Chapter 3

Conductive homogeneity and its consequences

3.1 Construction of p-energy: $p > \dim_{AR}(K, d)$

In this section, we are going to construct a p-energy on K as a scaling limit of the discrete counterparts $\mathcal{E}_p^n$'s step by step under Assumption 3.2, which consists of the following two requirements:

(3.1) Neighbor disparity constants and (conductance constants)$^{-1}$ have the same asymptotic behavior.

(3.2) Conductance constants have exponential decay.

Under these assumptions, the p-energy $\widehat{\mathcal{E}}_p$ is constructed in Theorem 3.21. Furthermore, in the case $p = 2$, we construct a local regular Dirichlet form in Theorem 3.23.

The question when Assumption 3.2 is fulfilled will be addressed in Section 3.3.

As in the previous sections, we continue to suppose that Assumptions 2.6, 2.7, 2.10 and 2.12 hold. Moreover, throughout this section, we fix $p \geq 1$.

Definition 3.1. For $M \geq 1, m \geq 0$ and $n \geq 1$, define

$$\mathcal{E}_{M,p,m,n} = \max_{v \in T_n} \mathcal{E}_{M,p,m}(v, T_n).$$

Remark. Theorem 6.3 shows that $\mathcal{E}_{M,p,m,n}$ is finite.

Assumption 3.2. *Let $\mathcal{J}$ be a covering system. There exist $c_1, c_2 > 0$ and $\alpha \in (0,1)$ such that*

$$c_1 \leq \mathcal{E}_{M_*,p,m,n}\sigma^{\mathcal{J}}_{p,m,n} \leq c_2 \tag{3.1}$$

and

$$\mathcal{E}_{M_*,p,m,n} \leq c_2 \alpha^m \tag{3.2}$$

for any $m \geq 0, n \geq 1$.

Hereafter in this section, we fix a covering system $\mathcal{J}$ with covering numbers (N_T, N_E) and use $\sigma_{p,m,n}$ (resp. $\sigma_{p,m}$) in place of $\sigma^{\mathcal{J}}_{p,m,n}$ (resp. $\sigma^{\mathcal{J}}_{p,m}$) for simplicity of notations.

By [34, Theorems 4.7.6 and 4.9.1], we have the following characterization of (3.2) under Assumption 2.15.

Proposition 3.3. *Under Assumption 2.15,*

$$\varlimsup_{m \to \infty} (\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} < 1 \quad \textit{if and only if} \quad p > \dim_{AR}(K, d).$$

In particular, (3.2) holds if and only if $p > \dim_{AR}(K, d)$.

Note that since K is assumed to be connected, we have $\dim_{AR}(K,d) \geq 1$, so that $p > 1$.

In the following definition, we introduce the principal notion of this paper called conductive homogeneity. Due to Theorem 3.5, conductive homogeneity yields (3.1).

Definition 3.4 (Conductive homogeneity). Define

$$\mathcal{E}_{M,p,m} = \sup_{w \in T, |w| \geq 1} \mathcal{E}_{M,p,m}(w, T_{|w|}).$$

A compact metric space K (with a partition $\{K_w\}_{w\in T}$ and a measure μ) is said to be *p-conductively homogeneous* if

$$\sup_{m \geq 0} \sigma_{p,m} \mathcal{E}_{M_*,p,m} < \infty. \tag{3.3}$$

Remark. As in the case of $\mathcal{E}_{M,p,m,n}$, $\mathcal{E}_{M,p,m}$ is always finite due to Theorem 6.3.

Remark. As we will see in Theorem 3.33, if $p > \dim_{AR}(K,d)$, then the conductive homogeneity is solely determined by the conductance constants. Consequently, it is independent of a choice of a covering system $\mathscr{J}$. So, in the case $p > \dim_{AR}(K,d)$, the covering system $\mathscr{J}_*$ is good enough in the end.

Theorem 3.5. *If K is p-conductively homogeneous, then* (3.1) *holds.*

A proof of Theorem 3.5 will be provided in Section 3.3.

Under conductive homogeneity, it will be shown in Theorem 3.30 that there exist $c_1, c_2 > 0$ and $\sigma > 0$ such that

$$c_1\sigma^m \leq \sigma_{p,m,n} \leq c_2\sigma^m \quad \text{and} \quad c_1\sigma^{-m} \leq \mathcal{E}_{M_*,p,m}(v, T_n) \leq c_2\sigma^{-m}$$

for any $m \geq 1$, $n \geq 0$ and $v \in T_n$. This is why we have given the name “homogeneity” to this notion.

Now we start to construct a p-energy under Assumption 3.2. An immediate consequence of Assumption 3.2 is the following multiplicative property of $\sigma_{p,m,n}$.

Lemma 3.6. *There exist $c_1, c_2 > 0$ such that*

$$c_1\sigma_{p,m,n+k}\sigma_{p,n,k} \leq \sigma_{p,n+m,k} \leq c_2\sigma_{p,m,n+k}\sigma_{p,n,k}$$

for any $k \geq 1$, and $m, n \geq 0$.

Proof. By (2.14), we have

$$\mathcal{E}_{M_*,p,n+m,k} \leq c\,\mathcal{E}_{M_*,p,m,n+k}\mathcal{E}_{M_*,p,n,k}.$$

This along with (3.1) shows

$$c_1\sigma_{p,m,n+k}\sigma_{p,n,k} \leq \sigma_{p,n+m,k}.$$

The other half of the desired inequality follows from Lemma 2.34. ∎

Next, we study some geometry associated with the partition $\{K_w\}_{w\in T}$.

Definition 3.7. Let $L \geq 1$. Define

$$n_L(x, y) = \max\{n \mid \text{there exist } w, v \in T_n \text{ such that } x \in K_w,\ y \in K_v \text{ and } v \in \Gamma_L(w)\}.$$

Furthermore, fix $r \in (0, 1)$ and define

$$\delta_L(x, y) = r^{n_L(x,y)}. \tag{3.4}$$

Recall that $h_r\colon T \to (0, 1]$ is given as $h_r(w) = r^{|w|}$. Since $\Lambda_s^{h_r} = T_n$ if $r^{n-1} > s \geq r^n$, where

$$\Lambda_s^{h_r} = \{w \mid w \in T,\ h_r(\pi(w)) > s \geq h_r(w)\},$$

δ_L is nothing but $\delta_L^{h_r}$ defined in [34, Definition 2.3.8].

By [34, Proposition 2.3.7] and the discussions in its proof, we have the following fact.

Proposition 3.8. *Suppose that d is a metric on K giving the original topology $\mathcal{O}$ of K. Let $L \geq 1$. There exists a monotonically non-decreasing function $\eta_L\colon [0, 1] \to [0, 1]$ satisfying $\lim_{t\downarrow} \eta_L(t) = 0$ and $\delta_L(x, y) \leq \eta_L(d(x, y))$ for any $x, y \in K$.*

Proof. Define

$$\Lambda_{s,0}^{h_r}(x) = \{v \mid v \in \Lambda_s^{h_r},\ x \in K_v\}, \quad U_0^{h_r}(x, s) = \bigcup_{v \in \Lambda_{s,0}^{h_r}(x)} K_v$$

and

$$U_1^{h_r}(x, s) = \bigcup_{y \in U_0^{h_r}(x,s)} U_0^{h_r}(y, s)$$

for $s \in (0, 1]$ and $x \in K$. First we show that for any $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that $\delta_L(x, y) \leq \varepsilon$ whenever $d(x, y) \leq \gamma_\varepsilon$. If this is not the case, then there exist $\varepsilon_0 > 0$, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ such that $d(x_n, y_n) \leq \frac{1}{n}$ and $\delta_L(x_n, y_n) > \varepsilon_0$. Since K is compact, choosing an adequate subsequence $\{n_k\}_{k\to\infty}$, we see that there exists $x \in K$ such that $x_{n_k} \to x$ and $y_{n_k} \to x$ for $k \to \infty$. By [34, Proposition 2.3.7], $U_0^{h_r}(x, \frac{\varepsilon_0}{2})$ is a neighborhood of x. Hence both x_{n_k} and y_{n_k} belong to $U_0^{h_r}(x, \frac{\varepsilon_0}{2})$ for sufficiently large k. So, there exist $w, v \in \Lambda_{\varepsilon_0/2,0}^{h_r}(x)$ such that $x_{n_k} \in K_w$ and $y_{n_k} \in K_v$. Since $x \in K_w \cap K_v$, we see that $y \in U_1^{h_r}(x, \frac{\varepsilon_0}{2})$, so that $\delta_L(x_{n_k}, y_{n_k}) \leq \frac{\varepsilon_0}{2}$. This contradicts the assumption that $\delta_L(x_n, y_n) \geq \varepsilon_0$. Thus our claim at the beginning of this proof is verified. Note that with a modification if necessary, we may assume that γ_ε is monotonically non-decreasing as a function of ε and $\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon = 0$. Define

$$\eta_L(t) = \inf\{\varepsilon \mid \varepsilon > 0, t \leq \gamma_\varepsilon\}.$$

