Appendix D

A product formula for modified Fredholm determinants

The purpose of this appendix is to prove a product formula for regularized (modified) Fredholm determinants extending the well-known Hilbert–Schmidt case.

The result we have in mind is a quantitative version of the following fact:

Theorem D.1. *Let $k \in \mathbb{N}$, and suppose $A, B \in \mathcal{B}_k(\mathcal{H})$. Then*

$$\begin{aligned} &\det_{\mathcal{H},k}\big((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)\big) \\ &\quad = \det_{\mathcal{H},k}(I_{\mathcal{H}} - A)\det_{\mathcal{H},k}(I_{\mathcal{H}} - B)\exp\big(\operatorname{tr}_{\mathcal{H}}\big(X_k(A, B)\big)\big), \end{aligned} \tag{D.1}$$

where $X_k(\cdot\,,\cdot) \in \mathcal{B}_1(\mathcal{H})$ is of the form

$$\begin{aligned} X_1(A, B) &= 0, \\ X_k(A, B) &= \sum_{j_1,\dots,j_{2k-2}=0}^{k-1} c_{j_1,\dots,j_{2k-2}} C_1^{j_1} \cdots C_{2k-2}^{j_{2k-2}}, \quad k \geq 2, \end{aligned}$$

with

$$c_{j_1,\dots,j_{2k-2}} \in \mathbb{Q}, \quad C_\ell = A \text{ or } B, \ 1 \leq \ell \leq 2k-2, \quad k \leq \sum_{\ell=1}^{2k-2} j_\ell \leq 2k-2, \ k \geq 2.$$

Explicitly, one obtains

$$\begin{aligned} X_1(A, B) &= 0, \\ X_2(A, B) &= -AB, \\ X_3(A, B) &= 2^{-1}\big[(AB)^2 - AB(A+B) - (A+B)AB\big], \\ X_4(A, B) &= 2^{-1}(AB)^2 - 3^{-1}\big[AB(A+B)^2 + (A+B)^2AB + (A+B)AB(A+B)\big] \\ &\quad + 3^{-1}\big[(AB)^2(A+B) + (A+B)(AB)^2 + AB(A+B)AB\big] - 3^{-1}(AB)^3, \end{aligned}$$

etc.

When taking traces (what is actually needed in (D.1)), this simplifies to

$$\begin{aligned} \operatorname{tr}_{\mathcal{H}}\big(X_1(A, B)\big) &= 0, \\ \operatorname{tr}_{\mathcal{H}}\big(X_2(A, B)\big) &= -\operatorname{tr}_{\mathcal{H}}(AB), \\ \operatorname{tr}_{\mathcal{H}}\big(X_3(A, B)\big) &= -\operatorname{tr}_{\mathcal{H}}\big(ABA + BAB - 2^{-1}(AB)^2\big), \\ \operatorname{tr}_{\mathcal{H}}\big(X_4(A, B)\big) &= -\operatorname{tr}_{\mathcal{H}}\big(A^3B + A^2B^2 + AB^3 + 2^{-1}(AB)^2 \\ &\qquad - (AB)^2A - B(AB)^2 + 3^{-1}(AB)^3\big), \end{aligned}$$

etc.

In the rest of this appendix, we will detail the characterization of $X_k(A, B)$ following the paper [36]. We also refer to [71, 90, 91] for related, but somewhat different product formulas for regularized determinants.

To prove a quantitative version of Theorem D.1 and hence derive a formula for $X_k(A, B)$, we first need to recall some facts on the commutator subspace of an algebra of noncommutative polynomials.

Let Pol_2 be the free polynomial algebra in 2 (noncommuting) variables, A and B. Let W be the set of noncommutative monomials (words in the alphabet $\{A, B\}$). (We recall that the set W is a semigroup with respect to concatenation, 1 is the neutral element of this semigroup, that is, 1 is an empty word in this alphabet.) Every $x \in \mathrm{Pol}_2$ can be written as a sum

$$x = \sum_{w \in W} \hat{x}(w) w.$$

Here the coefficients $\hat{x}(w)$ vanish for all but finitely many $w \in W$.

Let $[\mathrm{Pol}_2, \mathrm{Pol}_2]$ be the commutator subspace of Pol_2, that is, the linear span of commutators $[x_1, x_2]$, $x_1, x_2 \in \mathrm{Pol}_2$.

