n \geq 2. \end{aligned} \tag{C.1}$$

Due to the distinct difference in the behavior of the Hankel function $H^{(1)}_\nu$ for integer and fractional values of ν, we treat separately the cases where: Case (I) n is odd, Case (II) $n \in \mathbb{N}\backslash\{2\}$ is even, and Case (III) $n = 2$.

Case (I). If $n \in \mathbb{N}$ is odd, then $H^{(1)}_{(n/2)-1}$ and $H^{(1)}_{n/2}$ are fractional (half-integer) Hankel functions. Applying the identity (cf., e.g., [1, Equation 9.1.3])

$$H^{(1)}_\nu(\zeta) = i\big[\sin(\nu\pi)\big]^{-1}\big[e^{-\nu\pi i} J_\nu(\zeta) - J_{-\nu}(\zeta)\big], \quad \nu \in \mathbb{C},\ \zeta \in \mathbb{C}\backslash\{0\}, \tag{C.2}$$

one obtains

$$\begin{aligned} H^{(1)}_{(n/2)-1}(\zeta) &= i\big[\sin\big(((n/2)-1)\pi\big)\big]^{-1}\big[e^{-((n/2)-1)\pi i} J_{(n/2)-1}(\zeta) - J_{-(n/2)+1}(\zeta)\big] \\ &= i(-1)^{(n+1)/2}\big[-i(-1)^{(n+1)/2} J_{(n/2)-1}(\zeta) - J_{-(n/2)+1}(\zeta)\big] \\ &= (-1)^{n+1} J_{(n/2)-1}(\zeta) - i(-1)^{(n+1)/2} J_{-(n/2)+1}(\zeta) \\ &= J_{(n/2)-1}(\zeta) - i(-1)^{(n+1)/2} J_{-(n/2)+1}(\zeta). \end{aligned} \tag{C.3}$$

Similarly,

$$\begin{aligned} H^{(1)}_{n/2}(\zeta) &= i\big[\sin((n/2)\pi)\big]^{-1}\big[e^{-n\pi i/2} J_{n/2}(\zeta) - J_{-n/2}(\zeta)\big] \\ &= i(-1)^{(n-1)/2}\big[-i(-1)^{(n-1)/2} J_{n/2}(\zeta) - J_{-n/2}(\zeta)\big] \\ &= (-1)^{n-1} J_{n/2}(\zeta) + i(-1)^{(n+1)/2} J_{-n/2}(\zeta) \\ &= J_{n/2}(\zeta) + i(-1)^{(n+1)/2} J_{-n/2}(\zeta). \end{aligned} \tag{C.4}$$

The series representation for $J_\nu(\zeta)$ in (B.2) then yields the following expansion:

$$\begin{aligned}
G_0(z;x,y) &= i2^{-n}\pi^{1-(n/2)}\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k+n-1}|x-y|^{2k}}{k!\,\Gamma((n/2)+k)}I_N \\
&+ 4^{-1}(-1)^{(n+1)/2}\pi^{1-(n/2)}|x-y|^{2-n}\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k+1}|x-y|^{2k}}{k!\,\Gamma(-(n/2)+k+2)}I_N \\
&- 2^{-1-n}\pi^{1-(n/2)}|x-y|\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k+n}|x-y|^{2k}}{k!\,\Gamma((n/2)+k+1)}\alpha\cdot\frac{(x-y)}{|x-y|} \\
&- i2^{-1}(-1)^{(n+1)/2}\pi^{1-(n/2)}|x-y|^{1-n}\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k}|x-y|^{2k}}{k!\,\Gamma(-(n/2)+k+1)}\alpha\cdot\frac{(x-y)}{|x-y|}. \qquad \text{(C.5)}
\end{aligned}$$

The identity in (C.5) implies

$$\begin{aligned}
G_0(z;x,y) = &-\frac{i(-1)^{(n+1)/2}\pi^{1-(n/2)}}{2\,\Gamma\big(1-(n/2)\big)}|x-y|^{1-n}\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|} \\
&+\frac{(-1)^{(n+1)/2}\pi^{1-(n/2)}}{4\,\Gamma\big(2-(n/2)\big)}|x-y|^{2-n}z\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+\frac{i\pi^{1-(n/2)}}{2^n\Gamma(n/2)}z^{n-1}\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&-\frac{\pi^{1-(n/2)}}{2^{1+n}\Gamma\big(1+(n/2)\big)}|x-y|z^n\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|}, \\
&\text{as } z\to 0,\ z\in\overline{\mathbb{C}_+}\backslash\{0\},\ x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N}\backslash\{1\} \text{ odd.} \qquad \text{(C.6)}
\end{aligned}$$

One notes that (C.6) implies, together with the identity,

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)}, \quad z\in\mathbb{C}\backslash\mathbb{Z},$$

that

$$\begin{aligned}
\lim_{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}} G_0(z;x,y) &= -\frac{i(-1)^{(n+1)/2}\pi^{1-(n/2)}}{2\,\Gamma(1-(n/2))}|x-y|^{1-n}\alpha\cdot\frac{(x-y)}{|x-y|} \\
&= -\frac{i(-1)^{(n+1)/2}\pi^{1-(n/2)}}{2\sin(n\pi/2)}\Gamma(n/2)|x-y|^{1-n}\alpha\cdot\frac{(x-y)}{|x-y|} \\
&= i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha\cdot\frac{(x-y)}{|x-y|^n}, \\
&\quad x,y\in\mathbb{R}^n,\ x\neq y,\ n\in\mathbb{N}\backslash\{1\} \text{ odd,}
\end{aligned}$$

consistent with (5.10).