Now it is routine to see that η is the desired function. ∎

Let $T_n = \{w(1), \dots, w(l)\}$, where $l = \#(T_n)$. Inductively we define $\widetilde{K}_w$ by

$$\widetilde{K}_{w(1)} = K_{w(1)} \quad \text{and} \quad \widetilde{K}_{w(k+1)} = K_{w(k+1)} \backslash \Big(\bigcup_{i=1,\dots,k} \widetilde{K}_{w(i)} \Big).$$

Note that (2.4) implies that $\mu(B_w) = 0$ for any $w \in T_n$ and hence we have

$$\widetilde{K}_w \supseteq O_w \quad \text{and} \quad \mu(K_w \backslash \widetilde{K}_w) = 0$$

for any $w \in T_n$. The latter equality is due to (2.4). Now define $J_n \colon \ell(T_n) \to \mathbb{R}^K$ by

$$J_n f = \sum_{w \in T_n} f(w) \chi_{\widetilde{K}_w}. \tag{3.5}$$

Since $\widetilde{K}_w$ is a Borel set, $J_n f$ is μ-measurable for any $f \in \ell(T_n)$. The definitions of $\widetilde{K}_w$ and J_n depend on an enumeration of T_n but $J_n f$ stays the same in the μ-a.e. sense regardless of an enumeration.

Define

$$\widetilde{\mathcal{E}}_p^m(f) = \sigma_{p,m-1,1} \mathcal{E}_p^m(f). \tag{3.6}$$

The next lemma yields the control of the difference of values of $J_n f$ through $\widetilde{\mathcal{E}}_p^n(f)$.

Lemma 3.9. *Suppose that Assumption 3.2 holds. There exists $C > 0$ such that for any $n \geq 1$, $f \in \ell(T_n)$ and $x, y \in K$,*

$$|(J_n f)(x) - (J_n f)(y)| \leq C \alpha^{\frac{m}{p}} \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}},$$

where $m = \min\{n_L(x, y), n\}$.

Proof. Let $m = \min\{n_L(x, y), n\}$. Then there exist $w, w' \in T_m$, $v \in S^{n-m}(w)$ and $u \in S^{n-m}(w')$ such that $x \in K_v$, $y \in K_u$, $(J_n f)(x) = f(v)$, $(J_n f)(y) = f(u)$ and $w' \in \Gamma_{L+2}(w)$. By (2.19),

$$|f(u) - f(v)| \leq c \sum_{i=0}^{n-m} (\sigma_{p,n-m-i,m+i})^{\frac{1}{p}} \mathcal{E}_p^n(f)^{\frac{1}{p}}, \tag{3.7}$$

where $c = \max\{2(N_*)^2, N_E(L+2)\}$. Lemma 3.6 shows that

$$c_1 \sigma_{p,m+i-1,1} \sigma_{p,n-m-i,m+i} \leq \sigma_{p,n-1,1}.$$

Combining this with Assumption 3.2, we obtain

$$\sigma_{p,n-m-i,m+i} \leq c_3 \alpha^{m+i} \sigma_{p,n-1,1}.$$

Using (3.7), we see

$$|f(u) - f(v)| \leq c_4 \alpha^{\frac{m}{p}} \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}}. \qquad \blacksquare$$

By this lemma, the boundedness of $\widetilde{\mathcal{E}}_p^n(f_n)$ gives a kind of equicontinuity to the family $\{f_n\}_{n\geq 1}$ and hence an analogue of Arzelà–Ascoli theorem, which we present in Appendix 6.3, shows the existence of a uniform limit as follows.

Lemma 3.10. *Suppose that Assumption 3.2 holds. Define* $\tau = \frac{\log \alpha}{\log r}$. *Let* $f_n \in \ell(T_n)$ *for any* $n \geq 1$. *If*

$$\sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n) < \infty \quad \text{and} \quad \sup_{n\geq 1} |(f_n)_{T_n}| < \infty,$$

then there exist a subsequence $\{n_k\}_{k\geq 1}$ *and* $f \in C(K)$ *such that* $\{J_{n_k} f_{n_k}\}$ *converges uniformly to* f *as* $k \to \infty$, $\widetilde{\mathcal{E}}_p^{n_k}(f_{n_k})$ *is convergent as* $k \to \infty$ *and*

$$|f(x) - f(y)|^p \leq C\eta_L(d(x, y))^\tau \lim_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}),$$

where η_L *was introduced in Proposition 3.8.*

Proof. Set $C_* = \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$. By Lemma 3.9, if $n \geq n_L(x, y)$, then

$$|J_n f_n(x) - J_n f_n(y)| \leq C\alpha^{\frac{n_L(x,y)}{p}}(C_*)^{\frac{1}{p}} \leq C\eta_L(d(x, y))^{\frac{\tau}{p}}(C_*)^{\frac{1}{p}}. \tag{3.8}$$

In the case $n < n_L(x, y)$, there exist $w, w' \in T_n$ such that $x \in K_w$, $J_n f_n(x) = f(w)$, $y \in K_{w'}$, $J_n f_n(w') = f(w')$ and $w' \in \Gamma_{L+2}(w)$. So there exists an E_n^*-path $(w(0), \ldots, w(L + 2))$ satisfying

$$w(0) = w \quad \text{and} \quad w' = w(L + 2).$$

By Lemma A.1,

$$\begin{aligned} |f(w) - f(w')|^p &\leq (L + 2)^{p-1} \sum_{i=0}^{L+1} |f(w(i)) - f(w(i + 1))|^p \\ &\leq (L + 2)^{p-1}\mathcal{E}_p^n(f_n). \end{aligned}$$

On the other hand, since $\widetilde{\mathcal{E}}_p^n(f_n) \leq C_*$, Assumption 3.2 implies

$$\mathcal{E}_p^n(f_n) \leq (\sigma_{p,n-1,1})^{-1}C_* \leq c_2\mathcal{E}_{M_*,p,n-1,1}C_* \leq (c_2)^2\alpha^{n-1}C_*.$$

Thus we have

$$|J_n f_n(x) - J_n f_n(y)| \leq c\alpha^{\frac{n}{p}}(C_*)^{\frac{1}{p}}. \tag{3.9}$$

Making use of (3.8) and (3.9), we see that

$$|J_n f_n(x) - J_n f_n(y)| \leq C\eta_L(d(x, y))^{\frac{\tau}{p}}(C_*)^{\frac{1}{p}} + c\alpha^{\frac{n}{p}}(C_*)^{\frac{1}{p}}$$

for any $x, y \in K$. Applying Lemma D.1 with $X = K$, $Y = \mathbb{R}$, $u_i = J_i f_i$, we obtain the desired result. ■

Definition 3.11. Define $P_n\colon L^1(K,\mu) \to \ell(T_n)$ by

$$(P_n f)(w) = \frac{1}{\mu(w)} \int_{K_w} f\, d\mu$$

for any $n, m \geq 1$. For $f \in \ell(T_k)$, we define

$$P_n f = P_n J_k f.$$

The next lemma is one of the keys to the construction of a p-energy. A counterpart of this fact has already been used in Kusuoka–Zhou's construction of Dirichlet forms on self-similar sets in [36].

Lemma 3.12. *Under Assumption 3.2, there exists $C > 0$ such that for any $n, m \geq 1$ and $f \in L^1(K,\mu) \cup (\bigcup_{k\geq 1} \ell(T_k))$,*

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f) \leq \widetilde{\mathcal{E}}_p^{n+m}(P_{n+m} f). \tag{3.10}$$

In particular,

$$C \sup_{n\geq 0} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \varlimsup_{n\to\infty} \widetilde{\mathcal{E}}_p^n(P_n f) \leq \sup_{n\geq 0} \widetilde{\mathcal{E}}_p^n(P_n f) \tag{3.11}$$

for any $f \in L^1(K,\mu)$.

Remark. This lemma holds without (3.2).