Lemma D.2. *One has $x \in [\mathrm{Pol}_2, \mathrm{Pol}_2]$ provided that*

$$\sum_{m=1}^{L(w)} \hat{x}\big(\sigma^m(w)\big) = 0, \quad w \in W.$$

Here, $L(w)$ is the length of each word $w = w_1 w_2 \cdots w_{L(w)}$, σ is the cyclic shift given by $\sigma(w) = w_2 \cdots w_{L(w)} w_1$.

Proof. One notes that

$$x = \sum_{w \in W} \hat{x}(w) w = \hat{x}(1) + \sum_{w \neq 1} L(w)^{-1} \sum_{m=1}^{L(w)} \hat{x}\big(\sigma^m(w)\big) \sigma^m(w).$$

Obviously, $(\sigma^m(w) - w) \in [\mathrm{Pol}_2, \mathrm{Pol}_2]$ for each positive integer m and thus,

$$x \in \Big(\hat{x}(1) + \sum_{w \neq 1} L(w)^{-1} \sum_{m=1}^{L(w)} \hat{x}\big(\sigma^m(w)\big) w + [\mathrm{Pol}_2, \mathrm{Pol}_2]\Big).$$

By hypothesis, $\hat{x}(1) = 0$ and

$$\sum_{m=1}^{L(w)} \hat{x}\big(\sigma^m(w)\big) = 0, \quad 1 \neq w \in W,$$

completing the proof. ∎

Next, we need some notation. Let $k_1, k_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and set

$$z_{k_1,k_2} = \begin{cases} 0, & k_1 = k_2 = 0, \\ k_1^{-1} A^{k_1}, & k_1 \in \mathbb{N},\ k_2 = 0, \\ k_2^{-1} B^{k_2}, & k_1 = 0,\ k_2 \in \mathbb{N}, \\ \displaystyle\sum_{j=1}^{k_1+k_2} j^{-1} \sum_{\substack{\pi \in S_j,\, |\pi|=3 \\ |\pi_1|+|\pi_3|=k_1 \\ |\pi_2|+|\pi_3|=k_2}} (-1)^{|\pi_3|} z_\pi, & k_1, k_2 \in \mathbb{N}. \end{cases} \tag{D.2}$$

Here, S_j is the set of all partitions of the set $\{1, \dots, j\}$, $1 \le j \le k_1 + k_2$. (The symbol $|\cdot|$ abbreviating the cardinality of a subset of $\mathbb{Z}$.) The condition $|\pi| = 3$ means that π breaks the set $\{1, \dots, j\}$ into exactly 3 pieces denoted by π_1, π_2, and π_3 (some of them can be empty). The element z_π denotes the product

$$z_\pi = \prod_{m=1}^{j} z_{m,\pi}, \quad z_{m,\pi} = \begin{cases} A, & m \in \pi_1, \\ B, & m \in \pi_2, \\ AB, & m \in \pi_3. \end{cases}$$

Finally, let W_{k_1,k_2} be the collection of all words with k_1 letters A and k_2 letters B.

Using this notation we now establish a combinatorial fact.

Lemma D.3. *Let $k_1, k_2 \in \mathbb{N}$. Then*

$$z_{k_1,k_2} = \sum_{w \in W_{k_1,k_2}} \left(\sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right) w,$$

where

$$n(w) = \big|S(w)\big|, \quad S(w) = \big\{1 \le \ell \le L(w) - 1 \mid w_\ell = A,\ w_{\ell+1} = B\big\}.$$

Proof. For each $j \in \{1, \dots, k_1 + k_2\}$, let

$$\Pi_j = \big\{\pi \in S_j \mid |\pi| = 3,\ |\pi_1| + |\pi_3| = k_1,\ |\pi_2| + |\pi_3| = k_2\big\},$$
$$\Pi_{j,w} = \{\pi \in \Pi_j \mid z_\pi = w\}, \quad w \in W_{k_1,k_2}.$$

