Chapter 2

Basic frameworks and key constants

2.1 Framework

In this section, we are going to make our framework of this paper clear. It is based on the notion of partitions of compact metric spaces parametrized by rooted trees, which was introduced in [34]. Roughly speaking, a partition is successive divisions of a given space like the binary division of the unit interval. See [34] for examples. Since this notion is relatively new and unfamiliar to most readers, we will give a minimal but detailed account of its definition.

To start with, we present the basics of graphs and trees.

Definition 2.1. Let T be a countable set and let $\mathcal{A}: T \times T \to \{0, 1\}$ which satisfies $\mathcal{A}(w, v) = \mathcal{A}(v, w)$ and $\mathcal{A}(w, w) = 0$ for any $w, v \in T$. We call the pair $(T, \mathcal{A})$ a (non-directed) *graph* with the *vertices* T and the *adjacency matrix* $\mathcal{A}$. An element $(u, v) \in T \times T$ is called an *edge* of $(T, \mathcal{A})$ if $\mathcal{A}(u, v) = 1$. We often identify the adjacency matrix $\mathcal{A}$ with the collection of edges $\{(u, v) \mid u, v \in T, \mathcal{A}(u, v) = 1\}$.

(1) A graph $(T, \mathcal{A})$ is called *locally finite* if $\#(\{v \mid \mathcal{A}(w, v) = 1\}) < \infty$ for any $w \in T$, where $\#(A)$ is the number of elements of a set A.

(2) For $w_0, \dots, w_n \in T$, $(w_0, w_1, \dots, w_n)$ is called a *path* between w_0 and w_n if $\mathcal{A}(w_i, w_{i+1}) = 1$ for any $i = 0, 1, \dots, n-1$. A path $(w_0, w_1, \dots, w_n)$ is called *simple* if $w_i \neq w_j$ for any i, j with $0 \leq i < j \leq n$ and $|i - j| < n$.

(3) $(T, \mathcal{A})$ is called a *tree* if there exists a unique simple path between w and v for any $w, v \in T$ with $w \neq v$. For a tree $(T, \mathcal{A})$, the unique simple path between two vertices w and v is called the *geodesic* between w and v and denoted by $\overline{wv}$. We write $u \in \overline{wv}$ if $\overline{wv} = (w_0, w_1, \dots, w_n)$ and $u = w_i$ for some i.

Next, we define fundamental notions on trees.

Definition 2.2. Let $(T, \mathcal{A})$ be a tree and let $\phi \in T$. The triple $(T, \mathcal{A}, \phi)$ is called a *rooted tree* with *root* (or *reference point*, see, e.g., [45]) ϕ.

(1) Define $\pi: T \to T$ by

$$\pi(w) = \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, w_1, \dots, w_{n-1}, w_n), \\ \phi & \text{if } w = \phi \end{cases}$$

and, for $w \in T$, set

$$S(w) = \{v \mid \pi(v) = w\} \backslash \{w\}.$$

An element $v \in S(w)$ is thought of as a *child* of w. Moreover, for any $k \geq 1$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v),$$

which is the collection of descendants in the k-th generation from w.

(2) For $w \in T$ and $m \geq 0$, we define

$$|w| = \min\{n \mid n \geq 0, \pi^n(w) = \phi\} \quad \text{and} \quad T_m = \{w \mid w \in T, |w| = m\}.$$

(3) For any $w \in T$, define

$$T(w) = \{v \mid \text{there exists } n \geq 0 \text{ such that } \pi^n(v) = w\},$$

which is the collection of all the descendants of w.

(4) Define

$$\Sigma = \{(w(i))_{i \geq 0} \mid w(i) \in T_i \text{ and } w(i) = \pi(w(i+1)) \text{ for any } i \geq 0\}.$$

For $\omega = (\omega(i))_{i \geq 0} \in \Sigma$, set $[\omega]_m = \omega(m)$ for $m \geq 0$. An element $(w(i))_{i \geq 0} \in \Sigma$ is called a *geodesic ray* starting from ϕ in [45].

Remark. In [34], we have used $(T)_n$ and T_w in place of T_n and $T(w)$, respectively.

Throughout this paper, T is a countably infinite set and $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{v \mid (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$.

Next, we define partitions.

Definition 2.3 (Partition). Let $(K, \mathcal{O})$ be a compact metrizable topological space having no isolated point, where $\mathcal{O}$ is the totality of open sets.

A collection of non-empty compact subsets $\{K_w\}_{w \in T}$ is called a *partition* of K parametrized by $(T, \mathcal{A}, \phi)$ if it satisfies the following conditions (P1) and (P2):

(P1) $K_\phi = K$ and for any $w \in T$, K_w has no isolated point and

$$K_w = \bigcup_{v \in S(w)} K_v.$$

(P2) For any geodesic ray $\omega \in \Sigma$, $\bigcap_{m \geq 0} K_{[\omega]_m}$ is a single point.

Originally in [34], we did not assume that K is connected to include spaces like the Cantor set. In this paper, however, we will only deal with connected spaces. In such cases, the assumption that K has no isolated point is always satisfied unless K is a single point.

As an illustrative example of partitions, we present the case of the unit square $[-1, 1]^2$ as a self-similar set. This is an example of the general construction of partitions associated with self-similar sets discussed in Section 4.1.

Example 2.4 (The unit square). Let $K = [-1, 1]^2$ and let $S = \{1, 2, 3, 4\}$. Set $p_1 = [-1, -1]$, $p_2 = [1, -1]$, $p_3 = [1, 1]$ and $p_4 = [-1, 1]$. For $i \in S$, define $f_i(x) = \frac{1}{2}(x - p_i) + p_i$ for any $x \in \mathbb{R}^2$. Then it is obvious that

$$K = \bigcup_{i \in S} f_i(K).$$

This is the expression of the unit square as the self-similar set with respect to the collection of contractions $\{f_i\}_{i \in S}$. Let

$$T_n = S^n = \{i_1 \dots i_n \mid i_j \in S \text{ for any } j = 1, \dots, n\}.$$

In particular, let $T_0 = \{\phi\}$. Moreover, define $T = \bigcup_{m \geq 0} T_m$ and define $\pi : T \to T$ by

$$\pi(i_1 \dots i_n i_{n+1}) = i_1 \dots i_n$$

for any $i_1 \dots i_n i_{n+1} \in T_{n+1}$ for $n \geq 1$ and $\pi(\phi) = \phi$. Define $\mathcal{A}(w, v)$ for $w, v \in T$ as $\mathcal{A}(w, v) = 1$ if $\pi(w) = v$ or $\pi(v) = w$ except for $(w, v) = (\phi, \phi)$. Then $(T, \mathcal{A}, \phi)$ is a rooted tree. For $w = w_1 \dots w_n \in T_n$, define

$$f_w = f_{w_1} \circ \cdots \circ f_{w_n} \quad \text{and} \quad K_w = f_w(K).$$

Then $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$. See Figure 2.1.

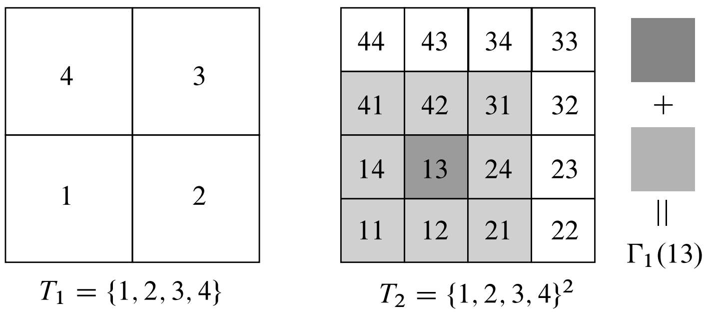

Figure 2.1. Partition of the unit square.

The following definition is a collection of notions concerning partitions.

Definition 2.5. Let $\{K_w\}_{w \in T}$ be a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

(1) Define O_w and B_w for $w \in T$ by

$$O_w = K_w \backslash \Big(\bigcup_{v \in T_{|w|} \backslash \{w\}} K_v \Big), \quad B_w = K_w \cap \Big(\bigcup_{v \in T_{|w|} \backslash \{w\}} K_v \Big).$$

If $O_w \neq \emptyset$ for any $w \in T$, then the partition K is called *minimal*.

