Appendix E

Notational conventions

For convenience of the reader we now summarize most of our notational conventions used throughout this manuscript.

Basic abbreviations

We employ the shortcut $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\lfloor \cdot \rfloor$ denotes the floor function on $\mathbb{R}$, that is, $\lfloor x \rfloor$ characterizes the largest integer less than or equal to $x \in \mathbb{R}$. Similarly, $\lceil \cdot \rceil$ denotes ceiling function, that is, $\lceil x \rceil$ characterizes the smallest integer larger than or equal to $x \in \mathbb{R}$.

We abbreviate $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im}(z) > 0\}$.

Vectors and matrices

Vectors in $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ (with $\top$ abbreviating the transpose operation) or $p = (p_1, \dots, p_n)^\top \in \mathbb{R}^n$, $n \in \mathbb{N}$. For $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ we abbreviate

$$\langle x \rangle = \big(1 + |x|^2\big)^{1/2},$$

where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denotes the standard Euclidean norm of $x \in \mathbb{R}^n$, $n \in \mathbb{N}$.

The dot symbol, "$\cdot$", is used in three different ways: First, it denotes the standard scalar product in $\mathbb{R}^n$,

$$x \cdot y = \sum_{j=1}^{n} x_j y_j, \quad x = (x_1, \dots, x_n)^\top,\ y = (y_1, \dots, y_n)^\top \in \mathbb{R}^n.$$

Second, we will also use it for n-vectors of operators, $\underline{A} = (A_1, \dots, A_n)^\top$ and $\underline{B} = (B_1, \dots, B_n)^\top$ acting in the same Hilbert space in the form

$$\underline{A} \cdot \underline{B} = \sum_{j=1}^{n} A_j B_j,$$

whenever it is obvious how to resolve the domain issues of the possibly unbounded operators involved.

For X a given space, T a linear operator in X, and $A = (a_{j,k})_{1 \le j,k \le N} \in \mathbb{C}^{N \times N}$ an $N \times N$ matrix with constant complex-valued entries acting in $\mathbb{C}^N$, $N \in \mathbb{N}$, we will avoid tensor product notation as in

$$T \otimes A \quad \text{in } X \otimes \mathbb{C}^N,$$

such that

$$X \otimes \mathbb{C}^N \text{ is identified with the symbol } X^N = (X, \dots, X)^\top,$$

and

$$T \otimes A \text{ is identified with } TA = (Ta_{j,k})_{1 \le j,k \le N} = (a_{j,k}T)_{1 \le j,k \le N} = AT. \quad \text{(E.1)}$$

That is, we interpret $T \otimes A$ as entrywise multiplication, resulting in an $N \times N$ block operator matrix $TA = AT$. Thus, if $T = (T_1, \dots, T_n)$, with T_j, $1 \le j \le n$, operators in $\mathcal{H}$, and $A = (A_1, \dots, A_n)$, with $A_j \in \mathbb{C}^{N \times N}$, $1 \le j \le n$, $N \times N$ matrices with constant, complex-valued entries acting in $\mathbb{C}^N$, we will employ the dot symbol also in the form

$$T \cdot A = \sum_{j=1}^{n} T_j A_j = \sum_{j=1}^{n} A_j T_j = A \cdot T,$$

where $T_j A_j = A_j T_j$, $1 \le j \le n$, are defined as in (E.1).

$A \in X^{m \times n}$, $m, n \in \mathbb{N}$, represents an $m \times n$ matrix $A = (A_{j,k})_{1 \le j \le m,\, 1 \le k \le n}$, with entries $A_{j,k}$ in X, $1 \le j \le m$, $1 \le k \le n$. In particular, $F = (F_1, \dots, F_n)^\top \in X^n$ is a vector with n components and $F_j \in X$ denotes its j-th component, $1 \le j \le n$.

The identity operator in $\mathbb{C}^n$ is represented by I_n, $n \in \mathbb{N}$.

Special functions and function spaces

For special functions such as the Gamma function $\Gamma(\cdot)$, the digamma function $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$, Bessel functions of order ν, $J_\nu(\cdot)$, $Y_\nu(\cdot)$, Hankel functions of the first kind and order ν, $H_\nu^{(1)}(\cdot)$, the Euler–Macheroni constant γ_{E-M}, etc., we refer to [1].

To simplify notation, we frequently omit Lebesgue measure whenever possible and simply use $L^p(\mathbb{R}^n)$ instead of $L^p(\mathbb{R}^n; d^n x)$, $p \in (0, \infty) \cup \{\infty\}$.