One observes that $|\pi_3| \le n(w) \le \min\{k_1, k_2\}$ and that

$$j = |\pi_1| + |\pi_2| + |\pi_3| = k_1 + k_2 - |\pi_3|.$$

For any partition $\pi \in \Pi_{j,w}$, let $I \subseteq S(w)$ indicate which subwords AB in w arise from elements in π_3. Then $|I| = |\pi_3| = k_1 + k_2 - j$. Therefore, each partition in $\pi \in \Pi_{j,w}$ is determined by a unique choice of I and each such choice of I determines the choice of π uniquely. This implies that

$$|\Pi_{j,w}| = \binom{n(w)}{k_1 + k_2 - j}.$$

Thus,

$$\begin{aligned} z_{k_1,k_2} &= \sum_{w \in W_{k_1,k_2}} \sum_{j=1}^{k_1+k_2} j^{-1} \sum_{\pi \in \Pi_{j,w}} (-1)^{|\pi_3|} w \\ &= \sum_{w \in W_{k_1,k_2}} \sum_{j=1}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} |\Pi_{j,w}| w \\ &= \sum_{w \in W_{k_1,k_2}} \sum_{j=1}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} \binom{n(w)}{k_1 + k_2 - j} w. \end{aligned}$$

Taking into account that

$$\binom{n(w)}{k_1 + k_2 - j} = 0, \quad k_1 + k_2 - j \notin \{0, \dots, n(w)\},$$

it follows that

$$\begin{aligned} z_{k_1,k_2} &= \sum_{w \in W_{k_1,k_2}} \sum_{j=k_1+k_2-n(w)}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} \binom{n(w)}{k_1 + k_2 - j} w \\ &= \sum_{w \in W_{k_1,k_2}} \left(\sum_{\ell=0}^{n(w)} \frac{(-1)^{\ell}}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right) w. \end{aligned}$$

■

We can now prove the main fact about the commutator subspace of Pol_2 needed later on.

Lemma D.4. *For every $k_1, k_2 \in \mathbb{N}$, $z_{k_1,k_2} \in [\mathrm{Pol}_2, \mathrm{Pol}_2]$.*

Proof. Let w be any element in W_{k_1,k_2} and let m be any positive integer. If $\sigma^m(w)$ starts with the subword AB, then $\sigma^{m+1}(w)$ has the form $B \cdots A$ and therefore has one fewer subwords AB than $\sigma^m(w)$; that is, $n(\sigma^{m+1}(w)) = n(\sigma^m(w)) - 1$. If, however, $\sigma^m(w)$ does not start with the subword AB, then the AB subwords of $\sigma^{m+1}(w)$ are precisely the AB subwords of $\sigma^m(w)$ each shifted once; hence, $n(\sigma^{m+1}(w)) = n(\sigma^m(w))$.

Now, to calculate $\sum_{m=1}^{L(w)} \widehat{z_{k_1,k_2}}(\sigma^m(w))$, one may assume, by applying cyclic shifts, that w starts with AB. Then there are $n(w)$ shifted words $\sigma^m(w)$ which start with the subword AB, and it follows that $n(w)$ of the numbers $\{n(\sigma^m(w)) : 1 \leq m \leq L(w)\}$ equal $n(w)-1$ and that the remaining $L(w)-n(w) = k_1+k_2-n(w)$ numbers equal $n(w)$. Lemma D.3 therefore implies that

$$\begin{aligned}\sum_{m=1}^{L(w)} \widehat{z_{k_1,k_2}}\big(\sigma^m(w)\big) &= \sum_{m=1}^{L(w)} \left(\sum_{\ell=0}^{n(\sigma^m(w))} \frac{(-1)^\ell}{k_1+k_2-\ell}\binom{n\big(\sigma^m(w)\big)}{\ell}\right)\\ &= n(w)\left(\sum_{\ell=0}^{n(w)-1} \frac{(-1)^\ell}{k_1+k_2-\ell}\binom{n(w)-1}{\ell}\right)\\ &\quad + \big(k_1+k_2-n(w)\big)\left(\sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1+k_2-\ell}\binom{n(w)}{\ell}\right).\end{aligned}$$

Since

$$\binom{n(w)-1}{n(w)} = 0,$$

it follows that

$$\begin{aligned}\sum_{m=1}^{L(w)} \widehat{z_{k_1,k_2}}\big(\sigma^m(w)\big) &= n(w)\left(\sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1+k_2-\ell}\binom{n(w)-1}{\ell}\right)\\ &\quad + \big(k_1+k_2-n(w)\big)\left(\sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1+k_2-\ell}\binom{n(w)}{\ell}\right)\\ &= \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1+k_2-\ell}\left(n(w)\binom{n(w)-1}{\ell}\right.\\ &\quad \left. + \big(k_1+k_2-n(w)\big)\binom{n(w)}{\ell}\right).\end{aligned}$$

