# Chapter 1

# Introduction

The main objective of this paper is to generalize the following elementary fact.

Let $I = [0, 1]$. Define

$$\mathcal{E}_p^n(f) = \sum_{i=1}^{2^n} \left| f\left(\frac{i-1}{2^n}\right) - f\left(\frac{i}{2^n}\right)\right|^p$$

for $n \geq 1$ and $f: I \to \mathbb{R}$. If $f$ is smooth or more generally $f \in W^{1,p}(I)$, which is the $(1, p)$-Sobolev space, then

$$(2^{p-1})^n \mathcal{E}_p^n(f) \to \int_0^1 |\nabla f|^p dx$$

as $n \to \infty$, where $\nabla f$ is the derivative of $f$.

Our naive question is what is a counterpart of this in the case of metric spaces. More precisely, our general strategy of the study is:

(1) To fix an adequate sequence of discrete graphs $\{(T_n, E_n^*)\}_{n\geq 1}$, where $T_n$ is a discrete approximation of the original metric space $(X, d)$ and $E_n^*$ is the collection of edges, i.e., pairs of points in $T_n$. For a function $f: T_n \to \mathbb{R}$, define

$$\mathcal{E}_p^n(f) = \frac{1}{2} \sum_{(x,y)\in E_n^*} |f(x) - f(y)|^p,$$

which is called the $p$-energy of the function $f$.

(2) To find a proper scaling constant $\sigma$ such that the space of functions

$$\{f: X \to \mathbb{R} \mid \sigma^n \mathcal{E}_p^n(P_n f) \text{ is “convergent” as } n \to \infty\},$$

where $P_n f$ is a suitable discrete approximation of $f$, is rich enough to be a “Sobolev” space in some sense. From our perspective, we do not care about the existence of a derivative $\nabla f$ but pursue the convergence of $\sigma^n \mathcal{E}_p^n(P_n f)$.

Actually, in the case $p = 2$, this strategy was employed to construct Dirichlet forms inducing diffusion processes on self-similar sets like the Sierpiński gasket[1] and the Sierpiński carpet. (See Figure 1.4.) For the sake of simplicity, we confine ourselves to non-finitely ramified self-similar sets. (This excludes post critically finite

[1] In many papers, people use “Sierpinski” in place of “Sierpiński”. Of course, originally “Sierpiński” is the correct one as a Polish family name but such a simplification often occurs when the subject becomes popular and a part of classics.

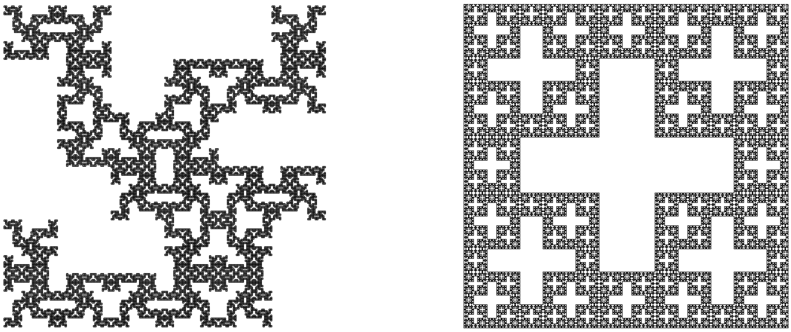


**Figure 1.1.** Square-based self-similar sets.

self-similar sets represented by the Sierpiński gasket.) Barlow and Bass constructed the Brownian motions on (generalized) Sierpiński carpets in [1–6] as scaling limits of the Brownian motions on regions approximating Sierpiński carpets. Later in [36], Kusuoka and Zhou employed the above strategy for $p = 2$ and directly constructed the Dirichlet form inducing the Brownian motion on the planar Sierpiński carpet. Note that all these works were done in the last century. Although more than 20 years have passed, no essential progress has been made on the construction of diffusion processes/Dirichlet forms on non-finitely ramified self-similar sets. In particular, no diffusion was constructed on square-based non-finitely ramified self-similar sets like those in Figure 1.1. The right-hand one is an example of rationally ramified Sierpiński crosses treated in Section 4.5. It has two different contraction ratios. The left-hand one is an example having no symmetry of the square. As a by-product of our results in this paper, we will construct non-trivial self-similar local regular Dirichlet forms on classes of square-based self-similar sets including those in Figure 1.1. See Sections 4.3, 4.4, and 4.5 for details.