Partial derivatives of $G_0(z;x,y)$ with respect to z may be computed by differentiating the series representations in (C.5) term-by-term. For $n \in \mathbb{N}\backslash\{1\}$ odd and $r \in \mathbb{N}$ with $1 \le r \le n$, one obtains

$$\begin{aligned}
&\frac{\partial^r}{\partial z^r} G_0(z;x,y) \\
&= i2^{-n}\pi^{1-(n/2)} \sum_{k=\delta_n(r)}^{\infty} \frac{(-4)^{-k}(2k+n-1)!\, z^{2k+n-1-r}|x-y|^{2k}}{k!\,(2k+n-1-r)!\,\Gamma\big((n/2)+k\big)} I_N \\
&\quad + 4^{-1}(-1)^{(n+1)/2}\pi^{1-(n/2)}|x-y|^{2-n} \\
&\quad \times \sum_{k=k_-(r)}^{\infty} \frac{(-4)^{-k}(2k+1)!\, z^{2k+1-r}|x-y|^{2k}}{k!\,(2k+1-r)!\,\Gamma\big(-(n/2)+k+2\big)} I_N \\
&\quad - 2^{-1-n}\pi^{1-(n/2)}|x-y| \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+n)!\, z^{2k+n-r}|x-y|^{2k}}{k!\,(2k+n-r)!\,\Gamma\big((n/2)+k+1\big)} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad - i2^{-1}(-1)^{(n+1)/2}\pi^{1-(n/2)}|x-y|^{1-n} \\
&\quad \times \sum_{k=k_+(r)}^{\infty} \frac{(-4)^{-k}(2k)!\, z^{2k-r}|x-y|^{2k}}{k!\,(2k-r)!\,\Gamma\big(-(n/2)+k+1\big)} \alpha \cdot \frac{(x-y)}{|x-y|},
\end{aligned} \tag{C.7}$$

where

$$k_\pm(r) := \begin{cases} (r \pm 1)/2, & r \text{ odd}, \\ r/2, & r \text{ even}, \end{cases} \quad 1 \le r \le n, \tag{C.8}$$

and δ_n is the Kronecker delta function,

$$\delta_n(r) = \begin{cases} 1, & r = n, \\ 0, & 1 \le r \le n-1, \end{cases} \quad 1 \le r \le n. \tag{C.9}$$

The expansion in (C.7) implies the following asymptotics of $\frac{\partial^r}{\partial z^r} G_0(z;x,y)$ as $z \to 0$:

(i) If $n \in \mathbb{N}$ is odd and $1 \le r \le n-2$ is odd, then

$$\begin{aligned}
\frac{\partial^r}{\partial z^r} G_0(z;x,y) &= \frac{i2^{-n}\pi^{1-(n/2)}(n-1)!}{(n-1-r)!\Gamma(n/2)} z^{n-1-r}\big[1+O\big(z^2|x-y|^2\big)\big] I_N \\
&+ \frac{(-1)^{(n+1)/2}\pi^{1-(n/2)}(-4)^{-(r+1)/2} r!}{[(r-1)/2]!\,\Gamma(-(n/2)+((r-1)/2)+2)} |x-y|^{1+r-n}\big[1+O\big(z^2|x-y|^2\big)\big] I_N \\
&- \frac{2^{-1-n}\pi^{1-(n/2)} n!}{(n-r)!\,\Gamma(1+(n/2))} |x-y| z^{n-r}\big[1+O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i2^{-1}(-1)^{(n+1)/2}\pi^{1-(n/2)}(-4)^{-(r+1)/2}(r+1)!}{[(r+1)/2]!\,\Gamma(-(n/2)+((r+1)/2)+1)} |x-y|^{2+r-n} z \\
&\times \big[1+O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|}, \text{ as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x,y \in \mathbb{R}^n,\ x \ne y.
\end{aligned} \tag{C.10}$$

(ii) If $n \in \mathbb{N}$ is odd and $1 \leq r \leq n-1$ is even, then

$$\frac{\partial^r}{\partial z^r} G_0(z; x, y) = \frac{i 2^{-n} \pi^{1-(n/2)} (n-1)!}{(n-1-r)! \Gamma(n/2)} z^{n-1-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N$$
$$+ \frac{4^{-1} (-1)^{(n+1)/2} \pi^{1-(n/2)} (-4)^{-r/2} (r+1)!}{(r/2)!\, \Gamma\big(-(n/2) + (r/2) + 2\big)} |x-y|^{2+r-n}$$
$$\times z \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N$$
$$- \frac{2^{-1-n} \pi^{1-(n/2)} n!}{(n-r)!\, \Gamma(1 + (n/2))} |x-y| z^{n-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}$$
$$- \frac{i 2^{-1} (-1)^{(n+1)/2} \pi^{1-(n/2)} (-4)^{-r/2} r!}{(r/2)!\, \Gamma\big(-(n/2) + (r/2) + 1\big)} |x-y|^{1+r-n}$$
$$\times \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \text{ as } z \to 0,\ z \in \overline{\mathbb{C}_+} \backslash \{0\},\ x, y \in \mathbb{R}^n,\ x \neq y. \quad \text{(C.11)}$$

(iii) If $n \in \mathbb{N}$ is odd, then

$$\frac{\partial^n}{\partial z^n} G_0(z; x, y) = -\frac{i 2^{-n} \pi^{1-(n/2)}}{4 \Gamma(1 + (n/2))} |x-y|^2 z \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N$$
$$+ \frac{2 (-1)^{(n+1)/2} \pi^{(1-n)/2} (-4)^{-(n+1)/2} n!}{[(n-1)/2]!} |x-y| \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N$$
$$- \frac{2^{-1-n} \pi^{1-(n/2)} n!}{\Gamma(1 + (n/2))} |x-y| \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}$$
$$- \frac{i (-1)^{(n+1)/2} \pi^{(1-n)/2} (-4)^{-(n+1)/2} (n+1)!}{[(n+1)/2]!} |x-y|^2 z$$
$$\times \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \text{ as } z \to 0,\ z \in \overline{\mathbb{C}_+} \backslash \{0\},\ x, y \in \mathbb{R}^n,\ x \neq y. \quad \text{(C.12)}$$

Case (II). If $n \in \mathbb{N}$ is even, then the indices of the Hankel functions $H^{(1)}_{(n/2)-1}$ and $H^{(1)}_{n/2}$ are nonnegative integers. Due to the difference in behavior of Y_n, $n \in \mathbb{N}$, and Y_0 (cf. (B.4) and (B.5)) we distinguish two cases: $n \geq 4$ and $n = 2$. First we treat the case $n \geq 4$.