Clearly,

$$n(w)\binom{n(w)-1}{\ell} + \big(k_1+k_2-n(w)\big)\binom{n(w)}{\ell} = (k_1+k_2-\ell)\binom{n(w)}{\ell},$$

and thus

$$\sum_{m=1}^{L(w)} \widehat{z_{k_1,k_2}}\big(\sigma^m(w)\big) = \sum_{\ell=0}^{n(w)} (-1)^\ell \binom{n(w)}{\ell} = 0.$$

Hence, Lemma D.2 completes the proof. ∎

Next, we introduce some further notation. Let $k \in \mathbb{N}$ and set

$$x_1 = 0, \quad x_k = \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A}\subseteq\{1,\ldots,j\}\\ j+|\mathcal{A}|\geq k}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}}, \; k \geq 2,$$

$$y_1 = 0, \quad y_k = \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A}\subseteq\{1,\dots,j\} \\ j+|\mathcal{A}|\leq k-1}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}}, \; k \geq 2,$$

$$y_{\mathcal{A}} = \prod_{m=1}^{j} y_{m,\mathcal{A}}, \quad y_{m,\mathcal{A}} = \begin{cases} A + B, & m \notin \mathcal{A}, \\ AB, & m \in \mathcal{A}. \end{cases} \tag{D.3}$$

In particular,

$$\sum_{j=1}^{k-1} j^{-1}(A + B - AB)^j = x_k + y_k, \tag{D.4}$$

and one notes that the length of the word $y_{\mathcal{A}}$ subject to $\mathcal{A} \subseteq \{1, \dots, j\}$, equals

$$L(y_{\mathcal{A}}) = \left|\mathcal{A}^c\right| + 2|\mathcal{A}| = j + |\mathcal{A}|, \quad 1 \leq j \leq k-1, \; k \geq 2 \tag{D.5}$$

(with $A^c = \{1, \dots, j\}\backslash\mathcal{A}$ the complement of $\mathcal{A}$ in $\{1, \dots, j\}$).

Using this notation we can now state the following fact:

Lemma D.5. *Let $k \in \mathbb{N}$, $k \geq 2$, then*

$$y_k \in \left(\sum_{j=1}^{k-1} \frac{1}{j}(A^j + B^j) + [\mathrm{Pol}_2, \mathrm{Pol}_2] \right).$$

Proof. Employing

$$y_k = \sum_{\substack{k_1,k_2\geq 0 \\ k_1+k_2\leq k-1}} z_{k_1,k_2}, \tag{D.6}$$

Lemma D.4 yields

$$z_{k_1,k_2} \in [\mathrm{Pol}_2, \mathrm{Pol}_2], \quad k_1, k_2 \in \mathbb{N}. \tag{D.7}$$

Since by (D.2),

$$z_{0,0} = 0, \quad z_{k_1,0} = k_1^{-1} A^{k_1}, \; k_1 \in \mathbb{N}, \quad z_{0,k_2} = k_2^{-1} B^{k_2}, \; k_2 \in \mathbb{N}, \tag{D.8}$$

combining (D.6)–(D.8) completes the proof. ∎

After these preparations we are ready to return to the product formula for regularized determinants and specialize the preceding algebraic considerations to the context of Theorem D.1.

First we recall that by (D.3) and (D.5),

$$x_k = \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A}\subseteq\{1,\cdots,j\} \\ j+|\mathcal{A}|\geq k}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}} := X_k(A, B) \in \mathcal{B}_1(\mathcal{H}), \quad k \geq 2,$$

since for $1 \le j \le k-1$, $L(y_{\mathcal{A}}) = j + |\mathcal{A}| \ge k$, and hence one obtains the inequality

$$\|x_k\|_{\mathcal{B}_1(\mathcal{H})} \le c_k \max_{\substack{0 \le k_1, k_2 < k \\ k_1 + k_2 \ge k}} \|A\|^{k_1}_{\mathcal{B}_k(\mathcal{H})} \|B\|^{k_2}_{\mathcal{B}_k(\mathcal{H})}, \quad k \in \mathbb{N},\ k \ge 2,$$

for some $c_k > 0$, $k \ge 2$. We also set (cf. (D.3)) $X_1(A, B) = 0$.