From the viewpoint of construction of Sobolev spaces on metric spaces, there have already been established theories based on upper gradients, which correspond to local Lipschitz constants of Lipschitz functions. Compared with our strategy above, this direction is to seek a counterpart of $\nabla f$ instead of the convergence of $\sigma^n \mathcal{E}_p^n(P_n f)$ like us. The pioneering works of this theory are Hajłasz [22], Cheeger [15] and Shanmugalingam [40]. One can find a panoramic view of this theory in [23]. Recent studies by Kajino and Murugan in [26, 27], however, have suggested that they may not cover all the interesting cases. So far examples in question are higher-dimensional Sierpiński gaskets, the Vicsek set, and the planar Sierpiński carpet. What they have shown in [26, 27] is that the Brownian motions on those examples will not have the Gaussian heat kernel estimate under any time change by a pair $(d, \mu)$, where $d$ is quasisymmetric to the Euclidean metric $d_E$ and $\mu$ has the volume doubling property with respect to $d_E$. On the other hand, under the established theory, the heat kernel associated with a $(1, 2)$-Sobolev space satisfying a $(2, 2)$-Poincaré inequality should satisfy the Gaussian estimate due to the results in [21, 39, 42]. Thus, the Dirichlet forms associated with the Brownian motions on the above-mentioned self-similar sets can hardly be one of $(1, 2)$-Sobolev spaces based on upper gradients. Note that, in these cases, there exist plenty of rectifiable curves with respect to (the restriction of) the Euclidean metrics, which are even quasiconvex. Partly motivated by such a situation, we will try to provide an alternative theory of function spaces, which may be called Sobolev spaces or else, on metric spaces, and to construct natural diffusion processes at the same time.

Getting straight to the conclusion, we are going to propose a condition called $p$-conductive homogeneity and show that under this condition, the strategy consisting of (1) and (2) succeeds for $p > \dim_{AR}(K, d)$, where $\dim_{AR}(K, d)$ is the *Ahlfors regular conformal dimension* of a compact metric space $(K, d)$. One can see a more precise and detailed exposition in what follows. The definition of the Ahlfors regular conformal dimension of $(K, d)$ is

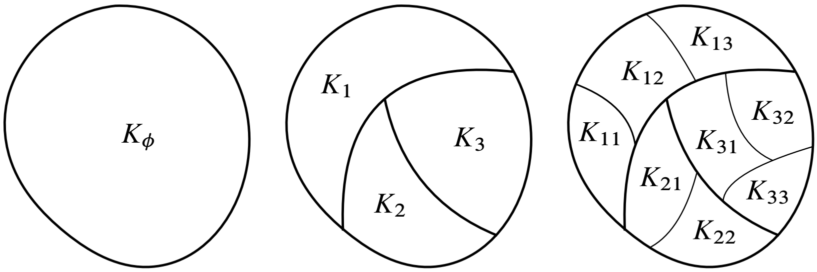
$$\begin{aligned}\dim_{AR}(K, d) = \inf\{\alpha \mid\ & \text{there exist a metric } \rho \text{ on } K \text{ which is} \\ & \text{quasisymmetric to } d \text{ and a Borel regular measure } \mu \\ & \text{which is } \alpha\text{-Ahlfors regular with respect to } \rho\},\end{aligned} \tag{1.1}$$

where the definition of Ahlfors regularity of a measure is given in (2.9). The notion of quasisymmetry was introduced in [43] as a certain generalization of quasiconformal maps of the complex plane. It is defined in the following way:

**Definition 1.1.** Let $(X, d)$ be a metric space. A metric $\rho$ on $X$ is said to be *quasisymmetirc* to $d$ if $(X, \rho)$ gives the same topology as $d$ and there exists a homeomorphism $h$ from $[0, \infty)$ to itself satisfying $h(0) = 0$ and for any $t > 0$, $\rho(x, z) \leq h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$.