Proof. Note that $P_n f = P_{n,m}(P_{n+m} f)$. Let $\mathcal{N} \subseteq \mathcal{J}$ be a covering of (T_n, E_n^*) with covering numbers (N_T, N_E). By Lemma 2.27,

$$\mathcal{E}_p^n(P_n f) \leq c_{2.27}\sigma_{p,m,n}\mathcal{E}_p^{n+m}(P_{n+m} f).$$

Hence

$$\frac{1}{\sigma_{p,n-1,1}}\widetilde{\mathcal{E}}_p^n(P_n f) \leq c_{2.27}\frac{\sigma_{p,m,n}}{\sigma_{p,n+m-1,1}}\widetilde{\mathcal{E}}_p^{n+m}(P_{n+m} f).$$

By Lemma 3.6, we have (3.10). ∎

By virtue of the last lemma, we have a proper definition of the domain $\mathcal{W}^p$ of a p-energy given in Theorem 3.21 and its semi-norm $\mathcal{N}_p$.

Lemma 3.13. *Define*

$$\mathcal{W}^p = \bigl\{f \mid f \in L^p(K,\mu), \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(P_n f) < +\infty\bigr\},$$

and

$$\mathcal{N}_p(f) = \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(P_n f)^{\frac{1}{p}}$$

for $f \in \mathcal{W}^p$. Then $\mathcal{W}^p$ is a normed linear space with norm $\|\cdot\|_{p,\mu} + \mathcal{N}_p(\cdot)$, where $\|\cdot\|_{p,\mu}$ is the L^p-norm. Moreover, for any $f \in \mathcal{W}^p$, there exists $f_* \in C(K)$ such that $f(x) = f_*(x)$ for μ-a.e. $x \in K$. In this way, $\mathcal{W}^p$ is regarded as a subset of $C(K)$ and

$$|f(x) - f(y)|^p \leq C\eta_L(d(x, y))^\tau \mathcal{N}_p(f)^p \tag{3.12}$$

for any $f \in \mathcal{W}^p$ and $x, y \in K$, where η_L was introduced in Proposition 3.8. In particular, $\mathcal{N}_p(f) = 0$ if and only if f is constant on K.

If no confusion may occur, we write $\|\cdot\|_p$ in place of $\|\cdot\|_{p,\mu}$ hereafter.
In fact, $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$ turns out to be a Banach space by Lemma 3.16.

Proof. Note that

$$\widetilde{\mathcal{E}}_p^n(f+g)^{\frac{1}{p}} \leq \widetilde{\mathcal{E}}_p^n(f)^{\frac{1}{p}} + \widetilde{\mathcal{E}}_p^n(g)^{\frac{1}{p}}$$

and so $\widetilde{\mathcal{E}}_p^n(\cdot)^{\frac{1}{p}}$ is a semi-norm. This implies that $\mathcal{N}_p(\cdot)$ is a semi-norm of $\mathcal{W}^p$.

For $f \in \mathcal{W}^p$, by Lemma 3.10, there exist $\{n_k\}_{k\geq 1}$ and $f_* \in C(K)$ such that

$$\|J_{n_k} P_{n_k} f - f_*\|_\infty \to 0$$

as $k \to \infty$ and

$$|f_*(x) - f_*(y)|^p \leq C\eta_L(d(x, y))^\tau \varlimsup_{n\to\infty} \mathcal{E}_p^n(P_n f).$$

Since $\int_{K_w} P_{n_k} f d\mu \to \int_{K_w} f_* \, d\mu$ as $k \to \infty$, it follows that $\int_{K_w} f d\mu = \int_{K_w} f_* \, d\mu$ for any $w \in T$. Hence $f = f_*$ for μ-a.e. $x \in K$. Thus we identify f_* with f and so $f \in C(K)$. Moreover, (3.12) holds for any $x, y \in K$. By (3.12), $\mathcal{N}_p(f) = 0$ if and only if f is constant on K. ∎

We now examine the properties of the normed space $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. The intermediate goals are to show its completeness (Lemma 3.16) and that it is dense in $C(K)$ with respect to the supremum norm (Lemma 3.19).

Lemma 3.14. *Suppose that Assumption 3.2 holds. The identity map*

$$I : (\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot)) \to (C(K), \|\cdot\|_\infty)$$

is continuous.

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Fix $x_0 \in K$ and set $g_n(x) = f_n(x) - f_n(x_0)$. Then

$$\begin{aligned}|g_n(x) - g_m(x)| &= |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| \\ &\leq C\eta_L(d(x, x_0))^{\frac{\tau}{p}} \mathcal{N}_p(f_n - f_m)\end{aligned}$$

for any $x \in K$ and $n, m \geq 1$. Thus $\{g_n\}_{n\geq 1}$ is a Cauchy sequence in $C(K)$ with the norm $\|\cdot\|_\infty$, so that there exists $g \in C(K)$ such that $\|g - g_n\|_\infty \to 0$ as $n \to \infty$. On the other hand, since $\{f_n\}_{n\geq 1}$ is a Cauchy sequence of $L^p(X, \mu)$, there exists $f \in L^p(X, \mu)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Thus $f_n(x_0) = f_n - g_n$ converges as $n \to \infty$ in $L^p(K, \mu)$. Let c be its limit. Then $f = g + c$ in $L^p(K, \mu)$. Therefore, $f \in C(K)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. ■

Define $\overline{\mathcal{W}}^p$ as the completion of $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Then the map I is extended to a continuous map from $\overline{\mathcal{W}}_p \to C(K)$, which is denoted by I as well for simplicity.

Lemma 3.15 (Closability). *Suppose that Assumption 3.2 holds. The extended map $I\colon \overline{\mathcal{W}}^p \to C(K)$ is injective. In particular, $\overline{\mathcal{W}}^p$ is identified with a subspace of $C(K)$.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{W}^p, \|\cdot\|_p + \mathcal{N}_p(\cdot))$. Suppose that $\lim_{n\to\infty} \|f_n\|_\infty = 0$. Note that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f_m) \leq \sup_{l\geq 1} \widetilde{\mathcal{E}}_p^l(P_l f_n - P_l f_m) = \mathcal{N}_p(f_n - f_m)^p$$

for any $k, n, m \geq 1$. Hence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f_m) \leq \varepsilon$$

for any $n, m \geq N$ and $k \geq 1$. As $\|f_m\|_\infty \to 0$ as $m \to \infty$, we see that

$$\widetilde{\mathcal{E}}_p^k(P_k f_n) \leq \varepsilon$$

for any $n \geq N$ and $k \geq 1$ and hence $\mathcal{N}_p(f_n)^p \leq \varepsilon$ for any $n \geq N$. Thus, $\mathcal{N}_p(f_n) \to 0$ as $n \to \infty$, so that $f_n \to 0$ in $\mathcal{W}^p$ as $n \to \infty$. ■

Lemma 3.16. *Suppose that Assumption 3.2 holds. Then*

$$\overline{\mathcal{W}}^p = \mathcal{W}^p.$$

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence of $\mathcal{W}^p$ and let f be its limit in $\overline{\mathcal{W}}^p$. It follows that $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Using the same argument as in the proof of Lemma 3.15, we see that for sufficiently large n,

$$C\,\widetilde{\mathcal{E}}_p^k(P_k f_n - P_k f) \leq \varepsilon$$

for any $k \geq 1$. Since

$$\widetilde{\mathcal{E}}_p^k(P_k f)^{\frac{1}{p}} \leq \widetilde{\mathcal{E}}_p^k(P_k f - P_k f_n)^{\frac{1}{p}} + \widetilde{\mathcal{E}}_p^k(P_k f_n)^{\frac{1}{p}},$$

it follows that $\sup_{k\geq 1} \widetilde{\mathcal{E}}_p^k(P_k f) < \infty$ and hence $f \in \mathcal{W}^p$. ■

Lemma 3.17. *Suppose that Assumption 3.2 holds.*

(1) *Let $\{n_k\}_{k\geq 1}$ be a monotonically increasing sequence of $\mathbb{N}$. Suppose that $f_{n_k} \in \ell(T_{n_k})$ for any $k \geq 1$, that $\sup_{k\geq 1} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}) < \infty$ and that there exists $f \in C(K)$ such that $\|J_{n_k} f_{n_k} - f\|_\infty \to 0$ as $k \to \infty$. Then $f \in \mathcal{W}^p$.*

(2) *Let $f, g \in \mathcal{W}^p$. Then $f \cdot g \in \mathcal{W}^p$.*

Proof. (1) Set $C_1 = \sup_{k\geq 1} \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k})$. By (3.10), if $n \leq n_l$, then

$$C\, \widetilde{\mathcal{E}}_p^n(P_n f_{n_l}) \leq \widetilde{\mathcal{E}}_p^{n_l}(f_{n_l}) \leq C_1.$$

Letting $l \to \infty$, we obtain

$$C\, \widetilde{\mathcal{E}}_p^n(P_n f) \leq C_1$$

for any $k \geq 1$. This implies $f \in \mathcal{W}^p$.