(2) For any $A \subseteq T_n$ and $w \in A$, define $\Gamma_M^A(w) \subseteq T_n$ as

$$\Gamma_M^A(w) = \{u \mid u \in A, \text{there exist } u(0), \dots, u(M) \in A \text{ such that } u(0) = w, u(M) = u \text{ and } K_{u(i)} \cap K_{u(i+1)} \neq \emptyset \text{ for any } i = 0, \dots, M-1\}.$$

For simplicity, for $w \in T_n$, we write $\Gamma_M(w) = \Gamma_M^{T_n}(w)$.

(3) $\{K_w\}_{w \in T}$ is called *uniformly finite* if

$$\sup_{w \in T} \#(\Gamma_1(w)) < +\infty.$$

If a partition is minimal, then O_w is actually the interior of K_w, and B_w is the topological boundary of K_w. See [34, Proposition 2.2.3] for details.

In the case of the unit square in Example 2.4, K_w is a square and O_w (resp. B_w) is the interior (resp. the boundary) of K_w. Therefore, it is minimal. Moreover,

$$\sup_{w \in T} \#(\Gamma_1(w)) \leq 8,$$

so that it is uniformly finite.

Now we give the first part of our framework in this paper.

As we declared partially before, through this paper, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

Assumption 2.6. (1) *For any $w \in T$, K_w is connected.*

(2) *There exist M_* and $k_* \in \mathbb{N}$ such that*

$$\pi^{k_*}(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi^{k_*}(w)) \tag{2.1}$$

for any $w \in T$.

(3) *There exists $M_0 \geq M_*$ such that*

$$\Gamma_{M_*}(u) \cap S^k(w) \subseteq \Gamma_{M_0}^{S^k(w)}(u) \tag{2.2}$$

for any $w \in T$, $k \geq 1$ and $u \in S^k(w)$.

See Figure 2.2 for an illustrative exposition of Assumption 2.6 in the case of the unit square.

Remark. As is explicitly mentioned in Proposition 2.16, Assumption 2.6 (2) is always satisfied under mild additional assumptions.

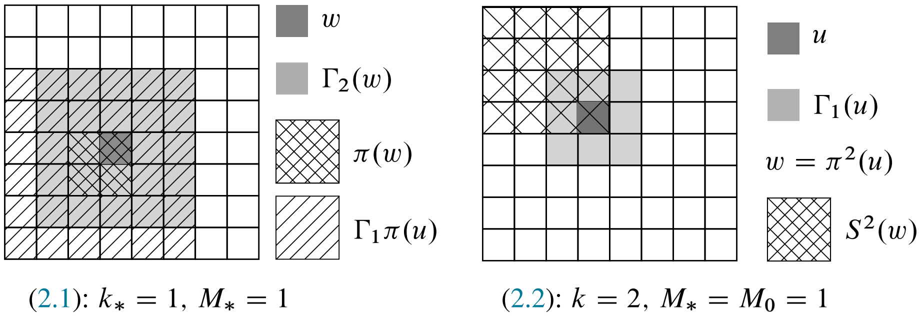

Figure 2.2. Assumption 2.6: the unit square.

Remark. If $M_* = 1$, then we have $\Gamma_{M_*}(w) \cap A = \Gamma^A_{M_*}(w)$ for any w and A. So in this case, by choosing $M_0 = M_* = 1$, Assumption 2.6 (3) is always satisfied.

Throughout this paper, we set

$$L_* = \sup_{w \in T} \#(\Gamma_1(w)). \tag{2.3}$$

Then, for any $m \in \mathbb{N}$,

$$\sup_{w \in T} \#(\Gamma_m(w)) \leq (L_*)^m.$$

Under Assumption 2.6 (2), if the partition $\{K_w\}_{w \in T}$ is replaced by the partition $\{K_w\}_{w \in T^{(k_*)}}$, where $T^{(k_*)} = \bigcup_{i \geq 0} T_{ik_*}$, the constant k_* can be regarded as 1. So doing such a replacement, we will adopt the following assumption.

Assumption 2.7. *The constant k_* appearing in (2.1) is* 1.

For a given partition $\{K_w\}_{w \in T}$, we always associate the following graph structure E^*_n on T_n.

Proposition 2.8. *For $n \geq 0$, define*

$$E^*_n = \{(w, v) \mid w, v \in T_n,\ w \neq v,\ K_w \cap K_v \neq \emptyset\}.$$

Then (T_n, E^*_n) is a non-directed graph. Under Assumption 2.6, (T_n, E^*_n) is connected for any $n \geq 0$, and

$$\Gamma_1(w) = \{v \mid v \in T_n,\ (w, v) \in E^*_n\}$$

for any $n \geq 0$ and $w \in T_n$.

Remark. In [34], E^*_n is denoted by $J^h_{1,n}$.

Definition 2.9. For $w \in T_n$, define

$$\partial S^m(w) = \{v \mid v \in S^m(w), \text{ there exists } v' \in T_{n+m} \text{ such that } (v, v') \in E^*_{n+m} \text{ and } \pi^m(v') \neq w\}.$$

The set $\partial S^m(w)$ is a kind of a boundary of $S^m(w)$. In fact, it is easy to see

$$\partial S^m(w) = \{v \mid v \in S^m(w),\ K_v \cap B_w \neq \emptyset\},$$

where B_w is the topological boundary of K_w as is mentioned above. So the next assumption means that the boundary is not the whole space.

Assumption 2.10. *There exists $m_0 \geq 1$ such that $S^m(w) \backslash \partial S^m(w) \neq \emptyset$ for any $w \in T$ and $m \geq m_0$.*

In Figure 2.3, we have an illustrative exposition of Assumption 2.10 in the case of the unit square.

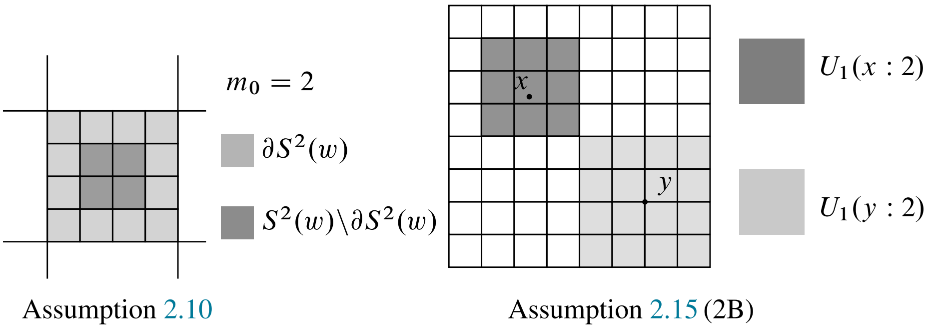

Figure 2.3. Assumptions 2.10 and 2.15 (2B); the unit square.

Definition 2.11. For $w \in T$, $M \geq 1$ and $k \geq 1$, define

$$B_{M,k}(w) = \left\{v \mid v \in S^k(w), \Gamma_{M-1}(v) \cap \partial S^k(w) \neq \emptyset\right\}.$$

Remark. $B_{1,k}(w) = \partial S^k(w)$.

The final assumption is an assumption on a measure μ on K.

Assumption 2.12. *The measure μ is a Borel regular probability measure on K satisfying*

$$\mu(K_w) = \sum_{v \in S(w)} \mu(K_v) \tag{2.4}$$

for any $w \in T$. There exists $\gamma \in (0, 1)$ such that

$$\mu(K_w) \geq \gamma \mu(K_{\pi(w)}) \tag{2.5}$$

for any $w \in T$. This property is called "super-exponential" in [34]. Moreover, there exists $\kappa > 0$ such that if $w, v \in T$, $|w| = |v|$ and $(w, v) \in E^*_{|w|}$, then

$$\mu(K_w) \leq \kappa \mu(K_v) \tag{2.6}$$

The above condition (2.6) corresponds to the gentleness of the measure μ introduced in [34]. Indeed, if μ has the volume doubling property, then this condition is satisfied. See Proposition 2.16 and its proof below for an exact statement.