Combining (B.1), (B.2), and (B.4), one obtains for $n \geq 4$:

$$H^{(1)}_{(n/2)-1}(\zeta) = J_{(n/2)-1}(\zeta) + i Y_{(n/2)-1}(\zeta)$$
$$= 2^{1-(n/2)} \zeta^{(n/2)-1} \sum_{k=0}^{\infty} \frac{(-4)^{-k} \zeta^{2k}}{k!\, \Gamma\big((n/2) + k\big)}$$
$$- i 2^{(n/2)-1} \pi^{-1} \zeta^{1-(n/2)} \sum_{k=0}^{(n/2)-2} \frac{\big((n/2) - k - 2\big)! 4^{-k} \zeta^{2k}}{k!}$$

$$+ i2\pi^{-1}\ln(\zeta/2)2^{1-(n/2)}\zeta^{(n/2)-1}\sum_{k=0}^{\infty}\frac{(-4)^{-k}\zeta^{2k}}{k!\,\Gamma\big((n/2)+k\big)}$$

$$- i2^{1-(n/2)}\pi^{-1}\zeta^{(n/2)-1}\sum_{k=0}^{\infty}\big[\psi(k+1)+\psi\big((n/2)+k\big)\big]\frac{(-4)^{-k}\zeta 2k}{k!\,\big((n/2)-1+k\big)!}$$

$$= 2^{1-n/2}\zeta^{(n/2)-1}\sum_{k=0}^{\infty}\frac{(-4)^{-k}\zeta^{2k}}{k!\,\big((n/2)+k-1\big)!}$$

$$- i2^{(n/2)-1}\pi^{-1}\zeta^{1-(n/2)}\sum_{k=0}^{(n/2)-2}\frac{\big((n/2)-k-2\big)!4^{-k}\zeta^{2k}}{k!}$$

$$+ i2\pi^{-1}\ln(\zeta/2)2^{1-(n/2)}\zeta^{(n/2)-1}\sum_{k=0}^{\infty}\frac{(-4)^{-k}\zeta^{2k}}{k!\,\big((n/2)+k-1\big)!}$$

$$- i2^{1-(n/2)}\pi^{-1}\zeta^{(n/2)-1}\sum_{k=0}^{\infty}\big[\psi(k+1)+\psi\big((n/2)+k\big)\big]\frac{(-4)^{-k}\zeta 2k}{k!\,\big((n/2)-1+k\big)!}. \quad \text{(C.13)}$$

Next, for any even $n \in \mathbb{N}$,

$$H^{(1)}_{(n/2)}(\zeta) = 2^{-n/2}\zeta^{n/2}\sum_{k=0}^{\infty}\frac{(-4)^{-k}\zeta^{2k}}{k!\,\big((n/2)+k\big)!}$$

$$- i2^{n/2}\pi^{-1}\zeta^{-n/2}\sum_{k=0}^{(n/2)-1}\frac{\big((n/2)-k-1\big)!4^{-k}\zeta^{2k}}{k!}$$

$$+ i2^{1-(n/2)}\pi^{-1}\ln(\zeta/2)\zeta^{n/2}\sum_{k=0}^{\infty}\frac{(-4)^{-k}\zeta^{2k}}{k!\,\big((n/2)+k\big)!}$$

$$- i2^{-n/2}\pi^{-1}\zeta^{n/2}\sum_{k=0}^{\infty}\big[\psi(k+1)+\psi(n/2+k+1)\big]\frac{(-4)^{-k}\zeta^{2k}}{k!\,\big((n/2)+k\big)!}. \quad \text{(C.14)}$$

Substitution of (C.13) and (C.14) into (C.1) then yields for even $n \geq 4$,

$$G_0(z;x,y) = i2^{-n}\pi^{1-(n/2)}\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k+n-1}|x-y|^{2k}}{k!\,\big((n/2)+k-1\big)!}I_N$$

$$+ 4^{-1}\pi^{-n/2}|x-y|^{2-n}\sum_{k=0}^{(n/2)-2}\frac{\big((n/2)-k-2\big)!\,4^{-k}z^{2k+1}|x-y|^{2k}}{k!}I_N$$

$$- 2^{1-n}\pi^{-n/2}\ln\big(z|x-y|/2\big)\sum_{k=0}^{\infty}\frac{(-4)^{-k}z^{2k+n-1}|x-y|^{2k}}{k!\,\big((n/2)+k-1\big)!}I_N$$

$$+ 2^{-n}\pi^{-n/2}\sum_{k=0}^{\infty}\big[\psi(k+1)+\psi\big((n/2)+k\big)\big]\frac{(-4)^{-k}z^{2k+n-1}|x-y|^{2k}}{k!\,\big((n/2)-1+k\big)!}I_N$$

$$
\begin{aligned}
&- 2^{-1-n}\pi^{1-(n/2)}|x-y| \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+n}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!}\alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ i2^{-1}\pi^{-n/2}|x-y|^{1-n} \sum_{k=0}^{(n/2)-1} \frac{\big((n/2)-k-1\big)!\,4^{-k} z^{2k}|x-y|^{2k}}{k!}\alpha \cdot \frac{(x-y)}{|x-y|} \\
&- i2^{-n}\pi^{-n/2}|x-y| \ln\big(z|x-y|/2\big) \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+n}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!}\alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ i2^{-1-n}\pi^{-n/2}|x-y| \sum_{k=0}^{\infty} \big[\psi(k+1) + \psi\big((n/2)+k+1\big)\big] \\
&\times \frac{(-4)^{-k} z^{2k+n}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!}\alpha \cdot \frac{(x-y)}{|x-y|}. \qquad \text{(C.15)}
\end{aligned}
$$