(2) For any $\varphi, \psi \in \ell(T_n)$,

$$\begin{aligned}\mathcal{E}_p^n(\varphi \cdot \psi) &= \frac{1}{2} \sum_{(w,v)\in E_n^*} |\varphi(w)\psi(w) - \varphi(v)\psi(v)|^p \\ &\leq 2^{p-1} \frac{1}{2} \sum_{(w,v)\in E_n^*} (|\varphi(w)|^p |\psi(w) - \psi(v)|^p + |\varphi(w) - \varphi(v)|^p |\psi(v)|^p) \\ &\leq 2^{p-1} (\|\varphi\|_\infty^p \mathcal{E}_p^n(\varphi) + \|\psi\|_\infty^p \mathcal{E}_p^n(\psi)).\end{aligned}$$

Hence if $h_n = P_n f \cdot P_n g$, then

$$\widetilde{\mathcal{E}}_p^n(h_n) \leq 2^{p-1} \big(\|f\|_\infty^p \widetilde{\mathcal{E}}_p^n(P_n f) + \|g\|_\infty^p \widetilde{\mathcal{E}}_p^n(P_n g)\big).$$

Since $f, g \in \mathcal{W}_p$, we see that $\sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(h_n) < \infty$. Moreover, $\|J_n h_n - fg\|_\infty \to 0$ as $n \to \infty$. Using (1), we conclude that $fg \in \mathcal{W}^p$. ∎

Lemma 3.18. *Suppose that Assumption 3.2 holds. There exist a monotonically increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ and $h^*_{M_*,w}, \varphi^*_{M_*,w} \in \mathcal{W}^p$ for $w \in T$ such that*

(a) *For any $w \in T$,*

$$\begin{aligned}&\lim_{j\to\infty} \|J_{m_j} h^*_{M_*,w,m_j-|w|} - h^*_{M_*,w}\|_\infty \\ &\quad = \lim_{j\to\infty} \|J_{m_j} \varphi^*_{M_*,w,m_j-|w|} - \varphi^*_{M_*,w}\|_\infty = 0,\end{aligned}$$

where $h^*_{M_*,w,m}$ and $\varphi^*_{M_*,w,m}$ are defined in Definition 2.20. For negative values of m, we formally define $h^*_{M_*.w,k-|w|} = P_k h^*_{M_*,w,0}$ and $\varphi^*_{M_*,w,k-|w|} = P_k \varphi^*_{M_*,w,0}$ for $k = 0, 1, \ldots, |w|$.

(b) *$\{\widetilde{\mathcal{E}}_p^{m_j}(h^*_{M_*,w,m_j-|w|})\}_{j\geq 1}$ and $\{\widetilde{\mathcal{E}}_p^{m_j}(\varphi^*_{M_*,w,m_j-|w|})\}_{j\geq 1}$ converge as $j \to \infty$.*

(c) *Set* $U_M(w) = \bigcup_{v \in \Gamma_M(w)} K_w$*. For any* $w \in T$*,* $h^*_{M_*,w}\colon K \to [0,1]$ *and*

$$h_{M_*,w}(x) = \begin{cases} 1 & \text{if } x \in K_w, \\ 0 & \text{if } x \notin U_{M_*}(w). \end{cases}$$

(d) *For any* $w \in T$*,* $\varphi^*_{M_*,w}\colon K \to [0,1]$*,* $\mathrm{supp}(\varphi^*_{M_*,w}) \subseteq U_{M_*}(w)$*, and*

$$\varphi^*_{M_*,w}(x) \geq (L_*)^{-M_*}$$

for any $x \in K_w$*. Moreover, for any* $n \geq 1$*,*

$$\sum_{w \in T_n} \varphi^*_{M_*,w} \equiv 1.$$

(e) *For any* $w \in T$ *and* $x \in K$*,*

$$\varphi^*_{M_*,w}(x) = \frac{h^*_{M_*,w}(x)}{\sum_{v \in T_{|w|}} h^*_{M_*,v}(x)}.$$

Remark. The family $\{\varphi^*_{M_*,w}\}_{w \in T_n}$ is a partition of unity subordinate to the covering $\{U_{M_*}(w)\}_{w \in T_n}$.

Proof. For ease of notation, write $\varphi^*_{w,m} = \varphi^*_{M_*,w,m}$ and $h^*_{w,m} = h^*_{M_*,w,m}$. By Lemma 2.19, (3.1) and Lemma 3.6, we see that

$$\begin{aligned}\widetilde{\mathcal{E}}_p^{|w|+m}(\varphi^*_{w,m}) &\leq ((L_*)^{2M+1} + 1)^p \sigma_{p,|w|+m-1,1} \mathcal{E}_{M,p,m}(w, T_{|w|}) \\ &\leq C\sigma_{p,|w|+m-1,1}\sigma^{-1}_{p,m,|w|} \leq C'\sigma_{p,|w|-1,1}\end{aligned}$$

for any $w \in T$ and $m \geq 0$. Similarly,

$$\widetilde{\mathcal{E}}_p^{|w|+m}(h^*_{w,m}) \leq C'\sigma_{p,|w|-1,1}.$$

Hence Lemma 3.10 shows that, for each w, there exists $\{n_k\}_{k\to\infty}$ such that the sequence $\{J_{|w|+n_k} h^*_{w,n_k}\}_{k \geq 1}$ (resp. $\{J_{|w|+m_j} \varphi^*_{w,n_k}\}_{k \geq 1}$) converges uniformly as $k \to \infty$. Let h^*_w (resp. φ^*_w) be its limit. Lemma 3.17 (1) implies that $h^*_w \in \mathcal{W}^p$ and $\varphi^*_w \in \mathcal{W}^p$. By the diagonal argument, we choose $\{m_j\}_{j \geq 1}$ such that (a) and (b) hold. Statements (c), (d) and (e) are straightforward from the properties of $h^*_{w,m}$ and $\varphi^*_{w,m}$. ∎

Lemma 3.19. *Under Assumption 3.2,* $\mathcal{W}^p$ *is dense in* $(C(K), \|\cdot\|_\infty)$*.*

Proof. Choose $x_w \in K_w$ for each $w \in T$. For $f \in C(K)$, define

$$f_n = \sum_{w \in T_n} f(x_w)\varphi^*_{M_*,w}.$$

Then by Lemma 3.18, it follows that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Hence $\mathcal{W}^p$ is dense in $C(K)$. ∎

Definition 3.20. For $f \in L^P(K,\mu)$, define $\overline{f}$ by

$$\overline{f}(x) = \begin{cases} 1 & \text{if } f(x) \geq 1, \\ f(x) & \text{if } 0 < f(x) < 1, \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

for $x \in K$.

Now we construct the p-energy $\widehat{\mathcal{E}}_p$ as a Γ-cluster point of $\widetilde{\mathcal{E}}_p^n(P_n\,\cdot)$. The use of Γ-convergence in the construction of Dirichlet forms on self-similar sets has been around for some time. See [13, 20] for example.

Theorem 3.21. *Suppose that Assumption 3.2 holds. Then there exist* $\widehat{\mathcal{E}}_p\colon \mathcal{W}^p \to [0,\infty)$ *and* $c > 0$ *such that*

(a) $(\widehat{\mathcal{E}}_p)^{\frac{1}{p}}$ *is a semi-norm on* $\mathcal{W}^p$ *and*

$$c\mathcal{N}_p(f) \leq \widehat{\mathcal{E}}_p(f)^{\frac{1}{p}} \leq \mathcal{N}_p(f) \tag{3.13}$$

for any $f \in \mathcal{W}^p$.

(b) *For any* $f \in \mathcal{W}^p$, $\overline{f} \in \mathcal{W}^p$ *and*

$$\widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f).$$

(c) *For any* $f \in \mathcal{W}^p$,

$$|f(x) - f(y)|^p \leq c\eta_L(d(x,y))^\tau \widehat{\mathcal{E}}_p(f).$$

In particular, for $p = 2$, $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ *is a regular Dirichlet form on* $L^2(K,\mu)$ *and the associated non-negative self-adjoint operator has compact resolvent.*

Property (b) in the above theorem is called the Markov property.

Theorem 3.22 (Shimizu [41]). *Suppose that Assumption 3.2 holds. Then the Banach space* $(\mathcal{W}^p, \|\cdot\|_p + \widehat{\mathcal{E}}_p(\cdot))$ *is reflexive and separable.*

Remark. In [41], the reflexivity and separability are shown in the case of the planar Sierpiński carpet. His method, however, can easily be extended to our general case and one has the above theorem.