Lemma 2.13. *Under Assumptions 2.6, 2.10 and 2.12,*

(1) *μ is exponential, i.e., μ satisfies (2.5) and there exist $m_1 \geq 1$ and $\gamma_1 \in (0, 1)$ such that $\mu(K_v) \leq \gamma_1 \mu(K_w)$ for any $w \in T$ and $v \in S^{m_1}(w)$.*

(2) $\sup_{w \in T} \#(S(w)) < \infty$.

Throughout this paper, we set

$$N_* = \sup_{w \in T} \#(S(w)). \tag{2.7}$$

Proof. (1) In fact, we set $m_1 = m_0$. For any w with $|w| \geq 1$ and $m \geq 0$, we see that $\partial S^m(w) \neq \emptyset$ because K is connected. Hence by Assumption 2.10, $\#(S^{m_0}(w)) \geq 2$ for any $w \in T$. Let $v \in S^{m_1}(w)$. Then there exists $u \in S^{m_1}(w)$ with $v \neq u$. By (2.5),

$$\mu(K_w) \geq \mu(K_v) + \mu(K_u) \geq \mu(K_v) + \gamma^{m_1} \mu(K_w),$$

so that $\mu(K_v) \leq (1 - \gamma^{m_1})\mu(K_w)$.

(2) $\mu(K_w) = \displaystyle\sum_{v \in S(w)} \mu(K_v) \geq \gamma \sum_{v \in S(w)} \mu(K_w) = \gamma \#(S(w))\mu(K_w)$.

Hence $\#(S(w)) \leq \frac{1}{\gamma}$. ■

Lemma 2.14. *Under Assumptions 2.6, 2.10 and 2.12,*

$$S^m(w) \backslash B_{M,m}(w) \neq \emptyset$$

for any $w \in T$, $M \geq 1$ and $m \geq M m_0$. Moreover,

$$\mu\Big(\bigcup_{v \in S^n(S^m(w) \backslash B_{M,m}(w))} K_v\Big) \geq \gamma^{m_0 M} \mu(K_w) \tag{2.8}$$

for any $w \in T$, $n \geq 0$ and $m \geq M m_0$.

Proof. By Assumption 2.10, we can inductively choose $v_i \in S^{i m_0}(w)$ for $i \geq 1$ such that $v_{i+1} \in S^{m_0}(v_i) \backslash \partial S^{m_0}(v_i)$ for any $i \geq 1$. At the same time, we see $v_i \notin B_{i, i m_0}(w)$. If $m_0 i < k \leq m_0(i+1)$, then $v \notin B_{i,k}(w)$ for $v = \pi^{m_0(i+1)-k}(v_{i+1})$. So the first part of the claim has been verified. Now if $v \in S^m(w) \backslash B_{M,m}(w)$, then

$$\mu\Big(\bigcup_{v \in S^n(S^m(w) \backslash B_{M,m}(w))} K_v\Big) \geq \mu(K_v) \geq \gamma^{m_0 M} \mu(K_w)$$

by Assumption 2.12. ■

Until now, we have not considered any metric of $(K, \mathcal{O})$, which was merely assumed to be compact and metrizable. The introduction of a metric d on K having suitable properties enables us to integrate the above assumptions into the following one.

Assumption 2.15. *The metric space (K, d) is a compact connected metric space and* $\mathrm{diam}(K, d) = 1$, *where*

$$\mathrm{diam}(A, d) = \sup_{x,y \in A} d(x, y)$$

for a subset $A \subseteq B$. The partition $\{K_w\}_{w \in T}$ is minimal and uniformly finite.

(1) *For any $w \in T$, K_w is connected.*

(2) *There exist $M_* \geq 1$ and $r \in (0, 1)$ such that the following properties hold:*

(2A) *Define $h_r : T \to (0, 1]$ as $h_r(w) = r^{|w|}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 h_r(w) \leq \mathrm{diam}(K_w, d) \leq c_2 h_r(w)$$

for any $w \in T$.

(2B) *For $x \in K$ and $n \geq 1$, define*

$$U_M(x : n) = \bigcup_{\substack{w \in T_n \\ x \in K_w}} \bigcup_{v \in \Gamma_M(w)} K_v.$$

(See Figure 2.3 for examples of $U_1(\cdot : 2)$ in the case of the unit square.) Then there exist $c_1, c_2 > 0$ such that

$$B_d(x, c_1 r^n) \subseteq U_{M_*}(x : n) \subseteq B_d(x, c_2 r^n)$$

for any $n \geq 1$ and $x \in K$, where $B_d(x, r) = \{y \mid d(x, y) < r\}$.

(2C) *There exist $c > 0$ such that, for any $n \geq 1$ and $w \in T_n$, there exists $x \in K_w$ such that*

$$K_w \supseteq B_d(x, cr^n).$$

(3) *μ is a Borel regular probability measure on K. Moreover, μ is exponential and has the volume doubling property with respect to the metric d. Furthermore, μ satisfies (2.4) for any $w \in T$.*

(4) *There exists M_0 such that (2.2) holds for any $w \in T$, $k \geq 1$ and $u \in S^k(w)$.*

(5) *For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.*

Remark. In the terminology of [34], (2A) corresponds to the bi-Lipschitz equivalence of d and h_r, (2B) says that the metric d is M_*-adapted to h_r and (2C) together with (2B) yields d being thick. The combination of (2A), (2B) and (2C) is equivalent to that of (BF1) and (BF2) in [34, Section 4.3].

Remark. Modifying the original partition $\{K_w\}_{w\in T}$, we may always obtain Assumption 2.15 (5) from Assumption 2.15 (1), (2), (3), and (4). Namely, by Proposition 2.16, we have k_* satisfying (2.1) under Assumption 2.15 (1), (2), (3) and (4). So, replacing the original partition $\{K_w\}_{w\in T}$ with $\{K_w\}_{w\in T^{(k_*)}}$, we may suppose $k_* = 1$.

Proposition 2.16. *Assumption* 2.15 (1), (2), (3) *and* (4) *suffice Assumptions* 2.6, 2.10 *and* 2.12.

Proof. About Assumption 2.6, (1) and (3) are included in Assumption 2.15. Since d is M_*-adapted, [34, Proposition 4.4.4] shows the existence of k_* required in Assumption 2.6 (2). By (2C) and (2B), there exists $m_0 \geq 1$ such that

$$K_w \supseteq B_d(x, cr^n) \supseteq U_{M_*}(x : n + m_0)$$

for any $n \geq 1$ and $w \in T_n$, where the point $x \in K_w$ is chosen as in (2C). So if $v \in T_{n+m_0}$ and $x \in K_v$, then $K_v \subseteq B_d(x, cr^n)$ and hence $K_v \cap B_w = \emptyset$. Therefore, Assumption 2.10 is satisfied. Assumption 2.15 includes (2.4) and (2.5) follows from the fact that μ is exponential. Finally, (2.6) is a consequence of the volume doubling property by [34, Theorem 3.3.4]. ∎

Under Assumption 2.15, we may suppose further properties of the metric d and the measure μ. Namely, if $\alpha > \dim_{AR}(K, d)$, then by (1.1), there exist an α-Ahlfors regular metric d_* which is quasisymmetric to d and a Borel regular measure ν which is α-Ahlfors regular with respect to d_*, i.e., there exist $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \nu(B_{d_*}(x, r)) \leq c_2 r^\alpha \tag{2.9}$$

for any $x \in K$ and $r \in (0, 2\mathrm{diam}(K, d)]$. Replacing d and μ by d_* and ν, respectively, we may assume that d is α-Ahlfors regular. Note that if μ is α-Ahlfors regular with respect to d, then α is the Hausdorff dimension of (K, d).

2.2 Conductance constant

In this section, we introduce the conductance constant $\mathcal{E}_{M,p,m}(w, A)$ and show the existence of a partition of unity whose p-energies are estimated by conductance constants from above. In the next section, using the method of combinatorial modulus, we will establish a sub-multiplicative inequality of conductance constants.

