Chapter 4

Conductive homogeneity of self-similar sets

4.1 Self-similar sets and self-similarity of energy

In this section, we consider the case where K is a self-similar set with rationally related contraction ratios and construct self-similar energies under conductive homogeneity. Throughout this section, we fix a self-similar structure

$$\mathcal{L} = (K, S, \{f_s\}_{s \in S}).$$

The notion of the self-similar structure was introduced to give a purely topological description of self-similar sets. See [29, Section 1.3] for details.

Definition 4.1. Let K be a compact metrizable space, let S be a finite set, and let $\{f_s\}_{s \in S}$ be a family of continuous injective maps from K to itself.

(1) The triple $(K, S, \{f_s\}_{s \in S})$ is called a *self-similar structure* if there exists a continuous surjective map $\chi\colon S^{\mathbb{N}} \to K$ such that

$$\chi(s_1 s_2 \ldots) = f_{s_1}(\chi(s_2 s_3 \ldots))$$

for any $s_1 s_2 \ldots \in S^{\mathbb{N}}$, where $S^{\mathbb{N}}$ is equipped with the product topology.

(2) Define $W_* = \bigcup_{n \geq 0} S^n$, where $S^0 = \{\phi\}$. An element $(w_1, \ldots, w_n) \in S^n$ is denoted by $w_1 \ldots w_n$. For $w_1 \ldots w_n \in S^n$, set

$$f_w = f_{w_1} \circ \cdots \circ f_{w_n} \quad \text{and} \quad K_w = f_w(K).$$

In particular, f_ϕ is an identity map and $K_\phi = K$.

Hereafter in this section, $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure.

By [29, Proposition 3.3], if $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure, $\chi\colon S^{\mathbb{N}} \to K$ is uniquely given by

$$\{\chi(s_1 s_2 \ldots)\} = \bigcap_{m \geq 0} K_{s_1 \ldots s_m}$$

for any $s_1 s_2 \ldots \in S^{\mathbb{N}}$.

Typically, an example of self-similar structures is given by a self-similar set with respect to a family of contractions. Let (X, d) be a complete metric spaces and let $\{f_i\}_{i=1,\ldots,N}$ be a family of contractions of (X, d), i.e., $f_i\colon X \to X$ and

$$\sup_{x,y \in X, x \neq y} \frac{d(f_i(x), f_i(y))}{d(x, y)} < 1$$

for any $i \in \{1, \dots, N\}$. Then it is known that there exists a unique non-empty compact subset K of X satisfying

$$K = \bigcup_{i=1}^{N} f_i(K). \tag{4.1}$$

See [29, Theorem 1.1.4] for example. The set K is called a self-similar set with respect to $\{f_i\}_{i=1,\dots,N}$. By [29, Theorem 1.2.3], if $S = \{1, \dots, N\}$, then $(K, S, \{f_i\}_{i \in S})$ is a self-similar structure.

Definition 4.2. Let $r \in (0, 1)$ and let $j_s \in \mathbb{N}$ for $s \in S$.

(1) Define

$$j(w) = \sum_{i=1}^{m} j_{w_i} \quad \text{and} \quad g(w) = r^{j(w)} \tag{4.2}$$

for $w = w_1 \dots w_m \in S^m$. (In particular, $j(\phi) = 0$, $g(\phi) = 1$.) Define $\widetilde{\pi}(w_1 \dots w_m) = w_1 \dots w_{m-1}$ for $w = w_1 \dots w_m \in S^m$ and

$$\Lambda^g_{r^n} = \{w \mid w = w_1 \dots w_m \in W_*,\ g(\widetilde{\pi}(w)) > r^n \geq g(w)\}. \tag{4.3}$$

(2) Set

$$T_n = \{(n, w) \mid w \in \Lambda_{r^n}\}, \quad T = \bigcup_{n \geq 0} T_n$$

and define $\iota \colon T \to W^*$ as $\iota(n, w) = w$. Moreover, define

$$\mathcal{A} = \{((n, v), (n+1, w)) \mid n \geq 0,\ v = w \text{ or } v = \widetilde{\pi}(w)\}.$$

Note that $\Lambda^g_{r^n} \cap \Lambda^g_{r^{n+1}}$ can be non-empty. (See Section 4.5 for example.) Thus to distinguish $w \in \Lambda^g_{r^n}$ and $w \in \Lambda^g_{r^{n+1}}$, we have introduced T_n in the above definition.