Proof of Theorem 3.21. Define $\widehat{\mathcal{E}}_p^n\colon L^p(K,\mu) \to [0,\infty)$ by $\widehat{\mathcal{E}}_p^n(f) = \widetilde{\mathcal{E}}_p^n(P_n f)$ for $f \in L^p(K,\mu)$. Then by [12, Proposition 2.14], there exists a Γ-convergent subsequence $\{\widehat{\mathcal{E}}_p^{n_k}\}_{k\geq 1}$. Define $\widehat{\mathcal{E}}_p$ as its limit. Let $f \in \mathcal{W}^p$. Then

$$\widehat{\mathcal{E}}_p(f) \leq \varliminf_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(f) \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(P_n f) = \mathcal{N}_p(f)^p.$$

Let $\{f_{n_k}\}_{k\geq 1}$ be a recovering sequence for f, i.e., $\|f - f_{n_k}\|_p \to 0$ as $k \to \infty$ and $\lim_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}) = \mathcal{E}_p(f)$. By (3.12), if $n_k \geq n$, then

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f_{n_k}) \leq \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} f_{n_k}) = \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}).$$

Letting $k \to \infty$, we obtain

$$C\,\widetilde{\mathcal{E}}_p^n(P_n f) \leq \widehat{\mathcal{E}}_p(f),$$

so that

$$C\,\mathcal{N}_p(f)^p \leq \widehat{\mathcal{E}}_p(f).$$

The semi-norm property of $\widehat{\mathcal{E}}_p(\cdot)^{\frac{1}{p}}$ is straightforward from basic properties of Γ-convergence.

Next we show that $\widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f)$ for any $f \in \mathcal{W}^p$. Define

$$Q_n f = \sum_{w\in T_n} (P_n f)(w)\chi_{K_w}. \tag{3.14}$$

Then

$$\begin{aligned}
&\int_K |f(y) - Q_n f(y)|^p \mu(dy) \\
&\qquad \leq \sum_{w\in T_n} \int_{K_w} \Big(\frac{1}{\mu(w)} \int_{K_w} |f(y) - f(x)|\mu(dx)\Big)^p \mu(dy) \\
&\qquad \leq \sum_{w\in T_n} \frac{1}{\mu(w)} \int_{K_w\times K_w} |f(y) - f(x)|^p \mu(dx)\mu(dy).
\end{aligned}$$

This shows that if $f \in C(K)$, then $\|f - Q_n f\|_p \to 0$ as $n \to \infty$. Let $\{f_{n_k}\}_{k\geq 1}$ be a recovering sequence for f. Since

$$\begin{aligned}
\|\overline{f} - \overline{Q_n g}\|_p &\leq \|\overline{f} - \overline{Q_n f}\|_p + \|\overline{Q_n f} - \overline{Q_n g}\|_p \\
&\leq \|f - Q_n f\|_p + \|Q_n f - Q_n g\|_p \\
&\leq \|f - Q_n f\|_p + \|f - g\|_p,
\end{aligned}$$

it follows that $\|\overline{f} - \overline{Q_{n_k} f_{n_k}}\|_p \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
\widehat{\mathcal{E}}_p(\overline{f}) &\leq \varliminf_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(\overline{Q_{n_k} f_{n_k}}) = \varliminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(\overline{P_{n_k} f_{n_k}}) \\
&\leq \varliminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} f_{n_k}) = \lim_{k\to\infty} \widehat{\mathcal{E}}_p^{n_k}(f_{n_k}) = \widehat{\mathcal{E}}_p(f).
\end{aligned}$$

Finally for $p = 2$, since a Γ-limit of quadratic forms is a quadratic form, we see that $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ is a regular Dirichlet form on $L^2(K,\mu)$. Since the inclusion map from $(\mathcal{W}^2, \|\cdot\|_2 + \mathcal{N}_p(\cdot))$ to $(C(K), \|\cdot\|_\infty)$ is a compact operator, by [17, Exercise 4.2], the non-negative self-adjoint operator associated with $(\mathcal{E}_2, \mathcal{W}^p)$ has compact resolvent. ∎

For the case $p = 2$, due to the above theorem, $\mathcal{W}^2$ is separable. Hence, we may replace Γ-convergence by point-wise convergence as seen in the following theorem. This enables us to obtain the local property of our Dirichlet form, which turns out to be a resistance form as well.

Theorem 3.23. *Suppose that Assumption 3.2 holds for $p = 2$. Then there exists a subsequence $\{m_k\}_{k\geq 1}$ such that $\{\mathcal{E}_2^{m_k}(P_{m_k} f, P_{m_k} g)\}_{k\geq 1}$ converges as $k \to \infty$ for any $f, g \in \mathcal{W}^2$. Furthermore, define $\mathcal{E}(f,g)$ as its limit. Then $(\mathcal{E}, \mathcal{W}^2)$ is a local regular Dirichlet form on $L^2(K,\mu)$, and there exist $c_1, c_2, c_3 > 0$ such that*

$$c_1 \mathcal{N}_2(f) \leq \mathcal{E}(f,f)^{\frac{1}{2}} \leq c_2 \mathcal{N}_2(f) \tag{3.15}$$

and

$$|f(x) - f(y)|^2 \leq c_3 \eta_L(d(x,y))^\tau \mathcal{E}(f,f) \tag{3.16}$$

for any $f \in \mathcal{W}^2$ and $x, y \in K$. In particular, $(\mathcal{E}, \mathcal{W}^2)$ is a resistance form on K and the associated resistance metric R gives the original topology $\mathcal{O}$ of K.

Proof. Existence of $\{m_k\}_{k\geq 1}$: By Lemma 3.21, the non-negative self-adjoint operator H associated with the regular Dirichlet form $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ has compact resolvent. Hence there exist a complete orthonormal basis $\{\varphi_i\}_{i\geq 1}$ of $L^2(K,\mu)$ and $\{\lambda_i\}_{i\geq 1} \subseteq [0,\infty)$ such that $H\varphi_i = \lambda_i \varphi_i$ and $\lambda_i \leq \lambda_{i+1}$ for any $i \geq 1$ and $\lim_{i\to\infty} \lambda_i = \infty$. Note that $\{\frac{\varphi_i}{\sqrt{1+\lambda_i}}\}_{i\geq 1}$ is a complete orthonormal system of $(\mathcal{W}^2, (\cdot,\cdot)_{2,\mu} + \widehat{\mathcal{E}}_p(\cdot,\cdot))$. Hence setting

$$\mathcal{F} = \{a_{i_1}\psi_{i_1} + \cdots + a_{i_m}\psi_{i_m} \mid m \geq 1,\, i_1, \ldots, i_m \geq 1,\, a_{i_1}, \ldots, a_{i_m} \in \mathbb{Q}\},$$

we see that $\mathcal{F}$ is a dense subset of $\mathcal{W}^p$. For any $f, g \in \mathcal{F}$, since

$$|\widetilde{\mathcal{E}}_2^n(P_n f, P_n g)| \leq \widetilde{\mathcal{E}}_2^n(P_n f)^{\frac{1}{2}} \widetilde{\mathcal{E}}_2^n(P_n g)^{\frac{1}{2}} \leq \mathcal{N}_2(f)\mathcal{N}_2(g),$$

some subsequence of $\{\widetilde{\mathcal{E}}_2^n(P_n f, P_n g)\}_{n\geq 1}$ is convergent. Since $\mathcal{F} \times \mathcal{F}$ is countable, the standard diagonal argument shows the existence of a subsequence $\{m_k\}_{k\geq 1}$ such that $\widetilde{\mathcal{E}}_2^{m_k}(P_{m_k} f, P_{m_k} g)$ converges as $k \to \infty$ for any $f, g \in \mathcal{F}$. Define $\mathcal{E}_2(f,g)$ as its limit. For $f, g \in \mathcal{W}^2$, choose $\{f_i\}_{i\geq 1} \subseteq \mathcal{F}$ and $\{g_i\}_{i\geq 1} \in \mathcal{F}$ such that $f_i \to f$ and $g_i \to g$ as $i \to \infty$ in $\mathcal{W}^2$. Write $\widetilde{\mathcal{E}}_k(u,v) = \widetilde{\mathcal{E}}_2^{m_k}(P_{m_k} u, P_{m_k} v)$ for ease of notation. Then

$$\begin{aligned}
|\widetilde{\mathcal{E}}_k(f,g) - \widetilde{\mathcal{E}}_l(f,g)| &\leq |\widetilde{\mathcal{E}}_k(f,g) - \widetilde{\mathcal{E}}_k(f_i,g)| + |\widetilde{\mathcal{E}}_k(f_i,g) - \widetilde{\mathcal{E}}_k(f_i,g_i)| \\
&\quad + |\widetilde{\mathcal{E}}_k(f_i.g_i) - \widetilde{\mathcal{E}}_l(f_i,g_i)| + |\widetilde{\mathcal{E}}_l(f_i,g_i) - \widetilde{\mathcal{E}}_l(f_i,g)| \\
&\quad + |\widetilde{\mathcal{E}}_l(f_i,g) - \widetilde{\mathcal{E}}_l(f,g)| \\
&\leq |\widetilde{\mathcal{E}}_k(f_i,g_i) - \widetilde{\mathcal{E}}_l(f_i,g_i)| + 2\mathcal{N}_2(f_i)\mathcal{N}_2(g - g_i) \\
&\quad + 2\mathcal{N}_2(f - f_i)\mathcal{N}_2(g).
\end{aligned}$$

This shows that $\{\widetilde{\mathcal{E}}_k(f,g)\}_{k\geq 1}$ is convergent as $k \to \infty$. The equivalence between $\mathcal{N}_2$ and $\mathcal{E}$, (3.15), is straightforward.