Through this section, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w\in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$. Moreover, hereafter in this paper, we always presume Assumptions 2.6, 2.7, 2.10 and 2.12.

To begin with, we define p-energies of functions on graphs (T_n, E_n^*) and the associated p-conductances between subsets.

Notation. Let A be a set. Set

$$\ell(A) = \{f \mid f\colon A \to \mathbb{R}\}. \tag{2.10}$$

Definition 2.17. (1) Let $A \subseteq T_n$. For $f \in \ell(A)$, define $\mathcal{E}^n_{p,A}(f)$ by

$$\mathcal{E}^n_{p,A}(f) = \frac{1}{2} \sum_{u,v\in A,(u,v)\in E_n^*} |f(u) - f(v)|^p.$$

In particular, if $A = T_n$, we define $\mathcal{E}^n_p(f) = \mathcal{E}^n_{p,T_n}(f)$ for $f \in \ell(T_n)$.

(2) Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$. Define

$$\mathcal{E}_{p,m}(A_1, A_2, A) = \inf\bigl\{\mathcal{E}^{n+m}_{p,S^m(A)}(f) \mid f \in \ell(S^m(A)), f|_{S^m(A_1)} \equiv 1, \\ f|_{S^m(A_2)} \equiv 0\bigr\}.$$

(3) Let $A \subseteq T_n$. For $w \in A$, define

$$\mathcal{E}_{M,p,m}(w, A) = \mathcal{E}_{p,m}(\{w\}, A\backslash\Gamma^A_M(w), A),$$

which is called the *p-conductance constant* of w in A at level m.

For simplicity, we often denote a set consisting of a single point, $\{w\}$, by w. For example, if A_1 and A_2 are single points u and v respectively, we sometimes write $\mathcal{E}_{p,m}(u, v, A)$ instead of $\mathcal{E}_{p,m}(\{u\}, \{v\}, A)$.

Remark. As we have mentioned in the introduction, the quantity $\mathcal{E}_{M,p,m}(w, A)$ can be regarded as "p-capacity" from the viewpoint of the potential theory.

Lemma 2.18. *For any $w \in T$, $k \geq 0$ and $u \in S^k(w)$,*

$$\mathcal{E}_{M_0,p,m}(u, S^k(w)) \leq \mathcal{E}_{M_*,p,m}(u, T_{|w|+k}).$$

Proof. This follows from Assumption 2.6 (3). ∎

Remark. In the case $M_* = 1$, we always have $\Gamma^A_1(w) = \Gamma_1(w) \cap A$. Hence even without (2.2),

$$\mathcal{E}_{1,p,m}(w, S^k(w)) \leq \mathcal{E}_{1,p,m}(w, T_{|w|+k})$$

for any $w \in T$, $k \geq 0$ and $u \in S^k(w)$.

The following lemma shows the existence of a partition of unity.

Lemma 2.19. *Let $p \geq 1$ and let $A \subseteq T_n$. For any $w \in A$, there exists $\varphi_w \colon S^m(A) \to [0,1]$ such that*

$$\sum_{w \in A} \varphi_w \equiv 1, \quad \varphi_w|_{S^m(w)} \geq (L_*)^{-M}, \quad \varphi_w|_{S^m(A) \setminus S^m(\Gamma_M^A(w))} \equiv 0$$

and

$$\mathcal{E}_{p,S^m(A)}^{n+m}(\varphi_w) \leq ((L_*)^{2M+1} + 1)^p \max_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{M,p,m}(w', A).$$

Proof. Choose $h_w \in \ell(S^m(A))$ such that $h_w|_{S^m(w)} \equiv 1$, $h_w|_{S^m(A)\setminus S^m(\Gamma_M^A(w))} \equiv 0$, and $\mathcal{E}_{M,p,m}(w, A) = \mathcal{E}_{p,S^m(A)}^{n+m}(h_w)$. Define $h \in \ell(S^m(A))$ as

$$h(v) = \sum_{w \in A} h_w(v)$$

for any $v \in S^m(A)$. Note that $1 \leq h(v) \leq (L_*)^M$. Set

$$\varphi_w = \frac{h_w}{h} \quad \text{and} \quad E_{n+m}(w) = E_{n+m}^* \cap S^m(\Gamma_{M+1}^A(w))^2.$$

It follows that $\varphi_w(u) = \varphi_w(v) = 0$ for any $(u,v) \notin E_{n+m}(w)$. Let $(u,v) \in E_{n+m}(w)$. Then $h_w(v)(h_{w'}(v) - h_{w'}(u)) = 0$ for any $w' \notin \Gamma_{2M+1}^A(w)$. Hence

$$\begin{aligned} |\varphi_w(u) - \varphi_w(v)| &= \left| \frac{1}{h(u)h(v)} (h(v)(h_w(u) - h_w(v)) + h_w(v)(h(v) - h(u))) \right| \\ &\leq |h_w(u) - h_w(v)| + \sum_{w' \in \Gamma_{2M+1}^A(w)} |h_{w'}(u) - h_{w'}(v)|. \end{aligned}$$

Set $C = (L_*)^{2M+1} + 1$. Then the last inequality yields

$$\begin{aligned} \mathcal{E}_p^{n+m}(\varphi_w) &= \frac{1}{2} \sum_{(u,v) \in E_{n+m}(w)} |\varphi_w(u) - \varphi_w(v)|^p \\ &\leq \frac{C^{p-1}}{2} \sum_{(u,v) \in E_{n+m}(w)} \Big(|h_w(u) - h_w(v)|^p \\ &\qquad + \sum_{w' \in \Gamma_{2M+1}^A(w)} |h_{w'}(u) - h_{w'}(v)|^p \Big) \\ &\leq C^{p-1} \Big(\mathcal{E}_{p,S^m(A)}^{n+m}(h_w) + \sum_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{p,S^m(A)}^{n+m}(h_{w'}) \Big) \\ &\leq C^p \max_{w' \in \Gamma_{2M+1}^A(w)} \mathcal{E}_{M,p,m}(w', A). \end{aligned}$$

∎

In particular, in the case $A = T_n$, the associated partition of unity defined below will be used to show the regularity of the p-energy constructed in Section 3.1.

Definition 2.20. For $w \in T$, define $h^*_{M,w,m} \in \ell(T_{|w|+m})$ as the unique function h satisfying $h|_{S^m(w)} = 1$, $h|_{T_{|w|+m} \setminus S^m(\Gamma_M(w))} = 0$ and

$$\mathcal{E}_p^{|w|+m}(h) = \mathcal{E}_{M,p,m}(w, T_{|w|}).$$

Moreover, define $\varphi^*_{M,w,m} \in \ell(T_{|w|+m})$ by

$$\varphi^*_{M,w,m} = \frac{h^*_{M,w,m}}{\sum_{v \in T_{|w|}} h^*_{M,v,m}}.$$

By the proof of Lemma 2.19,

$$\mathcal{E}_p^{n+m}(\varphi^*_{M,w,m}) \leq ((L_*)^{2M+1} + 1)^p \max_{v \in T_n} \mathcal{E}_{M,p,m}(v, T_n)$$

for any $w \in T_n$.

2.3 Combinatorial modulus

Another principal tool of this paper is the notion of combinatorial modulus of a path family of a graph introduced in [11]. The general theory will be briefly reviewed in Appendix 6.3. In this section, we introduce the notion of the p-modulus of paths between two sets and show a sub-multiplicative inequality for them. As in the last section, T is a countably infinite set, $\phi \in T$, $(T, \mathcal{A})$ is a locally finite tree satisfying $\#(\{w | (w, v) \in \mathcal{A}\}) \geq 2$ for any $w \in T$, $(K, \mathcal{O})$ is a compact connected metrizable space and $\{K_w\}_{w \in T}$ is a partition of K parametrized by $(T, \mathcal{A}, \phi)$.

Definition 2.21. Let $M, m \in \mathbb{N}$.

(1) Define

$$E^*_{M,m} = \{(w, v) \mid w, v \in T_m, v \in \Gamma_M(w)\}.$$

Note that $E^*_m = E^*_{1,m}$. Moreover, define

$$\theta_m(w, v) = \min\{M \mid v \in \Gamma_M(w)\}$$

for $w, v \in T_m$. $\theta_m(w, v)$ is called the *graph distance* of the graph (T_m, E^*_m).