In the direction of our study, Shimizu has done pioneering work for the case of the planar Sierpiński carpet, PSC for short, in the very recent paper [41]. Extending Kusuoka–Zhou's method, he has constructed a $p$-energy and the corresponding $p$-energy measure for $p > \dim_{AR}(\mathrm{PSC}, d_E)$, and done detailed analysis of those objects. In particular, he has shown that the collection of functions with finite $p$-energies is a Banach space that is reflexive and separable. His proof of reflexivity and separability can be easily extended to our general case as well. See Theorem 3.22 for details.

Our framework on metric spaces is the theory of partitions introduced in [34]. Let $(K, d)$ be a compact metric space. We always suppose that $(K, d)$ is connected in this paper. Roughly speaking, a partition of $K$ is a sequence of successive divisions of $K$ by some of its compact subsets. The idea is illustrated in Figure 1.2. Let $T_0 = \{\phi\}$




$T_0 = \{\phi\}$ $\quad$ $T_1 = \{1, 2, 3\}$ $\quad$ $S(1) = \{11, 12, 13\}$
$S(2) = \{21, 22\}$
$S(2) = \{31, 32, 33\}$

**Figure 1.2.** Partition.

and set $K_\phi = K$. Starting from $K$, we first divide $K$ into a finite number of children $K_w$ for $w \in T_1$, i.e.,

$$K = \bigcup_{w \in T_1} K_w.$$

The set $T_1$ is thought of as the collection of children of $T_0$ and denoted by $S(\phi)$. Then we repeat this process of division, i.e., each $w \in T_1$ has a collection of children, $S(w)$, such that

$$K_w = \bigcup_{v \in S(w)} K_v.$$

Define $T_2$ as the disjoint union of the $S(w)$'s for $w \in T_1$. So repeating this process inductively, we have $\{T_n\}_{n \geq 0}$ where each $w \in T_n$ has the collection of children $S(w) \subseteq T_{n+1}$. Set

$$T = \bigcup_{n \geq 0} T_n.$$

With several requirements described in Section 2.1, the family $\{K_w\}_{w \in T}$ is called a partition of $K$.

For each $n \geq 1$, $T_n$ has a natural graph structure associated with a given partition $\{K_w\}_{w \in T}$. Namely, if

$$E_n^* = \{(u, v) \mid u, v \in T_n, K_u \cap K_v \neq \emptyset\},$$

then $(T_n, E_n^*)$ is a connected graph, which is illustrated in Figure 1.3. To avoid technical complexity, we are going to explain our results under Assumption 2.15 hereafter in the introduction. In fact, if $(K, d)$ is $\alpha$-Ahlfors regular for some $\alpha$ and the metric $d$ is 1-adapted in the sense of [34], then Assumption 2.15 holds. So our setting should be broad enough.

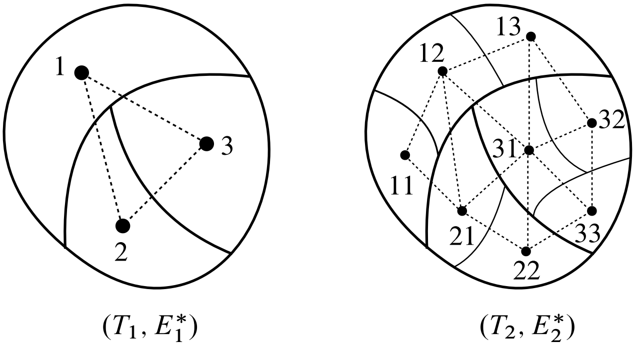



**Figure 1.3.** Graphs associated with a partition (dotted lines are vertices).

For $A \subseteq T_n$, we define the $p$-energy of a function on $A$ by

$$\mathcal{E}_{p,A}^n(f) = \frac{1}{2} \sum_{\substack{u,v \in A \\ (u,v) \in E_n^*}} |f(u) - f(v)|^p.$$