The following proposition is straightforward.

Proposition 4.3. *The triple $(T, \mathcal{A}, \phi)$ is a rooted tree and $\{K_w\}_{w \in T}$ is a minimal partition of K parametrized by $(T, \mathcal{A}, \phi)$.*

In the rest of this section, we fix $\{j_s\}_{s \in S}$ and the associated partition $(T, \mathcal{A}, \phi)$. Furthermore, we presume the following assumption.

Assumption 4.4. *There exists a metric d on K giving the original topology of K and Assumption 2.15 holds with the metric d.*

If this assumption is satisfied, we say that $\{f_s\}_{s \in S}$ has *rationally related contraction ratios $\{r^{j_s}\}_{s \in S}$.*

In fact, under this assumption, in particular, by Assumption 2.15 (3), there exist $c_1, c_2 > 0$ such that

$$c_1 r^{j(w)} \leq \mathrm{diam}(K_w, d) \leq c_2 r^{j(w)} \tag{4.4}$$

for any $w \in T$. This enable us to regard the contraction ratio of f_s as r^{j_s}. This is why we say that contraction ratios of $\{f_s\}_{s \in S}$ are rationally related.

Combining (4.4) with Assumption 2.15 (2B), we obtain the following proposition.

Proposition 4.5. *Define α_H to be the unique number satisfying*

$$\sum_{s \in S} r^{j_s \alpha_H} = 1$$

and let μ be the self-similar measure on K with weight $\{r^{j_s \alpha_H}\}_{s \in S}$. Then μ is α_H-Ahlfors regular with respect to the metric d and α_H coincides with the Hausdorff dimension of (K, d).

Under our assumptions, let σ be the same constant as in Theorem 3.30. Note that even if we replace the definition of $\widetilde{\mathcal{E}}_p^m(u)$, (3.6), by

$$\widetilde{\mathcal{E}}_p^m(u) = \sigma^m \mathcal{E}_p^m(u), \tag{4.5}$$

all the arguments in Section 3.1 work and the results are unchanged. Our goal in this section is the next theorem.

Theorem 4.6. *Let $(K, S, \{f_s\}_{s \in S})$ be a self-similar structure and let $(T, \mathcal{A}, \phi)$ be given in Definition 4.2. Suppose that Assumption 4.4 is satisfied and that K is p-conductively homogeneous for some $p \in (\dim_{AR}(K, d), \infty)$.*

(1) *For any $w \in W_*$ and $f \in \mathcal{W}^p$,*

$$f \circ f_w \in \mathcal{W}^p.$$

(2) *There exists $\mathcal{E}_p \colon \mathcal{W}^p \to [0, \infty)$ satisfying*

(a) *$(\mathcal{E}_p)^{\frac{1}{p}}$ is a semi-norm on $\mathcal{W}^p$ and there exist $c_1, c_2 > 0$ such that*

$$c_1 \mathcal{N}_p(f) \le \mathcal{E}_p(f)^{\frac{1}{p}} \le c_2 \mathcal{N}_p(f)$$

and

$$c_1 d(x, y)^{\tau_p} \le \sup_{f \in \mathcal{W}^2, \mathcal{E}_p(f) \neq 0} \frac{|f(x) - f(y)|^p}{\mathcal{E}_p(f)} \le c_2 d(x, y)^{\tau_p}$$

for any $f \in \mathcal{W}^p$ and $x, y \in K$.

(b) *For any $f \in \mathcal{W}^p$, $\overline{f} \in \mathcal{W}^p$ and*

$$\mathcal{E}_p(\overline{f}) \le \mathcal{E}_p(f).$$

(c) *For any $f \in \mathcal{W}^p$,*

$$\mathcal{E}_p(f) = \sum_{s \in S} \sigma^{j_s} \mathcal{E}_p(f \circ f_s).$$

In particular, for $p = 2$, $(\mathcal{E}_2, \mathcal{W}^2)$ is a local regular Dirichlet form on $L^2(K, \mu)$.