(2) Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$. For $k \geq 0$, define

$$\begin{aligned}\mathcal{C}^{(M)}_m(A_1, A_2, A) = \{&(v(1), \ldots, v(l)) \mid v(i) \in S^m(A) \text{ for any } i = 1, \ldots, l, \\ &\text{there exist } v(0) \in S^m(A_1) \text{ and } v(l+1) \in S^m(A_2) \text{ such} \\ &\text{that } (v(i), v(i+1)) \in E^*_{M,n+m} \text{ for any } i = 0, \ldots, l\}, \qquad (2.11)\\ \mathcal{A}^{(M)}_m(A_1, A_2, A) = \Big\{&f \mid f : T_{n+m} \to [0, \infty), \textstyle\sum_{i=1}^l f(w(i)) \geq 1 \\ &\text{for any } (w(1), \ldots, w(l)) \in \mathcal{C}^{(M)}_m(A_1, A_2, A)\Big\}\end{aligned}$$

and

$$\mathcal{M}_{p,m}^{(M)}(A_1, A_2, A) = \inf_{f \in \mathcal{A}_m^{(M)}(A_1,A_2,A)} \sum_{u \in T_{n+m}} f(u)^p. \tag{2.12}$$

(3) For $w \in T_n$, define

$$\mathcal{C}_{N,m}^{(M)}(w) = \mathcal{C}_m^{(M)}(\{w\}, \Gamma_N(w)^c, T_n), \mathcal{A}_{N,m}^{(M)}(w) = \mathcal{A}_m^{(M)}(\{w\}, \Gamma_N(w)^c, T_n)$$

and

$$\mathcal{M}_{N,p,m}^{(M)}(w) = \mathcal{M}_{p,m}^{(M)}(\{w\}, \Gamma_N(w)^c, T_n).$$

The quantity $\mathcal{M}_{p,m}^{(M)}(A_1, A_2, A)$ is called the *p-modulus* of the family of paths between A_1 and A_2 inside A.

Remark. In (2.11) and (2.12), the domain of f is T_{n+m}. However, since we only use $f(u)$ for $u \in S^m(A)$ in (2.11) and the sum in (2.12) becomes smaller by setting $f(u) = 0$ for $u \in T_{n+m} \backslash S^m(A)$, we may think of the domain of f as $S^m(A)$.

As in the case of conductances, if A_1 and A_2 consist of single points u and v, respectively, then we write $\mathcal{C}_m^{(M)}(u, v, A)$, $\mathcal{A}_m^{(M)}(u, v, A)$ and $\mathcal{M}_{p,m}^{(M)}(u, v, A)$ instead of $\mathcal{C}_m^{(M)}(\{u\}, \{v\}, A)$, $\mathcal{A}_m^{(M)}(\{u\}, \{v\}, A)$ and $\mathcal{M}_{p,m}^{(M)}(\{u\}, \{v\}, A)$, respectively.

In accordance with [34, Proposition 4.8.4], the following simple relation between $\mathcal{E}_{p,m}(A_1, A_2, A)$ and $\mathcal{M}_{p,m}^{(1)}(A_1, A_2, A)$ holds. Hence to know $\mathcal{M}_{p,m}^{(1)}(A_1, A_2, A)$ is essentially to know $\mathcal{E}_{p,m}(A_1, A_2, A)$.

Lemma 2.22. *Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Then for any $m \geq 1$ and $p > 0$,*

$$\begin{aligned} \frac{1}{L_*}\mathcal{E}_{p,m}(A_1, A_2, A) &\leq \mathcal{M}_{p,m}^{(1)}(A_1, A_2, A) \\ &\leq 2\max\{1, (L_*)^{p-1}\}\mathcal{E}_{p,m}(A_1, A_2, A). \end{aligned} \tag{2.13}$$

The following theorem is the main result of this section.

Theorem 2.23 (Sub-multiplicative inequality). *Let $k_0, L, M \in \mathbb{N}$. Suppose that*

$$\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$$

for any $u \in T$. Then

$$\mathcal{M}_{M,p,k+l}^{(1)}(w) \leq c_{2.23}\mathcal{M}_{M,p,k}^{(1)}(w) \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}_{L,p,l}^{(1)}(v)$$

for any $l \in \mathbb{N}$, $k \geq k_0$, $w \in T$ and $p > 0$, where $c_{2.23}$ depends only on p, L_* and L.

Remark. If $\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$, then $\pi^k(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^k(u))$ for any $k \geq k_0$.

Similar sub-multiplicative inequalities for moduli of curve families have been shown in [11, Proposition 3.6], [14, Lemma 3.8] and [34, Lemma 4.9.3].

By Assumption 2.7, the assumption $\pi^{k_0}(\Gamma_{L+1}(u)) \subseteq \Gamma_M(\pi^{k_0}(u))$ is satisfied with $M = L = M_*$ and $k_0 = 1$. This fact along with Lemma 2.22 shows the following sub-multiplicative inequality of conductance constants.

Corollary 2.24. *For any $n, k, l \geq 1$, $w \in T_n$ and $p \geq 1$.*

$$\mathcal{E}_{M_*,p,k+l}(w, T_n) \leq c_{2.24} \mathcal{E}_{M_*,p,k}(w, T_n) \max_{v \in S^k(\Gamma_M(w))} \mathcal{E}_{M_*,p,l}(v, T_{n+k}), \tag{2.14}$$

where the constant $c_{2.24} = c_{2.24}(p, L_*, M_*)$ depends only on p, L_* and M_*.

The rest of this section is devoted to a proof of Theorem 2.23.

Lemma 2.25. *Let $A \subseteq T_n$ and let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Assume that $\Gamma_M(u) \cap S^m(A)$ is connected for any $u \in S^m(A)$. Then*

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(M)}_{p,m}(A_1, A_2, A) \leq (L_*)^{(p+1)M} \mathcal{M}^{(1)}_{p,m}(A_1, A_2, A).$$

Proof. By definition,

$$\mathcal{C}^{(M)}_m(A_1, A_2, A) \supseteq \mathcal{C}^{(1)}_m(A_1, A_2, A) \quad \text{and} \quad \mathcal{A}^{(M)}_m(A_1, A_2, A) \subseteq \mathcal{A}^{(1)}_m(A_1, A_2, A).$$

This shows

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(M)}_{p,m}(A_1, A_2, A).$$

Define $H_u = \Gamma_M(u)$ for any $u \in T_{n+m}$. Then

$$\#(H_u) \leq (L_*)^M \quad \text{and} \quad \#(\{v \mid u \in H_v\}) \leq (L_*)^M.$$

Let $(u(1), \dots, u(l)) \in \mathcal{C}^{(M)}_m(A_1, A_2, A)$. Then there exist $u(0) \in S^m(A_1) \cap \Gamma_M(u(1))$ and $u(l+1) \in S^m(A_2) \cap \Gamma_M(u(l))$. Since $u(0)$ and $u(1)$ is connected by a chain in $\Gamma_M(u(1))$ and $u(i)$ and $u(i+1)$ is connected by a chain for $i = 1, \dots, l$ in $\Gamma_M(u(i))$, we have a chain belonging to $\mathcal{C}^{(1)}_m(A_1, A_2, A)$ and contained in $\bigcup_{i=1,\dots,n} H_{u(i)}$. Thus Lemma C.4 shows

$$\mathcal{M}^{(M)}_{p,m}(A_1, A_2, A) \leq (L_*)^{(p+1)M} \mathcal{M}^{(1)}_{p,m}(A_1, A_2, A). \qquad \blacksquare$$

Proof of Theorem 2.23. Let $f \in \mathcal{A}^{(L+1)}_{M,k}(w)$ and let $g_v \in \mathcal{A}^{(1)}_{L,l}(v)$ for any $v \in T_{|w|+k}$. Define $h\colon T_{|w|+k+l} \to [0, \infty)$ by

$$h(u) = \max\left\{f(v) g_v(u) \mid v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))\right\} \chi_{S^{k+l}(\Gamma_M(w))}(u).$$

Claim 1. $h \in \mathcal{A}^{(1)}_{M,k+l}(w)$.