The identity in (C.15) implies

$$
\begin{aligned}
G_0(z;x,y) &= \frac{i}{2^n \pi^{(n/2)-1}\big((n/2)-1\big)!} z^{n-1}\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&+ \frac{\big((n/2)-2\big)!}{4\pi^{n/2}} |x-y|^{2-n} z \big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&- \frac{1}{2^n \pi^{n/2}\big((n/2)-1\big)!} \ln\big(z|x-y|/2\big) z^{n-1}\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&+ \frac{\psi(1)+\psi(n/2)}{2^n\pi^{n/2}} z^{n-1}\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&- \frac{1}{2^{1+n}\pi^{(n/2)-1}(n/2)!} |x-y| z^n \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i\big((n/2)-1\big)!}{2\pi^{n/2}} |x-y|^{1-n} \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i}{2^n\pi^{n/2}(n/2)!} |x-y| z^n \ln\big(z|x-y|/2\big) \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i\big[\psi(1) + \psi\big((n/2)+1\big)\big]}{2^{1+n}\pi^{n/2}(n/2)!} |x-y| z^n \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \\
&\text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x,y \in \mathbb{R}^n,\ x \neq y, \text{ and } n \in \mathbb{N}\backslash\{2\} \text{ even.} \qquad \text{(C.16)}
\end{aligned}
$$

One notes that (C.16) implies, together with $(n/2-1)! = \Gamma(n/2)$, that

$$
\lim_{\substack{z\to 0,\\ z\in\overline{\mathbb{C}_+}\backslash\{0\}}} G_0(z;x,y) = i2^{-1}\pi^{-n/2}\Gamma(n/2)\alpha \cdot \frac{(x-y)}{|x-y|^n},
$$
$$
x,y \in \mathbb{R}^n,\ x \neq y,\ n \in \mathbb{N}\backslash\{2\} \text{ even}, \qquad \text{(C.17)}
$$

consistent with (5.10).

For $n \in \mathbb{N}\backslash\{2\}$ even and $r \in \mathbb{N}$ with $1 \leq r \leq n$, term-by-term differentiation of (C.15) implies

$$
\begin{aligned}
\frac{\partial^r}{\partial z^r} G_0(z;x,y) = \Bigg[& i 2^{-n} \pi^{1-(n/2)} \sum_{k=\delta_n(r)}^{\infty} \frac{(-4)^{-k}(2k+n-1)!\, z^{2k+n-1-r} |x-y|^{2k}}{k!\,(2k+n-1-r)!\,\big((n/2)+k-1\big)!} \\
& + \frac{|x-y|^{2-n}}{4\pi^{n/2}} \chi_{\leq n-3}(r) \sum_{k=k_-(r)}^{(n/2)-2} \frac{\big((n/2)-k-2\big)!\, 4^{-k}(2k+1)!\, z^{2k+1-r}|x-y|^{2k}}{k!\,(2k+1-r)!} \\
& - 2^{1-n}\pi^{-n/2} \ln\big(z|x-y|/2\big) \sum_{k=\delta_n(r)}^{\infty} \frac{(-4)^{-k}(2k+n-1)!\, z^{2k+n-1-r}|x-y|^{2k}}{k!\,(2k+n-1-r)!\,\big((n/2)+k-1\big)!} \\
& - 2^{1-n}\pi^{-n/2} \sum_{\ell=0}^{r-1}\sum_{k=0}^{\infty} \binom{r}{\ell} (-1)^{1+r-\ell}(r-\ell-1)! \\
& \times \frac{(-4)^{-k}(2k+n-1)!\, z^{2k+n-1-r}|x-y|^{2k}}{k!\,(2k+n-1-\ell)!\,\big((n/2)+k-1\big)!} \\
& + 2^{-n}\pi^{-n/2} \sum_{k=\delta_n(r)}^{\infty} \big[\psi(k+1)+\psi\big((n/2)+k\big)\big] \\
& \times \frac{(-4)^{-k}(2k+n-1)!\, z^{2k+n-1-r}|x-y|^{2k}}{k!\,(2k+n-1-r)!\,\big((n/2)+k-1\big)!} \Bigg] I_N \\
& + \Bigg[-2^{-1-n}\pi^{1-(n/2)}|x-y| \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+n)!\, z^{2k+n-r}|x-y|^{2k}}{k!\,(2k+n-r)!\,\big((n/2)+k\big)!} \\
& + i\frac{|x-y|^{1-n}}{2\pi^{n/2}} \chi_{\leq n-2}(r) \sum_{k=k_+(r)}^{(n/2)-1} \frac{\big((n/2)-k-1\big)!\, 4^{-k}(2k)!\, z^{2k-r}|x-y|^{2k}}{k!\,(2k-r)!} \\
& - i2^{-n}\pi^{-n/2}|x-y| \ln\big(z|x-y|/2\big) \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+n)!\, z^{2k+n-r}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!\,(2k+n-r)!} \\
& - i2^{-n}\pi^{-n/2}|x-y| \sum_{\ell=0}^{r-1}\sum_{k=0}^{\infty} \binom{r}{\ell} (-1)^{1+r-\ell}(r-\ell-1)! \\
& \times \frac{(-4)^{-k}(2k+n)!\, z^{2k+n-r}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!\,(2k+n-\ell)!} \\
& + i2^{-1-n}\pi^{-n/2}|x-y| \sum_{k=0}^{\infty} \big[\psi(k+1)+\psi\big((n/2)+k+1\big)\big] \\
& \times \frac{(-4)^{-k}(2k+n)! z^{2k+n-r}|x-y|^{2k}}{k!\,\big((n/2)+k\big)!} \Bigg] \alpha \cdot \frac{(x-y)}{|x-y|},
\end{aligned} \tag{C.18}
$$

where $\chi_{\leq a}$, $a \in \mathbb{R}$, denotes the characteristic (i.e., indicator) function of the interval $(-\infty, a]$. That is,

$$\chi_{\leq a}(x) = \begin{cases} 1, & x \in (-\infty, a], \\ 0, & x \in (a, \infty), \end{cases} \qquad x \in \mathbb{R}.$$

The expansion in (C.18) implies the following asymptotics of $\frac{\partial^r}{\partial z^r} G_0(z; x, y)$ as $z \to 0$:

