

Contents

Preface	vii
1 Non-linear statistical inverse problems	1
1.1 Model examples	2
1.1.1 Boundary measurements in tomography	2
1.1.2 Parameter identification for elliptic PDEs	3
1.1.3 Data assimilation	6
1.2 Bayesian regression	7
1.2.1 The forward map \mathcal{G}	7
1.2.2 A random design regression model with normal errors	7
1.2.3 The Bayesian prior and posterior distribution	9
1.2.4 Posterior computation by MCMC	10
1.3 The frequentist perspective	14
1.3.1 Information distances for random design regression	14
1.3.2 A first posterior contraction theorem	17
1.4 Notes	20
1.4.1 Exercises	20
1.4.2 Remarks and comments	20
2 Global stability and posterior consistency	23
2.1 Analytical hypotheses and PDEs	24
2.1.1 Forward regularity conditions for \mathcal{G}	24
2.1.2 Injectivity and stability estimates	27
2.2 Regularisation with Gaussian process priors	31
2.3 Convergence of posterior measure and mean	35
2.4 Notes	38
2.4.1 Exercises	38
2.4.2 Remarks and comments	39
3 Information operators and curvature	41
3.1 Information geometry	41
3.1.1 The LAN expansion	42
3.1.2 Cramer–Rao bounds and inverse information	45
3.2 Gradient stability and concentration	49
3.2.1 Convexity of $-\ell_N$ and the gradient of \mathcal{G}	49
3.2.2 A concentration result for the empirical Hessian	51

3.3	Information operators for elliptic PDEs	58
3.3.1	Schrödinger equation	58
3.3.2	Diffusion equation	59
3.3.3	Injectivity and local identifiability	61
3.4	Notes	63
3.4.1	Exercises	63
3.4.2	Remarks and comments	64
4	Bernstein–von Mises theorems	67
4.1	Gaussian asymptotics for cylindrical laws	68
4.1.1	Asymptotic normality of linear functionals of the posterior	68
4.1.2	Asymptotic distribution of the posterior mean	77
4.1.3	Applications to uncertainty quantification	80
4.2	Solving information equations in PDE models	81
4.2.1	A Bernstein–von Mises theorem for the Schrödinger equation	82
4.2.2	Impossibility of the BvM-phenomenon for Darcy’s problem	84
4.3	Notes	88
4.3.1	Exercises	88
4.3.2	Remarks and comments	89
5	Posteriors are probably log-concave	91
5.1	Wasserstein approximation of the posterior	92
5.1.1	Construction of a log-concave surrogate posterior	94
5.1.2	The log-concave approximation theorem	95
5.2	Computational complexity of MCMC in high dimensions	101
5.2.1	Gradient methods for approximately log-concave posteriors	101
5.2.2	On failure of ‘cold start’ MCMC in high dimensions	106
5.3	Application to PDE models	113
5.3.1	Darcy’s problem on the eigen-spaces of the Laplacian	114
5.3.2	Polynomial time computation of the posterior mean	119
5.4	Notes	121
5.4.1	Exercises	121
5.4.2	Remarks and comments	121
A	Analytical background	125
A.1	Sobolev and related spaces	125
A.2	Elliptic second-order differential operators	126
A.3	Orthonormal discretisation of L^2 and metric entropy	128
A.4	Feynman–Kac formulæ	132
A.5	Elliptic regularity estimates	133

B Further auxiliary results	139
B.1 Results from Gaussian process theory	139
B.2 A concentration inequality for empirical processes	143
B.3 Mixing time bounds for Langevin diffusions	145
B.4 A characterisation of vanishing efficient information	147
References	151
Index	159