To carry out our strategy, we introduce two key characteristic quantities: conductance and neighbor disparity constants. For $m \geq 0$, $A_1, A_2, A \subseteq T_n$ with $A_1, A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$, define

$$\mathcal{E}_{p,m}(A_1, A_2, A) = \inf \big\{ \mathcal{E}_{p,A}^{n+m}(f) \mid f \colon S^m(A) \to \mathbb{R}, f|_{S^m(A_1)} \equiv 1, f|_{S^m(A_2)} \equiv 0 \big\},$$

where $S^m(A) \subseteq T_{n+m}$ is the collection of the descendants in the $m$-th generation from $A$. The quantity $\mathcal{E}_{p,m}(A_1, A_2, A)$ is called the *p-conductance* between $A_1$ and $A_2$ within $A$ at the level $m$.

**Remark.** Attaching a resistor of resistance 1 to each edge $(u, v) \in E_{n+m}^*$, we may consider the graph $(T_{n+m}, E_{n+m}^*)$ as an electric network. In this respect, the reciprocal of $\mathcal{E}_{2,m}(A_1, A_2, A)$ is the effective resistance between $A_1$ and $A_2$ within $A$ and hence $\mathcal{E}_{2,m}(A_1, A_2, A)$ corresponds to the effective conductance. Such an analogy has been often used in the study of random walks. See [18] for a classical reference. In potential theory, the quantity $\mathcal{E}_{2,m}(A_1, A_2, A)$ is called "capacity" as well.

In particular, for $w \in T_n$, define

$$\mathcal{E}_{p,m}(w) = \mathcal{E}_{p,m}(\{w\}, \Gamma_1(w)^c, T_n),$$

where $\Gamma_1(w)$ is the collection of neighbors of $w$ in $T_n$ given by

$$\Gamma_1(w) = \{v \mid v \in T_n, (w, v) \in E_n^*\}.$$

The value $\mathcal{E}_{p,m}(w)$ represents the $p$-conductance between $w$ and the complement of its neighborhood $\Gamma_1(w)$ in the $m$-th generation from $w$. In [34], it was shown that

$$\overline{\lim_{m\to\infty}} \Big( \sup_{w\in T} \mathcal{E}_{p,m}(w)^{\frac{1}{m}} \Big) < 1 \quad \text{if and only if} \quad p > \dim_{AR}(K,d). \tag{1.2}$$

The other one, the neighbor disparity constant, is defined as

$$\sigma_{p,m,n} = \sup_{(w,v)\in E_n^*} \bigg( \sup_{f:S^m(w,v)\to\mathbb{R}} \frac{|(f)_{S^m(w)} - (f)_{S^m(v)}|^p}{\mathcal{E}^{n+m}_{p,S^m(w,v)}(f)} \bigg),$$

where $S^m(w,v) = S^m(w) \cup S^m(v)$ and $(f)_{S^m(w)}$ is the average of $f$ on $S^m(w)$ under a suitable measure $\mu$. (This definition of the neighbor disparity constant is a simplified version for introductory purposes. The full version will be presented in Section 2.4.) For the case $p = 2$, this constant was introduced in [36]. The neighbor disparity constant controls the difference of means of a function on neighboring cells via the $p$-energy.

And now, $p$-conductive homogeneity, which is the principal notion of this paper, is defined as follows.

**Definition 1.2.** A metric space $(K,d)$ is said to be *$p$-conductively homogeneous* if and only if there exists $c > 0$ such that

$$\sup_{w\in T} \mathcal{E}_{p,m}(w) \sup_{n\geq 1} \sigma_{p,m,n} \leq c$$

for any $m \geq 1$.