Proof. Define

$$\mathcal{U} = \{A(\cdot) \mid A(\cdot) \text{ is a semi-norm on } \mathcal{W}^p, \text{ there exist } c_1, c_2 > 0 \text{ such that } c_1\mathcal{N}_p(f) \le A(f) \le c_2\mathcal{N}_p(f) \text{ for any } f \in \mathcal{W}^p\}.$$

For $A_1, A_2 \in \mathcal{U}$, we write $A_1 \le A_2$ if and only if $A_1(f) \le A_2(f)$ for any $f \in \mathcal{W}^p$. We give $\mathcal{U}$ the point-wise convergence topology, i.e., $\{A_n\}_{n\ge 1} \subseteq \mathcal{U}$ is convergent to $A \in \mathcal{U}$ as $n \to \infty$ if and only if $A_n(f) \to A(f)$ as $n \to \infty$ for any $f \in \mathcal{W}^p$. Then due to the separability of $\mathcal{W}^p$ described in Theorem 3.22, $\mathcal{U}$ is an ordered topological cone in the sense of [28].

Let $w \in W_*$. For any $v = v_1 \dots v_k \in \Lambda_{r^{n-j(w)}}$, since

$$g(wv_1 \dots v_{k-1}) = g(w)g(v_1 \dots v_{k-1}) > g(w)r^{n-j(w)} = r^n \ge g(wv),$$

it follows that $wv \in \Lambda_{r^n}$. This shows that $\{(n, wv) | v \in \Lambda_{r^{n-j(w)}}\} \subseteq T_n$. In fact,

$$T_n = \bigcup_{w\in S^m} \{(n, wv) \mid v \in \Lambda_{r^{n-j(w)}}\},$$

which is a disjoint union. This yields

$$\sum_{w\in S^m} \mathcal{E}_p^{n-j(w)}(P_{n-j(w)}(f\circ f_w)) \le \mathcal{E}_p^n(P_n f)$$

for any $f \in L^p(K, \mu)$. Therefore,

$$\sum_{w\in S^m} \sigma^{j(w)}\widehat{\mathcal{E}}^{n-j(w)}(f\circ f_w) \le \widehat{\mathcal{E}}_p^n(f).$$

This inequality implies that $\sigma^{j(w)} \sup_{n\ge j(w)} \widehat{\mathcal{E}}^{n-j(w)}(f\circ f_w) \le \mathcal{N}_p(f)^p < \infty$ for any $f \in \mathcal{W}^p$, so that $f\circ f_w \in \mathcal{W}^p$. Thus we have verified the statement (1). Again by the above inequality,

$$\begin{aligned} c\sum_{w\in S^m} \sigma^{j(w)}\mathcal{N}_p(f\circ f_w)^p &\le \sum_{w\in S^m} \sigma^{j(w)} \varliminf_{n\to\infty} \widehat{\mathcal{E}}^{n-j(w)}(f\circ f_w) \\ &\le \sup_{n\ge 0} \widehat{\mathcal{E}}_p^n(f) = \mathcal{N}_p(f)^p. \end{aligned} \tag{4.6}$$

Note that

$$\sum_{(n,v)\in T_n} \sigma^{j(v)}\widehat{\mathcal{E}}_p^{k-j(v)}(f\circ f_v) \le \sum_{w\in S^m} \sigma^{j(w)}\widehat{\mathcal{E}}^{n+k-j(w)}(f\circ f_w).$$

By (3.11), taking $\varliminf$ in the left-hand side and sup in the right-hand side, we see that

$$c\sum_{(n,v)\in T_n} \sigma^{j(v)}\mathcal{N}_p(f\circ f_v)^p \le \sum_{w\in S^m} \sigma^{j(w)}\mathcal{N}_p(f\circ f_w)^p. \tag{4.7}$$