(i) If $n \in \mathbb{N}\backslash\{2\}$ is even and $1 \leq r \leq n-1$ is odd, then

$$\begin{aligned}
&\frac{\partial^r}{\partial z^r} G_0(z; x, y) = \frac{i 2^{-n} \pi^{1-(n/2)} (n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N \\
&+ \chi_{\leq n-3}(r) \frac{4^{-(1+r)/2} \pi^{-n/2} \big[(n-r-3)/2\big]!\, r!}{\big[(r-1)/2\big]!} |x-y|^{1+r-n} \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N \\
&- \frac{2^{1-n} \pi^{-n/2} (n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r} \ln\big(z|x-y|/2\big) \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N \\
&+ \frac{2^{1-n}}{\pi^{n/2}} \sum_{\ell=0}^{r-1} \binom{r}{\ell} (-1)^{r-\ell} \frac{(r-\ell-1)!\,(n-1)!}{(n-1-\ell)!\,\big((n/2)-1\big)!} z^{n-1-r} \\
&\quad \times \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N \\
&+ \frac{2^{-n} \pi^{-n/2} [\psi(1) + \psi(n/2)] (n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] I_N \\
&- \frac{2^{-(1+n)} \pi^{1-(n/2)} n!}{(n-r)!\,(n/2)!} |x-y| z^{n-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \chi_{\leq n-2}(r) \frac{i \pi^{-n/2} \big[(n-r-3)/2\big]!\,(r+1)!}{4^{1+(r/2)} \big[(r+1)/2\big]!} |x-y|^{r+2-n} z \\
&\quad \times \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad - \frac{i 2^{-n} \pi^{-n/2} n!}{(n/2)!(n-r)!} |x-y| z^{n-r} \ln(z|x-y|/2) \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad + i 2^{-n} \pi^{-n/2} \sum_{\ell=0}^{r-1} \binom{r}{\ell} \frac{(-1)^{r-\ell} (r-\ell-1)!\, n!}{(n/2)!\,(n-\ell)!} |x-y| z^{n-r} \\
&\quad \times \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i \big[\psi(1) + \psi\big((n/2)+1\big)\big] n!}{2^{1+n} \pi^{n/2} (n/2)!} |x-y| z^{n-r} \big[1 + O\big(z^2 |x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \\
&\quad \text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x, y \in \mathbb{R}^n,\ x \neq y. \qquad \text{(C.19)}
\end{aligned}$$

(ii) If $n \in \mathbb{N}\backslash\{2\}$ is even and $1 \le r \le n-2$ is even with $r \neq n$, then

$$
\begin{aligned}
\frac{\partial^r}{\partial z^r} G_0(z;x,y) &= \frac{i2^{-n}\pi^{1-(n/2)}(n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r}\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+ \chi_{\le n-3}(r)\frac{\big((n/2)-(r/2)-2\big)!\,(r+1)!}{4^{1+(r/2)}\pi^{n/2}(r/2)!}|x-y|^{2+r-n}z\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&- \frac{2^{1-n}\pi^{-n/2}(n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r}\ln\big(z|x-y|/2\big)\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+ \frac{2^{1-n}}{\pi^{n/2}}\sum_{\ell=0}^{r-1}\binom{r}{\ell}(-1)^{r-\ell}\frac{(r-\ell-1)!\,(n-1)!}{(n-1-\ell)!\,\big((n/2)-1\big)!} z^{n-1-r}\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+ \frac{2^{-n}\pi^{-n/2}\big[\psi(1)+\psi(n/2)\big](n-1)!}{(n-1-r)!\,\big((n/2)-1\big)!} z^{n-1-r}\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&- \frac{2^{-(1+n)}\pi^{1-(n/2)}n!}{(n-r)!\,(n/2)!}|x-y|z^{n-r}\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|} \\
&+ \chi_{\le n-2}(r)\frac{i\big((n/2)-(r/2)-1\big)!\,r!}{4^{(r+1)/2}\pi^{n/2}(r/2)!}|x-y|^{1+r-n}\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|} \\
&- \frac{i2^{-n}\pi^{-n/2}n!}{(n/2)!(n-r)!}|x-y|z^{n-r}\ln\big(z|x-y|/2\big)\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|} \\
&+ i2^{-n}\pi^{-n/2}\sum_{\ell=0}^{r-1}\binom{r}{\ell}\frac{(-1)^{r-\ell}(r-\ell-1)!\,n!}{(n/2)!\,(r-\ell)!}|x-y|z^{n-r} \\
&\quad\times\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|} \\
&+ \frac{i\big[\psi(1)+\psi\big((n/2)+1\big)\big]n!}{2^{1+n}\pi^{n/2}(n/2)!}|x-y|z^{n-r}\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|}, \\
&\quad \text{as } z\to 0,\ z\in\overline{\mathbb{C}_+}\backslash\{0\},\ x,y\in\mathbb{R}^n,\ x\neq y. \qquad \text{(C.20)}
\end{aligned}
$$

(iii) If $n \in \mathbb{N}\backslash\{2\}$ is even, then

$$
\begin{aligned}
\frac{\partial^n}{\partial z^n} G_0(z;x,y) &= -\frac{i\pi^{1-(n/2)}(n+1)!}{2^{n+2}(n/2)!}|x-y|^2 z\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+ \frac{2^{1-n}\pi^{-n/2}4^{-1}(n+1)!}{(n/2)!}|x-y|^2 z\ln\big(z|x-y|/2\big)\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&+ \frac{(n-1)!}{2^{n-1}\pi^{n/2}(\frac{n}{2}-1)!}\left(\sum_{\ell=0}^{n-1}\binom{n}{\ell}(-1)^{\ell}\right)z^{-1}\big[1+O\big(z^2|x-y|^2\big)\big]I_N \\
&- \frac{\big[\psi(2)+\psi\big((n/2)+1\big)\big](n+1)!}{2^{n+2}\pi^{n/2}(n/2)!}|x-y|^2 z\big[1+O\big(z^2|x-y|^2\big)\big]I_N
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\pi^{1-(n/2)} n!}{2^{n+1}(n/2)!} |x-y| \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i 2^{-n} \pi^{-n/2} n!}{(n/2)!} |x-y| \ln\big(z|x-y|/2\big) \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i 2^{-n} \pi^{-n/2} n!}{(n/2)!} \left(\sum_{\ell=0}^{n-1} \binom{n}{\ell} \frac{(-1)^\ell}{n-\ell} \right) |x-y| \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i\big[\psi(1) + \psi\big((n/2)+1\big)\big] n!}{2^{1+n}\pi^{n/2}(n/2)!} |x-y| \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \\
&\text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x, y \in \mathbb{R}^n,\ x \neq y.
\end{aligned} \tag{C.21}
$$