The above condition is essentially due to Kusuoka–Zhou [36] when $p = 2$. Cao and Qiu named this condition as condition (B) in [13], where they have constructed a diffusion process on so called unconstrained Sierpiński carpets by following the Kusuoka–Zhou's method.

At a glance, it does not quite look like "homogeneity". The following theorem, however, gives the legitimacy of the name.

**Theorem 1.3** (Theorem 3.30). *A metric space $(K,d)$ is $p$-conductively homogeneous if and only if there exist $\sigma > 0$ and $c_1, c_2 > 0$ such that*

$$c_1\sigma^{-m} \leq \mathcal{E}_{p,m}(w) \leq c_2\sigma^{-m}$$

*for any $w \in T\backslash\{\phi\}$ and $m \geq 1$ and*

$$c_1\sigma^{m} \leq \sigma_{p,m,n} \leq c_2\sigma^{m}$$

*for any $m, n \geq 1$.*

The next natural question is how the conductive homogeneity is related to the construction of a $p$-energy. The answer is the next theorem which follows by combining Theorems 3.5, 3.21, 3.23 and Lemma 3.34.

**Theorem 1.4.** *Suppose $p > \dim_{AR}(K,d)$ and $(K,d)$ is $p$-conductively homogeneous. Let $C(K)$ be the collection of continuous functions on $K$. Define*

$$\mathcal{N}_p(f) = \Big(\sup_{m\geq 0} \sigma^m \mathcal{E}_p^m(P_m f)\Big)^{\frac{1}{p}}$$

*for $f \in L^p(K,\mu)$, where*

$$(P_m f)(w) = \frac{1}{\mu(K_w)} \int_{K_w} f(x)\mu(dx),$$

*and*

$$\mathcal{W}^p = \{f \mid f \in L^p(K,\mu), \mathcal{N}_p(f) < \infty\}.$$

*Then*

(1) *$\mathcal{N}_p(f) = 0$ if and only if $f$ is constant on $K$.*

(2) *$\mathcal{N}_p$ is a semi-norm of $\mathcal{W}^p$.*

(3) *$(\mathcal{W}^p, \|\cdot\|_{p,\mu} + \mathcal{N}_p(\cdot))$ is a Banach space.*

(4) *$\mathcal{W}^p$ is a dense subset of $(C(K), \|\cdot\|_\infty)$.*

*Moreover, there exists $\widehat{\mathcal{E}}_p\colon \mathcal{W}^p \to [0,\infty)$ such that $\widehat{\mathcal{E}}_p^{\frac{1}{p}}$ is a semi-norm of $\mathcal{W}^p$ which is equivalent to $\mathcal{N}_p(\cdot)$, $\widehat{\mathcal{E}}_p$ satisfies the Markov property and there exist $\tau > 0$ and $c_1, c_2 > 0$ such that*

$$c_1 d(x,y)^\tau \leq \sup_{\substack{f\in\mathcal{W}^p\\ \widehat{\mathcal{E}}_p(f)\neq 0}} \frac{|f(x)-f(y)|^p}{\widehat{\mathcal{E}}_p(f)} \leq c_2 d(x,y)^\tau$$

*for any $x, y \in K$. In particular, for $p = 2$, one can choose $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ as a local regular Dirichlet form on $L^2(K,\mu)$.*

Note that by (1.2), the condition $p > \dim_{AR}(K,d)$ implies $\sigma > 1$. An explicit description of the constant $\tau$ is given in Lemma 3.34. In addition, we show a sub-Gaussian type heat kernel estimate for the diffusion process induced by the Dirichlet form $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ in Theorem 3.35. Moreover, if $(K,d)$ is a self-similar set with rationally related contraction ratios, then a self-similar $p$-energy which is equivalent to $\mathcal{N}_p$ will be constructed in Section 4.1.

Another important question is how to show conductive homogeneity. The following theorem provides an equivalent and useful condition for this purpose.