On the other hand, for any $(n, v) \in T_n$ and $x \in K_v$, the self-similarity of μ and (3.21) show

$$
\begin{aligned}
|(P_n f)(v) - f(x)| &\leq \int_K |f \circ f_v(y) - f \circ f_v(x_0)|\, \mu(dy) \\
&\leq c \int_K d(x_0, y)^{\frac{\tau *}{p}} \mu(dy) \mathcal{N}_p(f \circ f_v) \leq c' \mathcal{N}_p(f \circ f_v),
\end{aligned}
$$

where $x_0 = (f_v)^{-1}(x)$. Hence if $((n, v), (n, u)) \in E_n^*$, then

$$
|(P_n f)(v) - (P_n f)(u)| \leq c'(\mathcal{N}_p(f \circ f_v) + \mathcal{N}_p(f \circ f_w)).
$$

This along with (4.7) yields

$$
\begin{aligned}
\widehat{\mathcal{E}}_p^n(f) &= \frac{\sigma^n}{2} \sum_{((n,v),(n,u)) \in E_n^*} |(P_n f)(v) - (P_n f)(u)|^p \\
&\leq C \sum_{(n,v) \in T_n} \sigma^{j(w)} \mathcal{N}_p(f \circ f_v)^p \leq C' \sum_{w \in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p.
\end{aligned}
$$

Taking sup in the right-hand side, we have

$$
\mathcal{N}_p(f)^p \leq C' \sum_{w \in S^m} \sigma^{j(w)} \mathcal{N}_p(f \circ f_w)^p. \tag{4.8}
$$

Now for $A \in \mathcal{U}$, define $\mathcal{F}(A)$ by

$$
\mathcal{F}(A)(f) = \Big(\sum_{s \in S} \sigma^{j_s} A(f \circ f_s)^p \Big)^{\frac{1}{p}}.
$$

For any $A \in \mathcal{U}$, since $A \leq c_2 \mathcal{N}_p$, (4.6) implies

$$
\mathcal{F}(A) \leq c_2 \mathcal{F}(\mathcal{N}_p) \leq c' \mathcal{N}_p.
$$

On the other hand, the fact $c_1 \mathcal{N}_p \leq A$ and (4.8) yield

$$
\mathcal{F}(A) \geq c_1 \mathcal{F}(\mathcal{N}_p) \geq c'' \mathcal{N}_p.
$$

Thus $\mathcal{F}(A) \in \mathcal{U}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$. It is easy to see that $\mathcal{U}$ is continuous and

$$
\mathcal{F}(A + B) \leq \mathcal{F}(A) + \mathcal{F}(B).
$$

Combining (4.6) and (4.8), we see that there exist $C_1, C_2 > 0$ such that

$$
c_1 \mathcal{N}_p \leq \mathcal{F}^j(\mathcal{N}_p) \leq c_2 \mathcal{N}_p
$$

for any $j \geq 1$. So, by [28, Theorem 1.5], there exists $\mathcal{E}_* \in \mathcal{U}$ such that $\mathcal{F}(\mathcal{E}_*) = \mathcal{E}_*$. Define

$$
\mathcal{U}_M = \{ A \mid A \in \mathcal{U}, A(\overline{f}) \leq A(f) \text{ for any } f \in \mathcal{W}^p \}.
$$

Then $\widehat{\mathcal{E}}_p \in \mathcal{U}_M$ and $\mathcal{U}_M$ is a closed subset of $\mathcal{U}$. Hence by [28, Corollary 1.6], we see there exists $\mathcal{E}' \in \mathcal{U}_M$ such that $\mathcal{F}(\mathcal{E}') = \mathcal{E}'$. Letting $\mathcal{E} = (\mathcal{E}')^p$, we have the desired $\mathcal{E}$. In the case $p = 2$, define

$$\mathcal{U}_{DF} = \{A \mid A \in \mathcal{U}, A \text{ satisfies the parallelogram law, the resulting quadratic form has both Markov and local property}\}.$$

Then $\mathcal{U}_{DF}$ is a closed subspace of $\mathcal{U}$ and Theorem 3.23 ensures that $\mathcal{U}_{DF} \neq \emptyset$. So again by [28, Corollary 1.6], we have the desired local regular Dirichlet form. ∎

4.2 Conductive homogeneity of self-similar sets

In this section, we present a sufficient condition for conductive homogeneity of self-similar sets. The idea originated from [11], where the authors used symmetries of the spaces to show the combinatorial Loewner property of the Sierpiński carpet and the Menger curve, also known as the Menger sponge. Our sufficient condition, Theorem 4.8, will be used in Sections 4.3 and 4.6.