Proof. Let $(u(1), \ldots, u(m)) \in \mathcal{C}^{(1)}_{M,k+l}(w)$. There exist such $u(0) \in S^{k+l}(w)$ and $u(m+1) \in T_{|w|+k+l} \backslash S^{k+l}(\Gamma_M(w))$ that $u(0) \in \Gamma_1(u(1))$ and $u(m+1) \in \Gamma_1(u(m))$. Set $v(i) = \pi^l(u(i))$ for $i = 0, \ldots, m+1$. Let $v_*(0) = v(0)$ and let $i_0 = 0$. Define n_*, $v_*(n)$ and i_n for $i = 1, \ldots, n_*$ inductively as follows: If

$$\max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} = m,$$

then $n = n_*$. If

$$\max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} < m,$$

then define

$$i_{n+1} = \max\{j \mid i_n \le j \le m, v(j) \in \Gamma_L(v_*(n))\} + 1 \quad \text{and} \quad v_*(n+1) = v(i_{n+1}).$$

The fact that $\pi^k(\Gamma_{L+1}(v_*(0))) \subseteq \Gamma_M(\pi^k(v(0)))$ implies $n_* \ge 1$. Since $v(i_{n+1} - 1) \in \Gamma_L(v_*(n))$, we have $v_*(n+1) \in \Gamma_{L+1}(v_*(n))$. Hence

$$(v_*(1), \ldots, v_*(n_*)) \in \mathcal{C}^{(L+1)}_{M,k}(w).$$

Moreover, since $v_*(n-1) \notin \Gamma_L(v_*(n))$ for $n = 1, \ldots, n_*$, there exist j_n and m_n such that $i_{n-1} < j_n \le m_n < i_n$ and $(u(j_n), \ldots, u(m_n)) \in \mathcal{C}^{(1)}_{L,l}(v_*(n))$. Since $g_{v_*(n)} \in \mathcal{A}^{(1)}_{L,l}(v_*(n))$, we have

$$\sum_{i=j_n}^{m_n} h(u(i)) \ge \sum_{i=j_n}^{m_n} f(v_*(n)) g_{v_*(n)}(u(i)) \ge f(v_*(n)).$$

This and the fact that $(v_*(1), \ldots, v_*(n_*)) \in \mathcal{C}^{(L+1)}_{M,k}(w)$ yield

$$\sum_{i=1}^{m} h(u(i)) \ge \sum_{j=1}^{n_*} f(v_*(j)) \ge 1.$$

Thus Claim 1 has been verified. □

Set $C_0 = \max\{(L_*)^{L(p-1)}, 1\}$. Then by Lemma A.1, for $u \in S^{k+l}(\Gamma_M(w))$,

$$\begin{aligned} h(u)^p &\le \Bigg(\sum_{v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))} f(v) g_v(u) \Bigg)^p \\ &\le C_0 \sum_{v \in \Gamma_L(\pi^l(u)) \cap S^k(\Gamma_M(w))} f(v)^p g_v(u)^p. \end{aligned}$$

The above inequality and Claim 1 yield

$$\mathcal{M}^{(1)}_{M,p,k+l}(w) \le \sum_{u \in S^{k+l}(\Gamma_M(w))} h(u)^p \le C_0 \sum_{v \in S^k(\Gamma_M(w))} \sum_{u \in T_{|w|+k+l}} f(v)^p g_v(u)^p.$$

Taking infimum over $g_v \in \mathcal{A}^{(1)}_{L,l}(v)$ and $f \in \mathcal{A}^{(L+1)}_{M,k}(w)$, we have

$$\begin{aligned}\mathcal{M}^{(1)}_{M,p,k+l}(w) &\le C \sum_{v \in S^k(\Gamma_M(w))} f(v)^p \mathcal{M}^{(1)}_{L,p,l}(v) \\ &\le C_0 \sum_{v \in T_{|w|+k}} f(v)^p \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}^{(1)}_{L,p,l}(v) \\ &\le C_0 \mathcal{M}^{(L+1)}_{M,p,k}(w) \max_{v \in S^k(\Gamma_M(w))} \mathcal{M}^{(1)}_{L,p,l}(v).\end{aligned}$$

Finally, applying Lemma 2.25, we have the desired inequality. This completes the proof of Theorem 2.23. ∎

2.4 Neighbor disparity constant

Another important constant in this paper is $\sigma_{p,m}(\cdot)$, which is called the neighbor disparity constant. The neighbor disparity constant controls the differences between means of a function on several cells via the p-energy of the function. For $p = 2$, $\sigma_{2,m}$ was introduced in [36] for the case of self-similar sets.

Notation. For $A \subseteq T_n$ and $f \in \ell(A)$, define

$$(f)_A = \frac{1}{\sum_{v \in A} \mu(K_w)} \sum_{v \in A} f(w)\mu(K_w).$$

Furthermore, set

$$E^*_n(A) = (A \times A) \cap E^*_n. \tag{2.15}$$

Definition 2.26. Let $A \subseteq T_n$.

(1) Define $P_{n,m}\colon \ell(S^m(A)) \to \ell(A)$ by

$$(P_{n,m} f)(w) = (f)_{S^m(w)}$$

for any $f \in \ell(S^m(A))$ and $w \in A$.

(2) For $m \ge 0$ and $p \ge 1$, define

$$\sigma_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{\mathcal{E}^n_{p,A}(P_{n,m} f)}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)},$$

which is called the *p-neighbor disparity constant* of A at level m.

(3) Let $\{G_i\}_{i=1,\dots,k}$ be a collection of subsets of T_n. The family $\{G_i\}_{i=1,\dots,k}$ is called a *covering* of $(A, E^*_n(A))$ with *covering numbers* (N_T, N_E) if

$$A = \bigcup_{i=1}^{k} G_i, \quad \max_{x \in A} \#(\{i \mid x \in G_i\}) \le N_T,$$

and for any $(u, v) \in E_n^*(A)$, there exist $l \leq N_E$ and $\{w(1), \dots, w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1,\dots,k} E_n^*(G_j)$ for any $i = 1, \dots, l$.

Remark. The neighbor disparity constant $\sigma_{p,m}(w, v)$ defined in the introduction is equal to $\sigma_{p,m}(A)$ with $A = \{w, v\}$.

One of the advantages of neighbor disparity constants is their compatibility with the integral projection $P_{n,m}$ from $\ell(T_{n+m})$ to $\ell(T_n)$ as follows.

Lemma 2.27 ([36, Lemma 2.12]). *Let A be a connected subset of T_n, let $m \geq 0$ and let $\{G_i\}_{i=1}^k$ be a covering of $(A, E_n^*(A))$ with covering numbers (N_T, N_E). Then*

$$\mathcal{E}_{p,A}^n(P_{n,m} f) \leq c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i) \mathcal{E}_{p,S^m(A)}^{n+m}(f)$$

for any $f \in \ell(S^m(A))$, where $c_{2.27} = (L_*)^{N_E}(N_E)^{p-1} N_T$, and

$$\sigma_{p,m}(A) \leq c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i).$$

In particular, if $A_1, A_2 \subseteq A$, then

$$\mathcal{E}_{p,0}(A_1, A_2, A) \leq c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i) \mathcal{E}_{p,m}(A_1, A_2, A) \tag{2.16}$$

for any $m \geq 0$.