**Theorem 1.5** (Theorem 3.33). *Suppose that $p > \dim_{AR}(K, d)$. $(K, d)$ is $p$-conductively homogeneous if and only if, for any $k \geq 1$, there exists $c(k) > 0$ such that*

$$\sup_{z \in T} \mathcal{E}_{p,m}(z) \leq c(k)\mathcal{E}_{p,m}(u, v, S^k(w)) \tag{1.3}$$

*for any $m \geq 1$, $w \in T$ and $u, v \in S^k(w)$ with $u \neq v$.*

The condition in the above theorem, (1.3), which is the same as (3.20) in Theorem 3.33, is a relative of the "knight move" condition in [36] described in the terminology of random walks, although the word "knight move" does not make sense from the appearance of (1.3) any longer. The original "knight move" in [1] was the name of an argument based on the symmetry of the Sierpiński carpet to show a probabilistic counterpart of (1.3). With certain symmetries of the space, it is possible to show (1.3) by the method of combinatorial modulus in [11]. Applying Theorem 1.5, we are going to show the conductive homogeneity for examples like those in Figure 1.1 in Sections 4.4 and 4.5.

Besides applications, Theorem 1.5 has a remarkable theoretical consequence; conductive homogeneity is determined only by conductance constants and is independent of the neighbor disparity constants if $p > \dim_{AR}(K, d)$. This is the reason conductive homogeneity is called "*conductive*".

The major methodological backgrounds of this paper are Kusuoka–Zhou's arguments in [36] and combinatorial moduli of path families on graphs introduced in [11]. On many occasions, we will extend Kusuoka–Zhou's results to compact metric spaces and to general values of $p$. On such occasions, we will put a reference to the original result by Kusuoka and Zhou right behind the number of a proposition or a lemma like Lemma 2.27 [36, Lemma 2.12]. Beyond Kusuoka–Zhou's arguments, the notion of combinatorial modulus will play a crucial role on several occasions. The most important one is in the proof of a sub-multiplicative inequality of conductance constants, Corollary 2.24. Moreover, by Lemma C.4, one can compare moduli of different graphs and this lemma is indispensable for showing (1.3) in Sections 4.3 and 4.5.

Regrettably, we do not have much for the case $p \leq \dim_{AR}(K, d)$. In Section 3.2, we will construct a function space $\mathcal{W}^p$ and a semi-norm $\widehat{\mathcal{E}}_p$ on $\mathcal{W}^p$ under $p$-conductive homogeneity for $p \in [1, \dim_{AR}(K, d)]$. In this case, however, $\mathcal{W}^p$ is given as a subspace of $L^p(K, \mu)$ and we do not know whether $\mathcal{W}^p \cap C(K)$ is dense in $(C(K), \|\cdot\|_\infty)$ or not. This is due to the lack of an elliptic Harnack principle of $p$-harmonic functions on the corresponding graphs. In the case $p = 2$, using the coupling method, Barlow and Bass conquered this difficulty for higher-dimensional Sierpiński carpets in [5, 6]. We have little idea what is an analytic counterpart of the coupling method at this moment. It is a big open problem for future work. In particular, it is interesting to know whether the following naive conjecture is true or not.

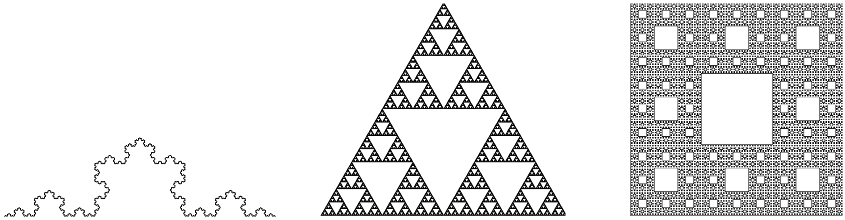


**Figure 1.4.** von Koch curve, Sierpiński gasket and Sierpiński carpet.

**Conjecture.** $\mathcal{W}^p \subseteq C(K)$ *if and only if* $p > \dim_{AR}(K, d)$.

Now we briefly explain what happens in the cases of familiar examples.