Case (III). If $n = 2$, then (B.1), (B.2), and (B.5) imply

$$
\begin{aligned}
H^{(1)}_{(n/2)-1}(\zeta) &= H^{(1)}_0(\zeta) = J_0(\zeta) + i Y_0(\zeta) \\
&= i \frac{2}{\pi} \big[\ln(\zeta/2) + \gamma_{E-M} - i(\pi/2) \big] J_0(\zeta) - \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k} \frac{1}{\ell} \right) \frac{(-4)^{-k} \zeta^{2k}}{(k!)^2} \\
&= i \frac{2}{\pi} \big[\ln(\zeta/2) + \gamma_{E-M} - i(\pi/2) \big] \sum_{k=0}^{\infty} \frac{(-4)^{-k} \zeta^{2k}}{(k!)^2} - \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k} \frac{1}{\ell} \right) \frac{(-4)^{-k} \zeta^{2k}}{(k!)^2}.
\end{aligned} \tag{C.22}
$$

Similarly, by combining (C.1), (C.14) (which is valid for $n = 2$), and (C.22), one obtains for $n = 2$:

$$
\begin{aligned}
G_0(z; x, y) &= i 4^{-1} z H^{(1)}_0\big(z|x-y|\big) I_N - 4^{-1} z H^{(1)}_1\big(z|x-y|\big) \alpha \cdot \frac{(x-y)}{|x-y|} \\
&= -\frac{1}{2\pi} \big[\ln\big(z|x-y|/2\big) + \gamma_{E-M} - i(\pi/2) \big] \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+1} |x-y|^{2k}}{(k!)^2} I_N \\
&\quad - \frac{i}{2\pi} \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k} \frac{1}{\ell} \right) \frac{(-4)^{-k} z^{2k+1} |x-y|^{2k}}{(k!)^2} I_N \\
&\quad - \frac{1}{8} \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+2} |x-y|^{2k+1}}{k!(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad + \frac{i}{2\pi} |x-y|^{-1} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad - \frac{i}{4\pi} \ln\big(z|x-y|/2\big) \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+2} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad + \frac{i}{8\pi} \sum_{k=0}^{\infty} \big[\psi(k+1) + \psi(k+2)\big] \frac{(-4)^{-k} z^{2k+2} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|}.
\end{aligned} \tag{C.23}
$$

The identity in (C.23) implies

$$
\begin{aligned}
G_0(z;x,y) = &-\frac{1}{2\pi} z \ln\big(z|x-y|/2\big)\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&- \frac{1}{2\pi}\big[\gamma_{E-M} - i(\pi/2)\big] z \big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&+ \frac{i}{8\pi} z^3 |x-y|^2 \big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&- \frac{1}{8} z^2 |x-y| \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i}{2\pi} |x-y|^{-1} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i}{4\pi} z^2 |x-y| \ln\big(z|x-y|/2\big)\big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i}{8\pi}\big[\psi(1) + \psi(2)\big] z^2 |x-y| \big[1 + O\big(z^2|x-y|^2\big)\big] \alpha \cdot \frac{(x-y)}{|x-y|}, \\
&\text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x, y \in \mathbb{R}^n,\ x \neq y, \text{ and } n = 2. \qquad \text{(C.24)}
\end{aligned}
$$

One notes that (C.24) implies

$$
\lim_{\substack{z \to 0, \\ z \in \overline{\mathbb{C}_+}\backslash\{0\}}} G_0(z;x,y) = \frac{i}{2\pi} \alpha \cdot \frac{(x-y)}{|x-y|^2}, \quad x, y \in \mathbb{R}^2,\ x \neq y, \qquad \text{(C.25)}
$$

which is consistent with (5.10).

Finally, one employs (C.23) to compute:

$$
\begin{aligned}
\frac{\partial}{\partial z} G_0(z;x,y) = &-\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k} |x-y|^{2k}}{(k!)^2} I_N \\
&- \frac{1}{2\pi}\big[\ln\big(z|x-y|/2\big) + \gamma_{E-M} - i(\pi/2)\big] \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+1) z^{2k} |x-y|^{2k}}{(k!)^2} I_N \\
&- \frac{i}{2\pi} \sum_{k=1}^{\infty} \Bigg(\sum_{\ell=1}^{k} \frac{1}{\ell}\Bigg) \frac{(-4)^{-k}(2k+1) z^{2k} |x-y|^{2k}}{(k!)^2} I_N \\
&- \frac{1}{8} \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+2) z^{2k+1} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i}{4\pi} \sum_{k=0}^{\infty} \frac{(-4)^{-k} z^{2k+1} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&- \frac{i}{4\pi} \ln\big(z|x-y|/2\big) \sum_{k=0}^{\infty} \frac{(-4)^{-k}(2k+2) z^{2k+1} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|} \\
&+ \frac{i}{8\pi} \sum_{k=0}^{\infty} \big[\psi(k+1) + \psi(k+2)\big] \frac{(-4)^{-k}(2k+2) z^{2k+1} |x-y|^{2k+1}}{k!\,(k+1)!} \alpha \cdot \frac{(x-y)}{|x-y|}, \qquad \text{(C.26)}
\end{aligned}
$$