Throughout this section, we assume that $(K, S, \{f_s\}_{s\in S})$ is a self-similar structure and adopt the setting in Section 4.1, i.e., let $(T, \mathcal{A}, \phi)$ be given in Definition 4.2 and we suppose that Assumption 4.4 is satisfied. For simplicity, we also assume that $j_s = 1$ for any $s \in S$, so that $g(w) = r^{|w|}$ and $T_m = S^m$.

Definition 4.7. (1) For any $e = (w, v) \in \bigcup_{m\geq 1} E_m^*$, define

$$X(e) = (f_w)^{-1}(f_w(K) \cap f_v(K))$$

and $\varphi_e\colon X(e) \to X(e^r)$ by $\varphi_e = (f_v)^{-1} \circ f_w|_{X(e)}$, where $e^r = (v, w)$ for $e = (w, v)$. Furthermore, define

$$\mathcal{IT}(K, T) = \{(X(e), X(e^r), \varphi_e) \mid m \geq 1, e \in E_m^*\}.$$

An element of $\mathcal{IT}(K, T)$ is called an *intersection type* of (K, T).

(2) A homeomorphism $g\colon K \to K$ is said to be a *symmetry* of (K, T) if there exists $g^*\colon T \to T$ such that $|g^*(w)| = |w|$ and $g(K_w) = K_{g^*(w)}$ for any $w \in T$. Define $\mathcal{G}_{(K,T)}$ as the collection of symmetries of (K, T).

(3) For any $n \geq 0$, define $\psi_n\colon \bigcup_{m\geq 0} T_{n+m} \to T$ by $\psi_n(v) = u$ if $v \in T_{n+m}$ and $v = \pi^m(v)u$.

Remark. The notion of intersection types and the set $\mathcal{IT}(K, T)$ were introduced in [31].

Note that $\psi_n(T_{n+m}) = T_m$ and $(f_{\pi^m(v)})^{-1}(K_v) = K_{\psi_n(v)}$ for any $v \in T_{n+m}$.

Notation. For $A \subseteq T$, set

$$K(A) = \bigcup_{v \in A} K_v. \tag{4.9}$$

Theorem 4.8. *Suppose that there exist a finite subset $\mathcal{I} \subseteq \mathcal{IT}(K,T)$ and finite subgroups $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{G}_{(K,T)}$ satisfying the following properties:*

(a) *$(T_m, E_m^{\mathcal{I}})$ is connected for any $m \geq 1$, where*

$$E_m^{\mathcal{I}} = \{e \mid e \in E_m^*, (X(e), X(e^r), \varphi_e) \in \mathcal{I}\}.$$

(b) *For any $(X, Y, \varphi) \in \mathcal{I}$ and $x \in X$, there exists $g \in \mathcal{G}_0$ such that $g(x) = \varphi(x)$.*

(c) *For any $n \geq 1$, $w \in T_n$ and $\mathbf{p} \in \mathcal{C}_{M,m}^{(1)}(w)$, there exists $\mathcal{U}_{\mathbf{p}} \subseteq \bigcup_{g \in \mathcal{G}_1} g^*(\psi_n(\mathbf{p}))$ such that $K(\mathcal{U}_p)$ is connected and $g(K(\mathcal{U}_p)) \cap X \neq \emptyset$ and $g(K(\mathcal{U}_p)) \cap Y \neq \emptyset$ for any $(X, Y, \varphi) \in \mathcal{I}$ and $g \in \mathcal{G}_0$.*

Then for any $p \geq 1$, $n, k \geq 1$, $m \geq 1$, $u_*, v_* \in T_k$, and $w \in T_n$,

$$\mathcal{M}_{M,p,m}^{(1)}(w) \leq (L_*)^M \#(\mathcal{G}_1)^{p+1} \#(T_k)^p \mathcal{M}_{p,m}^{(1)}(u_*, v_*, T_k). \tag{4.10}$$

Furthermore, if Assumption 4.4 holds with $M_* = M$, then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d)$.