Proof. For $(w, v) \in E_n^*$, set

$$\begin{aligned} D_l(w, v) = \{(u_1, u_2) \mid\ & (u_1, u_2) \in E_n^*, \text{there exists } (w(1), \dots, w(l), w(l+1)) \\ & \text{such that } w(1) = u_1, w(l+1) = u_2 \\ & \text{and } (w(i), w(i+1)) = (w, v) \text{ for some } i = 1, \dots, l\}. \end{aligned}$$

If $(u_1, u_2) \in D_l(w, v)$, then $u_1 \in \Gamma_{l-1}(w)$ and $u_2 \in \Gamma_1(u_1)$. Hence

$$\#(D_l(w, v)) \leq (L_*)^l.$$

Since $\{G_i\}_{i=1,\dots,k}$ is a covering of A with covering numbers (N_T, N_E), we have

$$\begin{aligned} \mathcal{E}_{p,A}^n(P_{n,m} f) &= \frac{1}{2} \sum_{(u_1,u_2) \in E_n^*(A)} |P_{n,m}(f)(u_1) - P_{n,m}(f)(u_2)|^p \\ &\leq (N_E)^{p-1} \max_{(w,v) \in E_n^*} \#(D_{N_E}(w, v)) \sum_{i=1}^k \mathcal{E}_{p,G_i}^m(P_{n,m} f) \\ &\leq (L_*)^{N_E}(N_E)^{p-1} \sum_{i=1}^k \sigma_{p,m}(G_i) \mathcal{E}_{p,S^m(G_i)}^{n+m}(f) \\ &\leq c_{2.27} \max_{i=1,\dots,k} \sigma_{p,m}(G_i) \mathcal{E}_{p,S^m(A)}^{n+m}(f). \end{aligned}$$

Next, choose f such that $f|_{A_1} \equiv 1$, $f|_{A_0} \equiv 0$ and $\mathcal{E}_{p,m}(A_1, A_2, A) = \mathcal{E}^{n+m}_{p,S^m(A)}(f)$. Then

$$\mathcal{E}_{p,0}(A_1, A_2, A) \leq \mathcal{E}^n_{p,A}(P_{n,m} f).$$

So we have (2.16). ■

Lemma 2.28 ([36, Proposition 2.13 (3)]). *Let $p \geq 1$ and let $A \subseteq T_k$. If $\{B_i\}_{i=1,\ldots,l}$ is a covering of $(S^n(A), E^*_{k+n}(S^n(A)))$ with covering number (N_T, N_E), then*

$$\sigma_{p,n+m}(A) \leq c_{2.27}\sigma_{p,n}(A) \max_{i=1,\ldots,l} \sigma_{p,m}(B_i).$$

Proof. By Lemma 2.27, for any $f \in \ell(T_{k+n+m})$,

$$\begin{aligned}\mathcal{E}^k_{p,A}(P_{k,n}(P_{k+n,m} f)) &\leq \sigma_{p,n}(A)\mathcal{E}^{k+n}_{p,S^n(A)}(P_{k+n,m} f)\\ &\leq \sigma_{p,n} c_{2.27}\sigma_{p,n}(A) \max_{i=1,\ldots,l} \sigma_{p,m}(B_i)\mathcal{E}^{k+n+m}_{p,S^{m+n}(A)}(f).\end{aligned}$$ ■

Due to Theorem 3.33, we will see that if $p > \dim_{AR}(K, d)$, then it is enough to consider neighbor disparity constants for a family $\mathcal{J}_* = \{\{w, v\} | (w, v) \in \bigcup_{n\geq 0} E^*_n\}$. As we will mention right after Example 2.30, however, allowing all the pairs in $\mathcal{J}_*$ might cause a trouble, so that we need the following notion of a covering system in general.

Definition 2.29. Let $\mathcal{J} \subseteq \bigcup_{n\geq 0}\{A \mid A \subseteq T_n\}$. The collection $\mathcal{J}$ is called a *covering system* with *covering numbers* (N_T, N_E) if the following conditions are satisfied:

(1) $\sup_{A\in\mathcal{J}} \#(A) < \infty$.

(2) For any $w \in T$ and $m \geq 1$, there exists a finite subset $\mathcal{N} \subseteq \mathcal{J}$ such that $\mathcal{N}$ is a covering of $(S^m(w), E^*_{n+m}(S^m(w)))$ with covering numbers (N_T, N_E).

(3) For any $G \in \mathcal{J}$ and $m \geq 0$, if $G \subseteq T_n$, then there exists a finite subset $\mathcal{N} \subseteq \mathcal{J}$ such that $\mathcal{N}$ is a covering of $(S^m(G), E^*_{n+m}(S^m(G)))$ with covering numbers (N_T, N_E).

For a covering system $\mathcal{J}$, set

$$\sigma^{\mathcal{J}}_{p,m,n} = \max\{\sigma_{p,m}(A) \mid A \in \mathcal{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathcal{J}}_{p,m} = \sup_{n\geq 0} \sigma^{\mathcal{J}}_{p,m,n}.$$

Remark. By (2.6), applying Theorem 6.10, we see that

$$0 < \sigma^{\mathcal{J}}_{p,m,n} < \infty \quad \text{and} \quad 0 < \sigma^{\mathcal{J}}_{p,m} < \infty.$$

Example 2.30. Define

$$\mathcal{J}_* = \{\{w, v\} \mid (w, v) \in E^*_n \text{ for some } n \geq 0\}.$$

Then $\mathcal{J}_*$ is a covering system with covering numbers $(L_*, 1)$.

If we allow all the pairs in $\mathcal{J}_*$, we may end up with the following situation.

Proposition 2.31. *Let $\mathcal{J}$ be a covering system and let $\{w, v\} \in \mathcal{J}$. Assume $K_w \cap K_v$ is a single point $\{x\}$, and for any $m \geq 0$, there exist $w' \in S^m(w)$ and $v' \in S^m(v)$ such that $\{w', v'\} = \{u \mid u \in T_{n+m}, x \in K_u\}$. Then*

$$\sigma^{\mathcal{J}}_{p,m,|w|} \geq 1 \quad \textit{and} \quad \sigma^{\mathcal{J}}_{p,m} \geq 1$$

for any $p > 0$ and $m \geq 0$.

Proof. Set $n = |w|$. Let $f = \chi_{S^m(w)}$. Then $P_{n,m} f = \chi_{\{w\}}$. Hence

$$\mathcal{E}^{n+m}_{p,S^m(w)\cup S^m(v)}(f) = 1 \quad \text{and} \quad \mathcal{E}^n_{p,\{w,v\}}(P_{n,m} f) = 1,$$

so that $\sigma_{p,m}(\{w, v\}) \geq 1$. ∎

As we will observe in the following sections, the consequence of the above proposition should be avoided if $p < \dim_{AR}(K, d)$ because we expect (but do not have a proof in general) that $\lim_{m\to 0} \sigma^{\mathcal{J}}_{p,m} = 0$ for $p < \dim_{AR}(K, d)$. For example, a suitable substitute of $\mathcal{J}_*$ for the unit square described in Example 2.4 is given as follows.

Example 2.32. Let K be the unit square $[-1, 1]^2$ treated in Example 2.4. Define

$$\mathcal{J}_\ell = \{\{w, v\} \mid \{w, v\} \in \mathcal{J}_*, K_w \cap K_v \text{ is a line segment}\},$$

where the subscript ℓ in $\mathcal{J}_\ell$ represents the word "line". Then $\mathcal{J}_\ell$ is a covering system with covering numbers $(4, 2)$. Note that no $\{w, v\} \in \mathcal{J}_\ell$ satisfies the assumption of Proposition 2.31.

Similar modification of $\mathcal{J}_*$ can be made in the case of subsystems of cubic tilings studied in Section 4.3 including the Sierpiński carpet. See (4.15) for details.

Now, we start to investigate the properties of the neighbor disparity constants of a fixed covering system.

The following lemma is a consequence of Lemma 2.27 connecting the conductance constants with the neighbor disparity constants.

Lemma 2.33. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $p \geq 1$ and let $w \in T$. For any $k \geq 1$, $m, l \geq 0$ and $v \in S^k(w)$,*

$$\mathcal{E}_{M,p,m}(v, S^k(w)) \leq c_{2.27} \sigma^{\mathcal{J}}_{p,l,|w|+k+m} \mathcal{E}_{M,p,m+l}(v, S^k(w)). \tag{2.17}$$

In particular, there exists $c_{2.33}$, depending only on N_T, N_E, M, p and L_*, such that if $S^k(w) \neq \Gamma^{S^k(w)}_M(v)$, then

$$c_{2.33} \leq \sigma^{\mathcal{J}}_{p,l,|w|+k} \mathcal{E}_{M,p,l}(v, S^k(w)) \tag{2.18}$$

for any $n \geq 1$ and $l \geq 0$.