1. *Unit (hyper)cube* $[-1, 1]^n$: In this case, for any $p > n$,

$$\mathcal{W}^p = W^{1,p}([-1, 1]^n)$$

and there exists $c > 0$ such that

$$c\widehat{\mathcal{E}}_p(f) \leq \int_{[-1,1]^n} |\nabla f|^p dx \leq c^{-1}\widehat{\mathcal{E}}_p(f)$$

for any $f \in W^{1,p}([-1, 1]^n)$. See Example 4.31 for details. Even if $p \in [1, n]$, the above results should be true but we do not have any proof for now.

2. *von Koch curve (Figure 1.4)*: The von Koch curve does not contain any rectifiable curve, so that the approaches using upper gradients do not work from the beginning. However, our theory does not distinguish metric spaces which are snowflake equivalent, i.e., two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are snowflake equivalent if there exist a homeomorphism $\varphi: X \to Y$, $c_1, c_2 > 0$ and $\alpha > 0$ such that

$$c_1 d_X(x_1, x_2)^\alpha \leq d_Y(\varphi(x_1), \varphi(x_2)) \leq c_2 d_X(x_1, x_2)^\alpha$$

for any $x_1, x_2 \in X$. Since the von Koch curve is snowflake equivalent to the unit interval $[0, 1]$, we see that $\mathcal{W}^p$ for the von Koch curve equals $W^{1,p}([0, 1])$ for any $p > 1$.

3. *Planar Sierpiński carpet (Figure 1.4)*: As is mentioned above, this is one of the original motivations of this paper and it is expected that our space $\mathcal{W}^p$ is quite different from what one may get from the upper gradient approaches. By Theorem 4.13, the planar Sierpiński carpet $K$ is shown to be $p$-conductive homogeneous for any $p > \dim_{AR}(K, d_*)$, where $d_*$ is the restriction of the Euclidean metric. Moreover, let

$$\alpha_H = \frac{\log 8}{\log 3} \quad \text{and} \quad \beta_p = \frac{\log 8\sigma}{\log 3},$$

where $\sigma$ is the exponent appearing in Theorem 1.3. Then by [41, Theorem 2.19], we have a fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ as follows:

$$\mathcal{W}^p = \left\{ f \mid f \in L^p(K,\mu), \varlimsup_{r \downarrow 0} \int_K \frac{1}{r^{\alpha_H}} \int_{B_{d_*}(x,r)} \frac{|f(x)-f(y)|^p}{r^{\beta_p}} \mu(dy)\mu(dx) < \infty \right\},$$

where $\mu$ is the normalized $\alpha_H$-dimensional Hausdorff measure. Furthermore, it is shown in [41] that $\beta_p > p$. This fact implies that $\mathcal{W}^p$ should not coincide with any of the spaces obtained by approaches using upper gradients.

4. *Sierpiński gasket (Figure 1.4)*: Let $K$ be the standard Sierpiński gasket and let $d_*$ be the restriction of the Euclidean metric. Since $K$ is one of nested fractals and

$$\dim_{AR}(K, d_*) = 1,$$

Theorem 4.50 yields that $K$ is $p$-conductively homogeneous for any $p > 1$. Arguments analogous to those in [41, Section 5.3] give the same fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ as the planar Sierpiński carpet. In this case,

$$\alpha_H = \frac{\log 3}{\log 2} \quad \text{and} \quad \beta_p = \frac{\log 3\sigma}{\log 2}.$$

We expect that $\beta_p > p$ for any $p > 1$. In fact, due to [8], we know

$$\beta_2 = \frac{\log 5}{\log 2} > 2.$$

Moreover, $\frac{\beta_p}{p}$ is monotonically decreasing by [34, Lemma 4.7.3]. So at least for $p \in (1, 2]$, $\beta_p > p$ and the space $\mathcal{W}^p$ does not seem to be obtained by the upper gradient approaches. However in this case, if we replace the Euclidean metric with the harmonic geodesic metric and the Hausdorff measure with the Kusuoka measure, then the heat kernel associated with the new pair of the metric and the measure has the Gaussian estimate. See [30] for details. Consequently, the Cheeger theory [15] is now in place for $\mathcal{W}^2$ at least. On the other hand, the replacement of the metric and the measure causes a change of the partition and, consequently, a change of the associated function space $\mathcal{W}^P$. So, we expect that $\mathcal{W}^p$ associated with the new pair may coincide with those obtained from the approaches based on upper gradients but we have no proof so far.