Theorem D.6. *Let $k \in \mathbb{N}$ and assume that $A, B \in \mathcal{B}_k(\mathcal{H})$. Then*

$$\begin{aligned} &\det_{\mathcal{H},k}\big((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)\big) \\ &\quad = \det_{\mathcal{H},k}(I_{\mathcal{H}} - A)\det_{\mathcal{H},k}(I_{\mathcal{H}} - B)\exp\big(\mathrm{tr}_{\mathcal{H}}\big(X_k(A,B)\big)\big). \end{aligned} \tag{D.9}$$

Proof. First, we suppose that $A, B \in \mathcal{B}_1(\mathcal{H})$. Then it is well known that

$$\det_{\mathcal{H},1}(I_{\mathcal{H}} - A)\det_{\mathcal{H},1}(I_{\mathcal{H}} - B) = \det_{\mathcal{H},1}\big((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)\big),$$

consistent with $X_1(A, B) = 0$. Without loss of generality we may assume that $k \in \mathbb{N}$, $k \ge 2$, in the following. Employing

$$\det_{\mathcal{H},k}(I_{\mathcal{H}} - T) = \det_{\mathcal{H}}(I_{\mathcal{H}} - T)\exp\Bigg(\mathrm{tr}_{\mathcal{H}}\bigg(\sum_{j=1}^{k-1} j^{-1} T^j\bigg)\Bigg), \quad T \in \mathcal{B}_1(\mathcal{H}),$$

one infers that

$$\begin{aligned} &\det_{\mathcal{H},k}\big((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)\big) = \det_{\mathcal{H},k}\big(I_{\mathcal{H}} - (A + B - AB)\big) \\ &\quad = \det_{\mathcal{H}}(I_{\mathcal{H}} - (A + B - AB))\exp\Bigg(\mathrm{tr}_{\mathcal{H}}\bigg(\sum_{j=1}^{k-1} j^{-1}(A + B - AB)^j\bigg)\Bigg) \\ &\quad = \det_{\mathcal{H}}(I_{\mathcal{H}} - A)\det_{\mathcal{H}}(I_{\mathcal{H}} - B)\exp\Bigg(\mathrm{tr}_{\mathcal{H}}\bigg(\sum_{j=1}^{k-1} j^{-1}(A + B - AB)^j\bigg)\Bigg) \\ &\quad = \det_{\mathcal{H},k}(I_{\mathcal{H}} - A)\det_{\mathcal{H},k}(I_{\mathcal{H}} - B) \\ &\qquad \times \exp\Bigg(\mathrm{tr}_{\mathcal{H}}\bigg(\sum_{j=1}^{k-1} j^{-1}\big[(A + B - AB)^j - A^j - B^j\big]\bigg)\Bigg). \end{aligned}$$

By (D.4) one concludes that

$$\begin{aligned} &\mathrm{tr}_{\mathcal{H}}\bigg(\sum_{j=1}^{k-1} j^{-1}\big[(A + B - AB)^j - A^j - B^j\big]\bigg) \\ &\qquad\qquad = \mathrm{tr}_{\mathcal{H}}(x_k) + \mathrm{tr}_{\mathcal{H}}\bigg(y_k - \sum_{j=1}^{k-1} j^{-1}(A^j + B^j)\bigg). \end{aligned}$$

By Lemma D.5,

$$y_k - \sum_{j=1}^{k-1} j^{-1}\big(A^j + B^j\big)$$

is a sum of commutators of polynomial expressions in A and B. Hence,

$$\left(y_k - \sum_{j=1}^{k-1} j^{-1}(A^j + B^j)\right) \subset \big[\mathcal{B}_1(\mathcal{H}), \mathcal{B}_1(\mathcal{H})\big],$$

and thus,

$$\operatorname{tr}_{\mathcal{H}}\left(y_k - \sum_{j=1}^{k-1} j^{-1}(A^j + B^j)\right) = 0,$$

proving assertion (D.9) for $A, B \in \mathcal{B}_1(\mathcal{H})$.

Since both, the right and left-hand sides in (D.9) are continuous with respect to the norm in $\mathcal{B}_k(\mathcal{H})$, and $\mathcal{B}_1(\mathcal{H})$ is dense in $\mathcal{B}_k(\mathcal{H})$, (D.9) holds for arbitrary $A, B \in \mathcal{B}_k(\mathcal{H})$. ∎