Strongly local property: Let $f, g \in \mathcal{W}^p$. Assume that there exists an open set $U \subseteq K$ such that $\mathrm{supp}(f) \subseteq U$ and $g|_U$ is a constant. Consequently, for sufficiently large k, $\widetilde{\mathcal{E}}_k(f,g) = 0$, so that $\mathcal{E}(f,g) = 0$.

Markov property: By (3.13) and (3.15),

$$0 \leq \mathcal{E}(f,f) \leq \widehat{\mathcal{E}}_2(f,f)$$

for any $f \in \mathcal{W}^2$. Since $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ is a regular Dirichlet form, by [16, Theorem 2.4.2], we see that $\mathcal{E}(f,g) = 0$ whenever

$$f, g \in \mathcal{W}^2 \quad \text{and} \quad f(x)g(x) = 0$$

for μ-a.e. $x \in K$. Now by the same argument as in the proof of [7, Theorem 2.1], we have the Markov property.

Resistance form: Among the conditions for a resistance form in [32, Definition 3.1], (RF1), (RF2), (RF3), and (RF5) are immediate from what we have already shown. (RF4) is deduced from (3.16). In fact, (3.16) yields that

$$R(x,y) \leq c\eta_L(d(x,y))^{\tau}$$

for any $x, y \in K$. Assume that $R(x_n, x) \to 0$ as $n \to \infty$ and $\overline{\lim}_{n\to\infty} d(x, x_n) > 0$. Note that the collection of

$$U_L^{h_r}(x, r^n) = \bigcup_{w \in T_n : x \in K_w} \Big(\bigcup_{v \in \Gamma_L(w)} K_v \Big)$$

for $n \geq 1$ is a fundamental system of neighborhoods of x by [34, Proposition 2.3.9]. Therefore, there exist $n \geq 1$ and $\{x_{m_k}\}_{k\geq 1}$ such that $x_{m_k} \notin U_L^{h_r}(x, r^n)$ for any $k \geq 1$. Choose $w \in T_n$ such that $x \in K_w$. Then x_{m_k} belongs to K_v for some $v \in \Gamma_L(w)^c$. So,

$$h^*_{L,w}(x) = 1 \quad \text{and} \quad h^*_{L,w}(x_{m_k}) = 0.$$

Hence

$$R(x_{m_k}, x) \geq \frac{1}{\mathcal{E}(h^*_{L,w})}$$

for any $k \geq 1$. This contradicts the fact that $R(x, x_{m_k}) \to 0$ as $k \to \infty$. Thus we have shown $d(x_n, x) \to 0$ as $n \to \infty$. Hence the topology induced by the resistance metric R is the same as the original topology $\mathcal{O}$. ■

3.2 Construction of p-energy: $p \leq \dim_{AR}(K,d)$

In this section, we will consider how much we can salvage the results in the previous section if $p \leq \dim_{AR}(K,d)$. Honestly, what we will have in this section is far from satisfactory mainly because we have no proof of the conjecture saying that $\mathcal{W}^p \cap C(K)$ is dense in $C(K)$ with respect to the supremum norm. In spite of this, we present what we have now for future study.

Throughout this section, we assume (3.1) and fix a covering system $\mathcal{J}$.

For $p < \dim_{AR}(K,d)$, a choice of a covering system really matters. As we have observed in Proposition 2.31, if $\{w,v\} \in \mathcal{J}$ and $K_w \cap K_v$ is a single point, then $\sigma^{\mathcal{J}}_{p,m,|w|} \geq 1$ for any $m \geq 1$. However, since we assume (3.1), this yields that $\mathcal{E}_{M_*,p,m,|w|} \leq c_2$ for any m, so that $\overline{\lim}_{m\to\infty}(\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} \leq 1$. As long as

$$p \geq \dim_{AR}(K,d),$$

this inequality does not cause any inconsistency with Proposition 3.3. On the contrary, if $p < \dim_{AR}(K,d)$, then this seems troublesome. For example, in the case of the unit square, a direct calculation shows that $\overline{\lim}_{m\to\infty}(\mathcal{E}_{M_*,p,m})^{\frac{1}{m}} > 1$ for any $p < \dim_{AR}([-1,1]^2) = 2$. A similar situation is expected in other cases including the Sierpiński carpet. So, for $p < \dim_{AR}(K,d)$, one should carefully choose $\mathcal{J}$ to avoid a pair sharing only a single point. In the case of the unit square, $\mathcal{J}_\ell$ given in Example 2.32 works for $p < 2$.

As in the previous section, we use $\sigma_{p,m}$ (resp. $\sigma_{p,m,n}$) in place of $\sigma^{\mathcal{J}}_{p,m}$ (reps. $\sigma^{\mathcal{J}}_{p,m,n}$).

Under (3.1), it is straightforward to see that Lemma 3.12 still holds. Replacing $(C(K), \|\cdot\|_\infty)$ by $(L^p(K,\mu), \|\cdot\|_p)$ in the statements and proofs of Lemmas 3.15 and 3.16, we have the following statement.

Lemma 3.24. *$\mathcal{W}^p$ is a Banach space with the norm $\|\cdot\|_p + \mathcal{N}_p(\cdot)$.*

Lemma 3.25. *Let $p > 1$. If $\{f_n\}_{n\geq 1}$ is a bounded sequence in the Banach space $\mathcal{W}^p$, then there exist $\{n_k\}_{k\geq 1}$ and $f \in \mathcal{W}^p$ such that f is the weak limit of $\{f_{n_k}\}_{k\geq 1}$ in $L^p(K,\mu)$,*

$$\|f\|_p \leq \sup_{n\geq 1} \|f_n\|_p \quad \textit{and} \quad \mathcal{N}_p(f) \leq \sup_{n\geq 1} \mathcal{N}_p(f_n).$$

Proof. Since $L^p(K,\mu)$ is reflexive, $\{f_n\}_{n\geq 1}$ contains a weakly convergent sub-sequence $\{f_{n_k}\}_{k\geq 1}$. (See [46, Section V.2].) Let $f \in L^p(K,\mu)$ be its weak limit. Since the map $f \to (P_m f)(w)$ is continuous, we see that $P_m f_{n_k} \to P_m f$ as $k \to \infty$ and hence

$$\widetilde{\mathcal{E}}^m_p(P_m f) = \lim_{k\to\infty} \widetilde{\mathcal{E}}^m_p(P_m f_{n_k}) \leq \sup_{k\geq 1} \mathcal{N}_p(f_{n_k})^{\frac{1}{p}}.$$ ∎

Lemma 3.26. *Let $p > 1$. Suppose that $f_n \in \ell(T_n)$ for any $n \geq 1$ and that*

$$\sup_{n\geq 1} \|J_n f_n\|_p < \infty \quad \text{and} \quad \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n) < \infty.$$

Then there exist a subsequence $\{n_k\}_{k\geq 1}$ and $f \in \mathcal{W}^p$ such that f is the weak limit of $\{J_{n_k} f_{n_k}\}_{k\geq 1}$ in $L^p(K,\mu)$ and

$$\|f\|_p \leq \sup_{n\geq 1} \|J_n f_n\|_p \quad \text{and} \quad C\mathcal{N}_p(f)^p \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n).$$

Proof. Since $L^p(K,\mu)$ is reflexive, $\{J_n f_n\}$ possesses a weak convergent sub-sequence $\{J_{n_k} f_{n_k}\}_{k\geq 1}$. (See [46, Section V.2].) Let $f \in L^p(K,\mu)$ be its weak limit. Lemma 3.12 shows that if $n_k \geq m$, then

$$C\widetilde{\mathcal{E}}_p^m(P_m J_{n_k} f_{n_k}) \leq \widetilde{\mathcal{E}}_p^{n_k}(P_{n_k} J_{n_k} f_{n_k}) = \widetilde{\mathcal{E}}_p^{n_k}(f_{n_k}) \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n).$$

Letting $k \to \infty$, we see

$$C\widetilde{\mathcal{E}}_p^m(P_m f) \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$$

for any $m \geq 1$. Thus $f \in \mathcal{W}^p$ and $C\mathcal{N}_p(f)^p \leq \sup_{n\geq 1} \widetilde{\mathcal{E}}_p^n(f_n)$. ∎

Using this lemma, we have a counterpart of Lemma 3.18 as follows.