Before the conclusion of the introduction, we mention two related works. The first one is [10], where the authors constructed another type of "Sobolev spaces" $\dot{A}_p(X)$ on a compact metric space $(Z, d)$ from its hyperbolic fillings $X$. The method is to construct a discretization $Pf$ on $X$ of $f \in L^1(Z)$, and to consider the weak $\ell^p$-norm of the gradient of $Pf$. Their space $\dot{A}^p(Z)$ seems closely related to our space $\mathcal{W}^p$ but we merely know that $\mathcal{W}^p \subseteq \dot{A}_p(X)$ under suitable assumptions at this point.

The second one is [24], where the authors constructed a $p$-energy on Sierpiński gasket type self-similar sets by extending the notion of harmonic structures in the case of $p = 2$ for post critically finite self-similar sets. Their $p$-energy should be equivalent to ours, although they did not show the completeness of the domain of their $p$-energy. Despite the fact that their method can work only for finitely ramified self-similar sets even if $p = 2$, their work is the first pioneering study to construct a $p$-energy by renormalizing discrete counterparts.

The organization of this paper is as follows.

In Section 2.1, we review the basics of partitions of compact metric spaces and then give a framework of this paper including standing assumptions, Assumptions 2.6, 2.7, 2.10 and 2.12. In the end, we present Assumption 2.15, which is stronger than the combination of all the assumptions above but more concise.

In Section 2.2, we introduce the notion of conductance constant which is one of two principal quantities of this paper and we show the existence of a partition of unity associated with the conductance constant.

In Section 2.3, we introduce the notion of combinatorial moduli of path families on graphs and show a sub-multiplicative inequality for conductance constants using them.

In Section 2.4, we introduce the other principal quantity, the neighbor disparity constant and show its relation with the conductance constant and a sub-multiplicative inequality of them.

In Section 3.1, we construct our function space $\mathcal{W}^p$ and the $p$-energy $\widehat{\mathcal{E}}_p$ under Assumption 3.2 and show Theorem 1.4. At the same time, we propose a condition called $p$-conducive homogeneity and show that the condition $p > \dim_{AR}(K, d)$ and $p$-conductive homogeneity imply Assumption 3.2 in Section 3.3.

In Section 3.2, we see what we can do for $p \le \dim_{AR}(K, d)$. In Section 3.3, we show Theorem 3.30 (= Theorem 1.3) and Theorem 3.33 (= Theorem 1.5). Moreover, in Theorem 3.35, we give a sub-Gaussian type heat kernel estimate for the diffusion process induced by the Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ given in Section 3.1.

In Section 4.1, we construct a self-similar $p$-energy for self-similar sets with rationally related contraction ratios. In Section 4.2, we give a sufficient condition for the conductive homogeneity for self-similar sets. Section 4.3 is devoted to a class of self-similar sets called subsystems of cubic tiling, for which conductive homogeneity is shown through Theorem 3.33. This class includes the Sierpiński carpets, the Menger curve, and the higher-dimensional hypercubes. In Section 4.4, we present examples of subsystems of cubic tiling having the conductive homogeneity. Also, Section 4.5 is devoted to showing conductive homogeneity of rationally ramified Sierpiński crosses.

In Sections 5.1, 5.2 and 5.3, we give a proof of Theorem 3.33. In Section 6.1, we show that conductance, Poincaré and neighbor disparity constants are uniformly bounded from below and above.

We will briefly discuss the modification of the graph structure in Section 6.2. Finally, in Section 6.3, we gather open problems and future directions of research. Appendices give basic facts used in this paper.