and

$$\frac{\partial^2}{\partial z^2} G_0(z;x,y) = -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{(-4)^{-k}(2k)z^{2k-1}|x-y|^{2k}}{(k!)^2} I_N$$
$$-\frac{1}{2\pi} z^{-1}\sum_{k=0}^{\infty}\frac{(-4)^{-k}(2k+1)z^{2k}|x-y|^{2k}}{(k!)^2} I_N$$
$$-\frac{1}{2\pi}\big[\ln\big(z|x-y|/2\big)+\gamma_{E-M}-i(\pi/2)\big]$$
$$\times\sum_{k=1}^{\infty}\frac{(-4)^{-k}(2k+1)(2k)z^{2k-1}|x-y|^{2k}}{(k!)^2} I_N$$
$$-\frac{i}{2\pi}\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{k}\frac{1}{\ell}\right)\frac{(-4)^{-k}(2k+1)(2k)z^{2k-1}|x-y|^{2k}}{(k!)^2} I_N$$
$$-\frac{1}{8}\sum_{k=0}^{\infty}\frac{(-4)^{-k}(2k+2)(2k+1)z^{2k}|x-y|^{2k+1}}{k!\,(k+1)!}\alpha\cdot\frac{(x-y)}{|x-y|}$$
$$-\frac{i}{4\pi}\sum_{k=0}^{\infty}\frac{(-4)^{-k}(2k+1)z^{2k}|x-y|^{2k+1}}{k!\,(k+1)!}\alpha\cdot\frac{(x-y)}{|x-y|}$$
$$-\frac{i}{4\pi}\sum_{k=0}^{\infty}\frac{(-4)^{-k}(2k+2)z^{2k}|x-y|^{2k+1}}{k!\,(k+1)!}\alpha\cdot\frac{(x-y)}{|x-y|}$$
$$-\frac{i}{4\pi}\ln\big(z|x-y|/2\big)\sum_{k=0}^{\infty}\frac{(-4)^{-k}(2k+2)(2k+1)z^{2k}|x-y|^{2k+1}}{k!\,(k+1)!}\alpha\cdot\frac{(x-y)}{|x-y|}$$
$$+\frac{i}{8\pi}\sum_{k=0}^{\infty}\big[\psi(k+1)+\psi(k+2)\big]\frac{(-4)^{-k}(2k+2)(2k+1)z^{2k}|x-y|^{2k+1}}{k!\,(k+1)!}$$
$$\times\alpha\cdot\frac{(x-y)}{|x-y|}. \tag{C.27}$$

Finally, the expansions in (C.26) and (C.27) imply the following asymptotics of $\frac{\partial^r}{\partial z^r}G_0(z;x,y)$, $1 \le r \le 2$, as $z \to 0$:

(i) If $n=2$, $r=1$, then

$$\frac{\partial}{\partial z}G_0(z;x,y)$$
$$= -\frac{1}{2\pi}\big[\ln\big(z|x-y|/2\big)+1+\gamma_{E-M}-i(\pi/2)\big]\big[1+O\big(z^2|x-y|^2\big)\big]I_N$$
$$+\frac{3i}{8\pi}z^2|x-y|^2\big[1+O\big(z^2|x-y|^2\big)\big]I_N$$
$$-\frac{1}{4\pi}\{\pi+i-i\big[\psi(1)+\psi(2)\big]\}z|x-y|\big[1+O\big(z^2|x-y|^2\big)\big]\alpha\cdot\frac{(x-y)}{|x-y|}$$

$$- \frac{i}{2\pi} z|x-y| \ln\big(z|x-y|/2\big)\big[1 + O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|},$$
$$\text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x,y \in \mathbb{R}^2,\ x \neq y. \tag{C.28}$$

(ii) If $n = 2$, $r = 2$, then

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2} G_0(z;x,y) \\
&= \frac{1}{4\pi}\big[1 + 3(\gamma_{E-M} + i) - (3i\pi/2)\big] z|x-y|^2 \big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&\quad - \frac{1}{2\pi} z^{-1}\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&\quad + \frac{3}{4\pi} z|x-y|^2 \ln\big(z|x-y|/2\big)\big[1 + O\big(z^2|x-y|^2\big)\big] I_N \\
&\quad - \frac{\pi + 3i}{4\pi}|x-y|\big[1 + O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad - \frac{i}{2\pi} \ln\big(z|x-y|/2\big)|x-y|\big[1 + O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|} \\
&\quad + \frac{i}{4\pi}\big[\psi(1) + \psi(2)\big]|x-y|\big[1 + O\big(z^2|x-y|^2\big)\big]\alpha \cdot \frac{(x-y)}{|x-y|}, \\
&\quad \text{as } z \to 0,\ z \in \overline{\mathbb{C}_+}\backslash\{0\},\ x,y \in \mathbb{R}^2,\ x \neq y.
\end{aligned} \tag{C.29}$$

Given the results in Appendices B and C, we can summarize the estimates on $G_0(z;\cdot,\cdot)$ as follows:

Theorem C.1. *Let $r \in \mathbb{N}_0$, $0 \le r \le n$, $z \in \overline{\mathbb{C}_+}$, and $x, y \in \mathbb{R}^n$, $x \neq y$.*

(i) *For $n \in \mathbb{N}$ odd, $n \ge 3$, one has the estimate*

$$\begin{aligned}
&\bigg\| \frac{\partial^r}{\partial z^r} G_0(z;x,y) \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\le \hat{c}_n \begin{cases} |x-y|^{r+1-n}, & |z||x-y| \le 1, \\ |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2} e^{-\operatorname{Im}(z)|x-y|}, & |z||x-y| \ge 1 \end{cases} \\
&\le \tilde{c}_n |x-y|^{r+1-n}\big[1 + |z|^{(n-1)/2}|x-y|^{(n-1)/2}\big] \\
&\le \widetilde{C}_n \big\{|x-y|^{r+1-n}\chi_{[0,1]}\big(|z||x-y|\big) \\
&\quad + |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2}\chi_{[1,\infty)}\big(|z||x-y|\big)\big\} \\
&\le C_n \big\{|x-y|^{r+1-n}\chi_{[0,1]}\big(|z||x-y|\big) \\
&\quad + |z|^{(n-1)/2}\big[|x|^{(2r+1-n)/2} + |y|^{(2r+1-n)/2}\big]\chi_{[1,\infty)}\big(|z||x-y|\big)\big\} \\
&\le c_n \big\{|x-y|^{r+1-n}\chi_{[0,1]}\big(|z||x-y|\big) \\
&\quad + |z|^{(n-1)/2}\big[1+|x|\big]^{(2r+1-n)/2}\big[1+|y|\big]^{(2r+1-n)/2}\chi_{[1,\infty)}\big(|z||x-y|\big)\big\},
\end{aligned} \tag{C.30}$$

where $\hat{c}_n, \tilde{c}_n, \widetilde{C}_n, C_n, c_n \in (0,\infty)$ are appropriate constants.