Weak L^p-spaces (i.e., Lorentz spaces $L^{p,q}(\mathbb{R}^n; d\rho)$ with $q = \infty$), are denoted by $L^p_{\text{weak}}(\mathbb{R}^n; d\rho)$, $p \in (0, \infty)$. Here $(\mathbb{R}^n, d\rho)$ represents a separable measure space and the measure ρ is assumed to be σ-finite. The seminorm on $L^p_{\text{weak}}(\mathbb{R}^n; d\rho)$ is abbreviated by

$$\|f\|_{L^p_{\text{weak}}(\mathbb{R}^n; d\rho)} := \sup_{t>0} \big(t\big[\mu_f(t)\big]^{1/p}\big), \quad p \in (0, \infty),$$

where

$$\mu_f(t) = \rho\big(\big\{x \in \mathbb{R}^n \,\big|\, |f(x)| > t\big\}\big).$$

In particular,

$$\|f + g\|_{L^p_{\text{weak}}(\mathbb{R}^n; d\rho)} \le c_p\big[\|f\|_{L^p_{\text{weak}}(\mathbb{R}^n; d\rho)} + \|g\|_{L^p_{\text{weak}}(\mathbb{R}^n; d\rho)}\big],$$
$$c_p = \max(2, 2^{1/p}), \ p \in (0, \infty).$$

Again, we omit the measure ρ and just employ the notation $L^p_{\text{weak}}(\mathbb{R}^n)$ in case ρ equals Lebesgue measure on $\mathbb{R}^n$.

If $n \in \mathbb{N}$ and $N \in \mathbb{N}\backslash\{1\}$, we set

$$[L^2(\mathbb{R}^n)]^N := L^2(\mathbb{R}^n;\mathbb{C}^N), \quad [W^{1,2}(\mathbb{R}^n)]^N := W^{1,2}(\mathbb{R}^n;\mathbb{C}^N), \quad \text{etc.} \tag{E.2}$$

The symbol $\mathcal{F}$ is used to denote the Fourier transform, $f^\wedge := \mathcal{F} f$, similarly $f^\vee := \mathcal{F}^{-1} f$, $f \in \mathcal{S}'(\mathbb{R}^n)$, with $\mathcal{S}(\mathbb{R}^n)$ the Schwartz test function space, and $\mathcal{S}'(\mathbb{R}^n)$ its dual with elements the tempered distributions. In particular,

$$\begin{aligned} f^\wedge(p) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n x\, e^{-ip\cdot x} f(x), \\ f^\vee(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n p\, e^{ip\cdot x} f(p), \end{aligned} \qquad p, x \in \mathbb{R}^n,\ f \in \mathcal{S}(\mathbb{R}^n).$$

If $-\infty \le a < b \le \infty$ and F maps $\mathbb{C}\backslash(a,b)$ to a normed linear space, then the normal boundary values of F at $\lambda \in (a,b)$ (when these values exist) are denoted by $F(\lambda \pm i0) := \lim_{\varepsilon\downarrow 0} F(\lambda \pm i\varepsilon)$.

For $n, k \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}^n$, $C^k(\Omega)$ denotes the set of all $f : \Omega \to \mathbb{C}$ that are k times continuously differentiable.

Linear operators in Hilbert spaces

Let $\mathcal{H}$, $\mathcal{K}$ be separable, complex Hilbert spaces, $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), $\|\cdot\|_{\mathcal{H}}$ the norm on $\mathcal{H}$, and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator A is denoted by $\overline{A}$. The set of closed linear operators with domain contained in $\mathcal{H}$ and range contained in $\mathcal{K}$ is denoted by $\mathcal{C}(\mathcal{H}, \mathcal{K})$ (or simply by $\mathcal{C}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$).

The resolvent set, spectrum, and point spectrum (i.e., the set of eigenvalues) of a closed operator T are denoted by $\rho(T)$, $\sigma(T)$, and $\sigma_{\mathrm{p}}(T)$, respectively.

If S is self-adjoint in $\mathcal{H}$, the family of strongly right-continuous spectral projections associated with S is denoted by $E_S(\lambda) = E_S((-\infty, \lambda])$, $\lambda \in \mathbb{R}$, moreover, the singular, discrete, essential, absolutely continuous, and singularly continuous spectrum of S are denoted by $\sigma_{\mathrm{s}}(S)$, $\sigma_{\mathrm{d}}(S)$, $\sigma_{\mathrm{ess}}(S)$, $\sigma_{\mathrm{ac}}(S)$, and $\sigma_{\mathrm{sc}}(S)$, respectively.

For a densely defined closed operator S in $\mathcal{H}$ we employ the abbreviation $\langle S\rangle := (I_{\mathcal{H}} + |S|^2)^{1/2}$, and similarly, if $T = (T_1, \dots, T_n)^\top$, with T_j densely defined and closed in $\mathcal{H}$, $1 \le j \le n$,

$$\langle T\rangle = \big(I_{\mathcal{H}} + |T|^2\big)^{1/2}, \quad |T| = \big(|T_1|^2 + \cdots + |T_n|^2\big)^{1/2},$$

whenever it is obvious how to define $|T_1|^2 + \cdots + |T_n|^2$ as a self-adjoint operator.

The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based Schatten–von Neumann ideals are denoted by $\mathcal{B}_p(\mathcal{H})$, with associated norm abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p \geq 1$.

Following a standard practice in mathematical physics, we simplify the notation of operators of multiplication by a scalar or matrix-valued function V and hence use V rather than the more elaborate symbol M_V throughout this manuscript.