Remark. Strictly, a path $\mathbf{p} = (w(1), \dots, w(k))$ of a graph is not a subset of vertices but a sequence of them. However, we use $\mathbf{p}$ to denote a subset $\{w(1), \dots, w(k)\}$ if no confusion may occur. For example, in the expression $\psi_n(\mathbf{p})$ above, we regard $\mathbf{p}$ as a subset of T_{n+m}.

Proof of Theorem 4.8. For $u \in S^m(\Gamma_M(w))$, define $H_u \subseteq T_{k+m}$ by

$$H_u = \{vg^*(\psi_n(u)) \mid g \in \mathcal{G}_1, v \in T_k\}.$$

Then we have that $\#(H_u) \leq \#(T_k)\#(\mathcal{G}_1)$ for any $u \in S^m(\Gamma_1(w))$ and $\#(\{u \mid v \in H_u\}) \leq \#(\Gamma_M(w))\#(\mathcal{G}_1)$ for any $v \in T_{k+m}$.

Now, since $(T_k, E_k^{\mathcal{I}})$ is connected, there exists $(w(0), w(1), \dots, w(l), w(l+1)) \in (T_k)^{l+2}$ such that $w(0) = u_*$, $w(l+1) = v_*$, $(w(i), w(i+1)) \in E_k^{\mathcal{I}}$ for any $i = 0, 1, \dots, l$. Set $e_i = (w(i), w(i+1))$. Then $(X(e_i), X((e_i)^r), \varphi_{e_i}) \in \mathcal{I}$.

Claim. *There exist $\mathcal{A}_i \subseteq T_m$, $x_i \in K$ and $g_i, h_i \in \mathcal{G}_0$ for $i = 1, 2, \dots, l$ such that*

(i) $\mathcal{A}_i = (h_i)^*(\mathcal{U}_{\mathbf{p}})$ *and* $K(\mathcal{A}_i) \cap X(e_i) \neq \emptyset$,

(ii) $x_i \in K(\mathcal{A}_i) \cap X(e_i)$ *and* $g_i(x_i) = \varphi_{e_i}(x_i)$,

(iii) $\mathcal{A}_{i+1} = (g_i)^*(\mathcal{A}_i)$.

Proof. For $i = 1$, let h_1 be the identity map. Then $\mathcal{A}_1 = \mathcal{U}_{\mathbf{p}}$. Since by (c) $K(\mathcal{A}_1) \cap X(e_1) \neq \emptyset$, we may choose $x_1 \in K(\mathcal{A}_1) \cap X(e_1)$. By (b), there exists $g_1 \in \mathcal{G}_0$ such that $g_1(x_1) = \varphi_{e_1}(x_1)$.

Assume that we have the desired objects for $i \in \{1, \ldots, l-1\}$. Letting $h_{i+1} = g_i \circ h_i \in \mathcal{G}_0$ and $\mathcal{A}_{i+1} = (g_i)^*(\mathcal{A}_i)$, we obtain

$$\mathcal{A}_{i+1} = (g_i)^*(h_i)^*(\mathcal{U}_{\mathbf{p}}) = (h_{i+1})^*(\mathcal{U}_{\mathbf{p}}).$$

Using (c), we see that $K(\mathcal{A}_{i+1}) \cap X(e_{i+1}) \neq \emptyset$. Choose $x_{i+1} \in K(\mathcal{A}_{i+1}) \cap X(e_{i+1})$. By (b), there exists $g_{i+1} \in \mathcal{G}_0$ such that $g_{i+1}(x_{i+1}) = \varphi_{e_{i+1}}(x_{i+1})$.

Thus by induction, the claim has been proven. □

Now, by (c), $X(e_0) \cap K(\mathcal{A}_1) \neq \emptyset$ and $X((e_l)^r) \cap K(\mathcal{A}_l) \neq \emptyset$. This implies

$$f_{w(1)}(K(\mathcal{A}_1)) \cap K_{w(0)} \neq \emptyset \quad \text{and} \quad f_{w(l)}(K(\mathcal{A}_l)) \cap K_{w(l+1)} \neq \emptyset. \tag{4.11}$$