Lemma 3.27. *There exist $\{h_w^*\}_{w\in T}$ and $\{\varphi_w^*\}_{w\in T} \subseteq \mathcal{W}^p$ such that*

(a) *Set $U_{M_*}(w) = \bigcup_{v\in\Gamma_{M_*}(w)} K_v$. For any $w \in T$, $h_w^*\colon K \to [0,1]$ and*

$$h_w^*(x) = \begin{cases} 1 & \text{if } x \in K_w, \\ 0 & \text{if } x \notin U_{M_*}(w). \end{cases}$$

(b) *For any $w \in T$, $\varphi_w^*\colon K \to [0,1]$, $\mathrm{supp}(\varphi_w^*) \subseteq U(w)$, and*

$$\varphi_w^*(x) \geq (L_*)^{-M_*}$$

for any $x \in K_w$. Moreover, for any $n \geq 1$,

$$\sum_{w\in T_n} \varphi_w^* \equiv 1.$$

(c) *For any $w \in T$ and $x \in K$,*

$$\varphi_w^*(x) = \frac{h_w^*(x)}{\sum_{v\in T_{|w|}} h_v^*(x)}.$$

By the above lemma, we have the next statement.

Lemma 3.28. *$\mathcal{W}^p$ is dense in $L^p(K, \mu)$.*

Finally, we have the following result on the construction of a p-energy.

Lemma 3.29. *There exist $\widehat{\mathcal{E}}_p\colon \mathcal{W}^p \to [0, \infty)$ and $c_1, c_2 > 0$ such that $\widehat{\mathcal{E}}_p^{\frac{1}{p}}$ is a semi-norm,*

$$c_1 \mathcal{N}_p(f)^p \leq \widehat{\mathcal{E}}_p(f) \leq c_2 \mathcal{N}_p(f)^p \quad \textit{and} \quad \widehat{\mathcal{E}}_p(\overline{f}) \leq \widehat{\mathcal{E}}_p(f)$$

for any $f \in \mathcal{W}^p$. In particular, for $p = 2$, $(\widehat{\mathcal{E}}_2, \mathcal{W}^2)$ is a Dirichlet form on $L^2(K, \mu)$.

3.3 Conductive homogeneity

In this section, we study the notion of conductive homogeneity, namely, its consequence and how one can show it.

Throughout this section, we suppose that Assumptions 2.6, 2.7, 2.10 and 2.12 hold. Moreover, we fix a covering system $\mathcal{J}$ with covering numbers (N_T, N_E). As in the previous sections, we omit $\mathcal{J}$ in the notations of $\sigma^{\mathcal{J}}_{p,m,n}$ and $\sigma^{\mathcal{J}}_{p,m}$ and use $\sigma_{p,m,n}$ and $\sigma_{p,m}$, respectively. In the end, we will see by Theorem 3.33 that the conductive homogeneity is solely determined by the conductance constants and a choice of $\mathcal{J}$ makes no difference.

The first theorem explains the reason why it is called “homogeneity”.

Theorem 3.30. *A metric space A is p-conductively homogeneous if and only if there exist $c_1, c_2 > 0$ and $\sigma > 0$ such that*

$$c_1 \sigma^{-m} \leq \mathcal{E}_{M_*,p,m}(v, T_n) \leq c_2 \sigma^{-m}, \tag{3.17}$$

and

$$c_1 \sigma^m \leq \sigma_{p,m,n} \leq c_2 \sigma^m$$

for any $m \geq 0$, $n \geq 1$ and $v \in T_n$.

An immediate corollary of this theorem is Theorem 3.5.

Corollary 3.31 (Theorem 3.5). *If K is p-conductively homogeneous, then* (3.1) *holds.*

Proof of Theorem 3.30. Assume that K is p-conductively homogeneous. Then by formula (2.18), there exists $c_1 > 0$ such that

$$c_1 \leq \sigma_{p,m} \mathcal{E}_{M_*,p,m}.$$

Also by Lemma 2.34, there exists $c_2 > 0$ such that

$$\sigma_{p,m+n} \leq c_2 \sigma_{p,m} \sigma_{p,n} \tag{3.18}$$

for any $n, m \geq 0$. Moreover, by (2.14), there exists $c_3 > 0$ such that

$$\mathcal{E}_{M_*,p,m+n} \leq c_3 \mathcal{E}_{M_*,p,m} \mathcal{E}_{M_*,p,n}$$

for any $n, m \geq 0$. These inequalities along with (3.3) shows that there exist $c_4, c_5 > 0$ such that

$$c_4 \sigma_{p,m} \sigma_{p,n} \leq \sigma_{p,m+n} \leq c_5 \sigma_{p,m} \sigma_{p,n} \quad \text{and} \quad c_4 \leq \sigma_{p,m} \mathcal{E}_{M_*,p,m} \leq c_5$$

for any $m, n \geq 0$. From these, there exist $c_6, c_7 > 0$ and $\sigma > 0$ such that

$$c_6 \sigma^m \leq \sigma_{p,m} \leq c_7 \sigma^m \quad \text{and} \quad c_6 \sigma^m \leq (\mathcal{E}_{M_*,p,m})^{-1} \leq c_7 \sigma^m$$

for any $m \geq 0$. Hence for any $w \in T$ and $n \geq 1$,

$$c_6 \sigma^m \leq (\mathcal{E}_{p,m})^{-1} \leq (\mathcal{E}_{M_*,p,m}(w, T_n))^{-1} \quad \text{and} \quad \sigma_{p,m,n} \leq c_7 \sigma^m.$$

Making use of (2.18), we see that there exists $c_8 > 0$ such that

$$c_6 \sigma^m \leq (\mathcal{E}_{M_*,p,m}(w, T_n))^{-1} \leq c_8 \sigma_{p,m,n} \leq c_8 c_7 \sigma^m$$

for any $m \geq 0$, $n \geq 1$ and $w \in T_n$.

The converse direction is straightforward. ∎

Next, we show another consequence of conductive homogeneity. For simplicity, we set $\mathcal{E}_{p,m}(u, v, S^k(w)) = \mathcal{E}_{p,m}(\{u\}, \{v\}, S^k(w))$. (In other words, we deliberately confuse u with $\{u\}$.)

Lemma 3.32. *If K is p-conductively homogeneous, then there exists $c_{3.32} > 0$, depending only on p, L_*, N_*, M_*, k, N_T, N_E, such that*

$$\mathcal{E}_{M_*,p,m} \leq c_{3.32} \mathcal{E}_{p,m}(u, v, S^k(w))$$

for any $m \geq 0$, $w \in T$ and $u, v \in S^k(w)$ with $u \neq v$.

Proof. By (2.16), we see that

$$\mathcal{E}_{p,0}(u, v, S^k(w)) \leq c_{2.27} \sigma_{p,m} \mathcal{E}_{p,m}(u, v, S^k(w)).$$

Using Theorem 6.3, it follows that

$$\underline{c}_{\mathcal{E}}(L_*, (N_*)^k, p) \leq \mathcal{E}_{p,0}(u, v, S^k(w)) \leq c_{2.27} \sigma_{p,m} \mathcal{E}_{p,m}(u, v, S^k(w)).$$

Now Theorem 3.30 suffices. ∎

When $p > \dim_{AR}(K, d)$, the converse direction of the above lemma is actually true.

Theorem 3.33. *Assume that there exist $c > 0$ and $\alpha \in (0, 1)$ such that*

$$\mathcal{E}_{M_*,p,m} \le c\alpha^m \tag{3.19}$$

for any $m \ge 0$. Then K is p-conductively homogeneous if and only if for any $k \ge 1$, there exists $c(k) > 0$ such that

$$\mathcal{E}_{M_*,p,m} \le c(k)\mathcal{E}_{p,m}(u, v, S^k(w)) \tag{3.20}$$

for any $m \ge 0$, $w \in T$ and $u, v \in S^k(w)$ with $u \ne v$. In particular, under Assumption 2.15, if $p > \dim_{AR}(K, d)$, then whether K is p-conductively homogeneous or not is independent of neighbor disparity constants and hence a choice of a covering system $\mathcal{J}$.