(ii) *For $n \in \mathbb{N}$ even, one has the following estimate. For every $\delta \in (0,1)$,*

$$
\begin{aligned}
&\left\| \frac{\partial^r}{\partial z^r} G_0(z;x,y) \right\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq c_n \begin{cases} |x-y|^{r+1-n}\big[1+\big|\ln\big(z|x-y|/2\big)\big|\big], & |z||x-y|\leq 1,\ r\neq n, \\ |x-y|\big[1+\big|\ln\big(z|x-y|/2\big)\big|\big]+|z|^{-1}, & |z||x-y|\leq 1,\ r=n, \\ |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2}e^{-\operatorname{Im}(z)|x-y|}, & |z||x-y|\geq 1 \end{cases} \\
&\leq \tilde{c}_{n,\delta} \begin{cases} |z|^{-\delta}|x-y|^{r+1-\delta-n}, & |z||x-y|\leq 1,\ r\neq n, \\ |z|^{-\delta}|x-y|^{1-\delta}+|z|^{-1}, & |z||x-y|\leq 1,\ r=n, \\ |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2}e^{-\operatorname{Im}(z)|x-y|}, & |z||x-y|\geq 1 \end{cases} \\
&= \tilde{c}_{n,\delta} \begin{cases} |z|^{-\delta}|x-y|^{r+1-\delta-n}, & |z||x-y|\leq 1,\ r\neq n, \\ |z|^{-1}\big[|z|^{1-\delta}|x-y|^{1-\delta}+1\big], & |z||x-y|\leq 1,\ r=n, \\ |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2}e^{-\operatorname{Im}(z)|x-y|}, & |z||x-y|\geq 1 \end{cases} \\
&\leq C_{n,\delta} \begin{cases} |z|^{-\delta}|x-y|^{r+1-\delta-n}, & |z||x-y|\leq 1,\ r\neq n, \\ |z|^{-1}, & |z||x-y|\leq 1,\ r=n, \\ |z|^{(n-1)/2}|x-y|^{(2r+1-n)/2}e^{-\operatorname{Im}(z)|x-y|}, & |z||x-y|\geq 1, \end{cases} \qquad \text{(C.31)}
\end{aligned}
$$

where $c_n, \tilde{c}_{n,\delta}, C_{n,\delta} \in (0,\infty)$ are appropriate constants.

Proof. The first estimate in (C.30) (resp., in (C.31)) follows immediately in the regime $|z||x-y| \leq 1$ from (C.10), (C.11), and (C.12) (resp., (C.19), (C.20), and (C.21) and (C.24), (C.28), and (C.29)). One employs Lemma B.6 in conjunction with (C.1) to obtain the first estimate in (C.30), and (C.31) in the regime $|z||x-y| \geq 1$. In fact, by Lemma B.6 and (C.1), $G_0(z;\cdot,\cdot)$ is of the form

$$
\begin{aligned}
G_0(z;x,y) &= c_1 z^{n/2}|x-y|^{1-(n/2)}H^{(1)}_{(n/2)-1}\big(z|x-y|\big)I_N \\
&\quad + c_2 z^{n/2}|x-y|^{1-(n/2)}H^{(1)}_{n/2}\big(z|x-y|\big)\alpha\cdot\frac{(x-y)}{|x-y|} \\
&= z^{n/2}|x-y|^{1-(n/2)}e^{iz|x-y|} \\
&\quad \times \left[c_1\omega_{(n/2)-1}\big(z|x-y|\big)I_N + c_2\omega_{n/2}\big(z|x-y|\big)\alpha\cdot\frac{(x-y)}{|x-y|}\right], \\
&\qquad x,y\in\mathbb{R}^n,\ x\neq y,\ z\in\mathbb{C}_+, \qquad \text{(C.32)}
\end{aligned}
$$

for an appropriate pair of constants $c_1, c_2 \in \mathbb{C}$. The constants c_1 and c_2 are independent of (z,x,y), and their precise values are immaterial for the purpose at hand.

Differentiating throughout (C.32) with respect to z, one obtains

$$\frac{\partial^r}{\partial z^r} G_0(z;x,y) = \sum_{\substack{j,k,\ell \in \mathbb{N}_0 \\ j+k+\ell=r}} c_{j,k,\ell} z^{(n/2)-j} |x-y|^{r-j+1-(n/2)} e^{iz|x-y|} \times \bigg[c_1 \omega_{(n/2)-1}^{(\ell)}\big(z|x-y|\big) I_N + c_2 \omega_{n/2}^{(\ell)}\big(z|x-y|\big) \alpha \cdot \frac{(x-y)}{|x-y|} \bigg], \quad x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \mathbb{C}_+, \tag{C.33}$$

where the $c_{j,k,\ell}$ are constants which do not depend upon (z,x,y). By (C.33),

$$\begin{aligned}
&\bigg\| \frac{\partial^r}{\partial z^r} G_0(z;x,y) \bigg\|_{\mathcal{B}(\mathbb{C}^N)} \\
&\leq \sum_{\substack{j,k,\ell \in \mathbb{N}_0 \\ j+k+\ell=r}} \tilde{c}_{j,k,\ell} |z|^{(n/2)-j} |x-y|^{r-j+1-(n/2)} e^{-\operatorname{Im}(z)|x-y|} |z|^{-(1/2)-\ell} |x-y|^{-(1/2)-\ell} \\
&\leq \sum_{\substack{j,k,\ell \in \mathbb{N}_0 \\ j+k+\ell=r}} \tilde{c}_{j,k,\ell} |z|^{[(n-1)/2]-(j+\ell)} |x-y|^{r-(j+\ell)+(1/2)-(n/2)} e^{-\operatorname{Im}(z)|x-y|} \\
&\leq \widetilde{C} |z|^{[(n-1)/2]} \big[|z|^{j+\ell} |x-y|^{j+\ell} \big]^{-1} |x-y|^{(2r+1-n)/2} e^{-\operatorname{Im}(z)|x-y|} \\
&\leq \widetilde{C} |z|^{[(n-1)/2]} |x-y|^{(2r+1-n)/2} e^{-\operatorname{Im}(z)|x-y|}, \\
&\quad x,y \in \mathbb{R}^n,\ x \neq y,\ z \in \mathbb{C}_+,\ |z||x-y| \geq 1,
\end{aligned}$$

where the $\tilde{c}_{j,k,\ell}$ are constants which do not depend upon (z,x,y). ∎