Proof. Let $A = S^{k+m}(w)$ and choose a covering $\mathcal{N} \subseteq \mathcal{J}$ of $S^{k+m}(w)$ with covering number (N_T, N_E). Then by (2.16),

$$\begin{aligned}&\mathcal{E}_{p,0}(S^m(v), S^m(\Gamma_M^{S^k(w)}(v)^c), S^{m+k}(w))\\&\qquad \le c_{2.27} \max_{G\in\mathcal{N}} \sigma_{p,l}(G)\mathcal{E}_{p,l}(S^m(v), S^m(\Gamma_M^{S^k(w)}(v)^c), S^{m+k}(w)).\end{aligned}$$

This implies (2.17). To obtain (2.18), letting $m = 0$ in (2.17), we have

$$\mathcal{E}_{M,p,0}(v, S^k(w)) \le c_{2.27}\sigma^{\mathcal{J}}_{p,l,|w|+k}\mathcal{E}_{M,p,l}(v, S^k(w)).$$

According to Theorem 6.3, $\underline{c}_{\mathcal{E}}(L_*, (L_*)^{M-1}, p) \le \mathcal{E}_{M,p,0}(v, S^k(w))$. This immediately implies (2.18). ■

Another important consequence of Lemma 2.27 is a sub-multiplicative inequality of neighbor disparity constants.

Lemma 2.34. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E) and let $p \ge 1$. Then*

$$\sigma^{\mathcal{J}}_{p,n+m,k} \le c_{2.27}\sigma^{\mathcal{J}}_{p,n,k}\sigma^{\mathcal{J}}_{p,m,k+n}$$

for any $n, m, k \in \mathbb{N}$.

Proof. This is straightforward by Lemma 2.28. ■

In the rest of this section, we study an estimate of the difference $f(u) - f(v)$ for $f\colon T_n \to \mathbb{R}$ and $u, v \in T$ by means of the p-energy $\mathcal{E}^n_p(f)$ and neighbor disparity constants.

Lemma 2.35. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $w \in T$ and let $m \ge 1$. For any $f \in \ell(S^m(w))$ and $u \in S(w)$,*

$$|(f)_{S^m(w)} - (f)_{S^{m-1}(u)}| \le N_*(\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.$$

Proof. Let $\mathcal{N} \subseteq \mathcal{J}$ be a covering of $(S(w), E^*_{|w|+1}(S(w)))$ with covering numbers (N_T, N_E). For any $v \in S(w)$, there exist $v_1, v_2, \dots, v_k \in S(w)$ and $G_1, \dots, G_k \in \mathcal{N}$ such that $k \le N_*$, $v_1 = v$, $v_k = u$ and $(v_i, v_{i+1}) \in E^*_n(G_i)$ for any $i = 1, \dots, k-1$. Hence

$$\begin{aligned}&|(f)_{S^{m-1}(v)} - (f)_{S^{m-1}(u)}|\\&\quad\le \sum_{i=1}^{k-1} |(f)_{S^{m-1}(v_i)} - (f)_{S^{m-1}(v_{i+1})}| \le \sum_{i=1}^{k-1} \mathcal{E}^{|w|+1}_{p,G_i}(P_{|w|+1,m-1}f)^{\frac{1}{p}}\\&\quad\le (\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\sum_{i=1}^{k-1}\mathcal{E}^{|w|+m}_{p,S^{m-1}(G_i)}(f)^{\frac{1}{p}} \le N_*(\sigma^{\mathcal{J}}_{p,m-1,|w|+1})^{\frac{1}{p}}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.\end{aligned}$$

Combining this with

$$(f)_{S^m(w)} - (f)_{S^{m-1}(u)} = \frac{1}{\mu(w)} \sum_{v \in S(w)} ((f)_{S^{m-1}(v)} - (f)_{S^{m-1}(u)})\mu(v),$$

we obtain the desired inequality. ■

Lemma 2.36. *Suppose that $\mathcal{J}$ is a covering system with covering numbers (N_T, N_E). For any $u, v \in T_n$ and $f \in \ell(T_{n+m})$,*

$$|(f)_{S^m(u)} - (f)_{S^m(v)}| \leq N_E \theta_n(u,v) \big(\sigma^{\mathcal{J}}_{p,m,n} \mathcal{E}^{n+m}_p(f)\big)^{\frac{1}{p}}.$$

Proof. Suppose that $\mathcal{N} \subseteq \mathcal{J}$ is a covering of T_n with covering number (N_T, N_E). Set $N = \theta_n(u,v)$ and $g = P_{n,m} f$. There exists $(u(1), \dots, u(N+1)) \subseteq T_n$ such that $u(1) = u, u(N+1) = v$ and $(u(i), u(i+1)) \in E^*_n$ for any $i = 1, \dots, N$. For any i, there exist $G_{i,1}, \dots, G_{i,N_E} \in \mathcal{H}$ and $(u(i,1), \dots, u(i, N_E+1))$ such that $u(i,1) = u(i), u(i, N_E+1) = u(i+1)$ and $(u(i,j), u(i,j+1)) \in E^*_n(G_{i,j})$ for any $j = 1, \dots, N_E$. Then,

$$\begin{aligned}
|g(u) - g(v)| &\leq \sum_{i=1}^{N} |g(u(i)) - g(u(i+1))| \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sum_{j=1}^{N_E} |g(u(i,j)) - g(u(i,j+1))|^p\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sum_{j=1}^{N_E} \mathcal{E}^n_{p,G_{i,j}}(P_{n,m} f)\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \Big((N_E)^{p-1} \sigma^{\mathcal{J}}_{p,m,n} \sum_{j=1}^{N_E} \mathcal{E}^n_{p,S^m(G_{i,j})}(f)\Big)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{N} \big((N_E)^{p-1} \sigma^{\mathcal{J}}_{p,m,n} N_E \mathcal{E}^{n+m}_{p,T_{n+m}}(f)\big)^{\frac{1}{p}} \\
&\leq N N_E \big(\sigma^{\mathcal{J}}_{p,m,n} \mathcal{E}^{n+m}_p(f)\big)^{\frac{1}{p}}.
\end{aligned}$$

■

Lemma 2.37. *Let $\mathcal{J}$ be a covering system with covering numbers (N_T, N_E). Let $n \geq m$. Then, for any $u, v \in T_n$ and $f \in \ell(T_n)$,*

$$\begin{aligned}
|f(u) - f(v)| \leq \Big(&N_E \theta_m(\pi^{n-m}(u), \pi^{n-m}(v))(\sigma^{\mathcal{J}}_{p,n-m,m})^{\frac{1}{p}} \\
&+ 2N_* \sum_{i=1}^{n-m} (\sigma^{\mathcal{J}}_{p,n-m-i,m+i})^{\frac{1}{p}}\Big) \mathcal{E}^n_p(f)^{\frac{1}{p}}. \qquad (2.19)
\end{aligned}$$

Proof. Set $v(i) = \pi^{n-m-i}(v)$ for $i = 0, \ldots, n-m$. Then by Lemma 2.35,

$$\begin{aligned} |f(v) - (f)_{S^{n-m}(\pi^{n-m}(v))}| &\leq \sum_{i=1}^{n-m} |(f)_{S^{n-m-i}(v(i))} - (f)_{S^{n-m-i+1}(v(i-1))}| \\ &\leq N_* \sum_{i=1}^{n-m} (\sigma^{\mathcal{J}}_{p,n-m-i,m+i})^{\frac{1}{p}} \mathcal{E}^n_p(f)^{\frac{1}{p}}. \end{aligned} \tag{2.20}$$

The same inequality holds if we replace v by u. Set $v' = \pi^{n-m}(v)$ and $u' = \pi^{n-m}(u)$. Applying Lemma 2.36, we obtain

$$|(f)_{S^{n-m}(u')} - (f)_{S^{n-m}(v')}| \leq N_E \theta_m(u', v')(\sigma^{\mathcal{J}}_{p,n-m,m})^{\frac{1}{p}} \mathcal{E}^n_p(f)^{\frac{1}{p}}. \tag{2.21}$$

By (2.20) and (2.21), we have (2.19). ■