The last part of the theorem justifies the name "*conductive*" homogeneity.

In fact, (3.19) is the same as (3.2). Recall that, by Proposition 3.3, (3.19) holds if and only if $p > \dim_{AR}(K, d)$ under Assumption 2.15.

As was mentioned in the introduction, (3.20) is an analytic relative of the "knight move" condition described in probabilistic terminologies in [36]. The name "knight move" originated from the epoch-making paper [1] where Barlow and Bass constructed the Brownian motion on the Sierpiński carpet.

The proof of the "only if" part of the above theorem is Lemma 3.32. A proof of the "if" part will be given in Chapter 5.

In the next chapter, we are going to give examples for which one can show p-conductive homogeneity by Theorem 3.33.

In the rest of this section, we study asymptotic behaviors of the heat kernel associated with the diffusion process induced by the Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ under Assumption 2.15. The next lemma shows that the associated resistance metric is bi-Lipschitz equivalent to a power of the original metric.

Lemma 3.34. *Suppose that Assumption 2.15 holds, $p > \dim_{AR}(K, d)$ and K is p-conductively homogeneous. Let σ be the same as in Theorem 3.30 and set $\tau_p = -\frac{\log \sigma}{\log r}$. Then there exist $c_1, c_2 > 0$ such that*

$$c_1 d(x, y)^{\tau_p} \le \sup_{f \in \mathcal{W}^p, \widehat{\mathcal{E}}_p(f) \ne 0} \frac{|f(x) - f(y)|^p}{\widehat{\mathcal{E}}_p(f)} \le c_2 d(x, y)^{\tau_p} \tag{3.21}$$

for any $x, y \in K$. In particular, if $2 > \dim_{AR}(K, d)$, then

$$c_1 d(x, y)^{\tau_2} \le R(x, y) \le c_2 d(x, y)^{\tau_2} \tag{3.22}$$

for any $x, y \in K$, where $R(x, y)$ is the resistance metric associated with the resistance form $(\mathcal{E}, \mathcal{W}^2)$.

Proof. Since $\mathcal{E}_p^m(h^*_{M_*,w,m-|w|}) = \mathcal{E}_{M_*,p,m-|w|}(w, T_{|w|})$, we have

$$c_1\sigma^{-m+|w|} \le \mathcal{E}_p^m(h^*_{M_*,w,m-|w|}) \le c_2\sigma^{-m+|w|}$$

by (3.17). This shows

$$c_1\sigma^{|w|} \le \widehat{\mathcal{E}}_p(h^*_{M_*,w}) \le c_2\sigma^{|w|}.$$

Note that d is M_*- adapted to h_r by Assumption 2.15. Hence by [34, (2.4.1)],

$$c_1 d(x,y) \le \delta_{M_*}(x,y) \le c_2 d(x,y) \tag{3.23}$$

for any $x, y \in K$. Choose $n = n_{M_*}(x,y) + 1$. Let $w \in T_n$ satisfying $x \in K_w$. Since $n > n_{M_*}(x,y)$, it follows that if $v \in T_n$ and $y \in K_v$, then $v \notin \Gamma_{M_*}(w)$. Hence

$$h^*_{M_*,w}(x) = 1 \quad \text{and} \quad h^*_{M_*,w}(y) = 0.$$

Therefore (3.4) and (3.23) yield

$$\sup_{f\in\mathcal{W}^p, \widehat{\mathcal{E}}_p(f)\neq 0} \frac{|f(x)-f(y)|^p}{\widehat{\mathcal{E}}_p(f)} \ge \frac{1}{\widehat{\mathcal{E}}_p(h^*_{M_*,w})}$$
$$\ge c(\sigma_p)^{-n} \ge c' r^{n_{M_*}(x,y)\tau_p} \ge c'' d(x,y)^{\tau_p}.$$

On the other hand in this case, $\eta_{M_*}(t) = t$ by (3.23). Hence Theorem 3.21 (c) implies the other side of the desired inequality. ∎

Due to the general theory of resistance forms in [32], once we have (3.22), it is straightforward to obtain asymptotic estimates of the heat kernel.

Theorem 3.35. *Suppose that Assumption 2.15 holds,* $2 > \dim_{AR}(K,d)$ *and* K *is 2-conductively homogeneous. Set* $\tau_* = \tau_2$. *Then there exists a jointly continuous hear kernel* $p_\mu(t,x,y)$ *on* $(0,\infty)\times K\times K$ *associated with the diffusion process induced by the local regular Dirichlet form* $(\mathcal{E}, \mathcal{W}^2)$ *on* $L^2(K,\mu)$. *Moreover,*

(1) *There exist* $\beta \ge 2$, *a metric* ρ, *which is quasisymmetric to* d, *and positive constants* c_1, c_2, c_3, c_4 *such that*

$$p_\mu(t,x,y) \le \frac{c_1}{\mu(B_\rho(x,t^{\frac{1}{\beta}}))}\exp\left(-c_2\left(\frac{\rho(x,y)^\beta}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{3.24}$$

for any $(t,x,y) \in (0,\infty)\times K\times K$ *and*

$$\frac{c_3}{\mu(B_\rho(x,t^{\frac{1}{\beta}}))} \le p_\mu(t,x,y) \tag{3.25}$$

for any $y \in B_\rho(x, c_4 t^{\frac{1}{\beta}})$.

(2) *Suppose that* μ *is* α_H*-Ahlfors regular with respect to the metric* d. *Set*

$$\beta_* = \tau_* + \alpha_H.$$

Then $\beta_* \geq 2$ *and there exist* $c_7, c_8, c_9, c_{10} > 0$ *such that*

$$p_\mu(t,x,y) \leq c_6 t^{-\frac{\alpha_H}{\beta_*}} \exp\Big(-c_7\Big(\frac{d(x,y)^{\beta_*}}{t}\Big)^{\frac{1}{\beta_*-1}}\Big) \tag{3.26}$$

for any $(t,x,y) \in (0,\infty) \times K \times K$ *and*

$$c_9 t^{-\frac{\alpha_H}{\beta_*}} \leq p_\mu(t,x,y) \tag{3.27}$$

for any $y \in B_d(x, c_{10} t^{\frac{\alpha_H}{\beta_*}})$. *In addition, suppose that* d *has the chain condition, i.e., for any* $x, y \in K$ *and* $n \in \mathbb{N}$, *there exist* $x_0, \ldots, x_n \in K$ *such that* $x_0 = x, x_n = y$ *and*

$$d(x_i, x_{i+1}) \leq \frac{Cd(x,y)}{n},$$

where the constant $C > 0$ *is independent of* x, y *and* n. *Then there exist* c_{11}, $c_{12} > 0$ *such that*

$$c_{11} t^{-\frac{\alpha_H}{\beta_*}} \exp\Big(-c_{12}\Big(\frac{d(x,y)^{\beta_*}}{t}\Big)^{\frac{1}{\beta_*-1}}\Big) \leq p_\mu(t,x,y). \tag{3.28}$$

The exponent α_H above is in fact the Hausdorff dimension of (K, d). The exponents β and β_* are called the walk dimensions.

Proof. We make use of [32, Theorems 15.10 and 15.11]. Since μ has the volume doubling property with respect to d, (3.22) shows that μ has the volume doubling property with respect to R as well. Since K is connected, (K, R) is uniformly perfect. Moreover, since $(\mathcal{E}, \mathcal{W}^2)$ has the local property, the annulus comparable condition (ACC) holds by [32, Proposition 7.6]. Thus, (C1) of [32, Theorem 15.11] is verified and so is (C3) of [32, Theorem 15.11]. Using [32, Theorem 15.11], we have (3.24). Consequently, by [32, Theorem 15.10], we see (3.25). Thus we have shown the first part of the statement. The fact that $\beta \geq 2$, which is beyond the reach of [32, Theorem 15.10], is due to [25]. See also [33, Theorem 22.2].

About the second part, assuming α_H-Ahlfors regularity, i.e., (2.9), we see that

$$h_d(x,s) = s^{\tau_* + \alpha_H} = s^{\beta_*},$$

where $h_d(x,s)$ is defined as

$$h_d(x,s) = \sup_{y \in B_d(x,s)} R(x,y) \cdot \mu(B_d(x,s)).$$

Hence following the flow of exposition of [32, Theorem 15.10], we have

$$g(s) = s^{\beta_*} \quad \text{and} \quad \Phi(s) = s^{\beta_* - 1},$$

where g and Φ appear in the statement of [32, Theorem 15.10]. Consequently, by [32, Theorem 15.10], we obtain (3.26), (3.27) and (3.28). The fact that $\beta_* \geq 2$ can be shown in the same way as we did for β above. ■