Next, (ii) yields $f_{w(i+1)}(g_i(x_i)) = f_{w(i)}(x_i)$. Since

$$g_i(x_i) \in K((g_i)^*(\mathcal{A}_i)) = K(\mathcal{A}_{i+1}),$$

we have

$$f_{w(i)}(K(\mathcal{A}_i)) \cap f_{w(i+1)}(K(\mathcal{A}_{i+1})) \neq \emptyset \tag{4.12}$$

for $i = 1, \ldots, l$. Since $\mathcal{A}_i = (h_i)^*(\mathcal{U}_{\mathbf{p}}) \subseteq \bigcup_{g \in \mathcal{G}_1} g^*(\psi_n(\mathbf{p}))$, we see that

$$\bigcup_{i=1}^{l} w(i)\mathcal{A}_i \subseteq \bigcup_{u \in \mathbf{p}} H_u.$$

Note that $K(\bigcup_{i=1}^{l} w(i)\mathcal{A}_i) = \bigcup_{i=1}^{l} f_{w(i)}(\mathcal{A}_i)$. By formulas (4.12) and (4.11), we see that $K(\bigcup_{i=1}^{l} w(i)\mathcal{A}_i)$ is connected and intersects with $K_{w(0)}$ and $K_{w(l+1)}$. Therefore, there exists $\mathbf{p}_0 \in \mathcal{C}_m^{(1)}(u_*, v_*, T_k)$ included in $\bigcup_{i=1}^{l} w(i)\mathcal{A}_i \subseteq \bigcup_{u \in \mathbf{p}} H_u$. Consequently, Lemma C.4 shows (4.10). The conductive homogeneity follows from Lemma 2.22 and Theorem 3.33. ■

4.3 Subsystems of (hyper)cubic tiling

In this section, we present three classes of hypercube-based self-similar sets as examples of conductively homogeneous spaces. The first one given in Theorem 4.13 includes generalized Sierpiński carpets studied in the series of papers [1–6] by Barlow and Bass, the Menger curves (also known as the Menger sponge), and the hypercubes $[-1, 1]^L$ for $L \geq 1$. Unlike those examples, however, our examples also contain self-similar sets with fewer, or even no, symmetries of a hypercube. See Section 4.4, where we present explicit examples of self-similar sets belonging to the classes given in this section.

We start with basic notations on the hypercube $[-1, 1]^L$ and its symmetry group.

Definition 4.9. Let $L \in \mathbb{N}$ and let $C_*^L = [-1, 1]^L$. Moreover, let $\mathbb{B}_L$ be the L-dimensional *hyperoctahedral group*, that is,

$$\mathbb{B}_L = \{g \mid g \in O(L),\ g(C_*^L) = C_*^L\},$$

where $O(L)$ is the collection of orthogonal transformations of $\mathbb{R}^L$. For the case $L = 2$, $\mathbb{B}_2$ is often denoted by D_4 in a literature. Define

$$B_{j,i} = \{(x_1, \ldots, x_L) \mid (x_1, \ldots, x_L) \in [-1, 1]^L,\ x_j = i\}$$

for $j = \{1, \ldots, L\}$ and $i \in \{-1, 0, 1\}$. Then the boundary of $[-1, 1]^L$ consists of $\{B_{j,i}\}_{j \in \{1,\ldots,L\}, i \in \{1,-1\}}$. For $s = (s_1, \ldots, s_L) \in \{1, \ldots, N\}^L$, define

$$C_s^{L,N} = \prod_{i=1}^{L} \Big[\frac{2s_i - 2 - N}{N}, \frac{2s_i - N}{N}\Big],$$
$$c_s^{L,N} = \Big(\frac{2s_1 - 1 - N}{N}, \ldots, \frac{2s_L - 1 - N}{N}\Big).$$

If no confusion may occur, we use C_*, C_s and c_s instead of C_*^L, $C_s^{L,N}$ and $c_s^{L,N}$ respectively hereafter.

In the course of this section, we are going to deal with particular elements of $\mathbb{B}_L$.

Definition 4.10. Define $R_j \in \mathbb{B}_L$ as the reflection in the hyperplane $B_{j,0}$ for $j \in \{1, \ldots, L\}$. Furthermore, define $R^i_{j_1,j_2}$ as the reflection in the hyperplane

$$\mathcal{H}^i_{j_1,j_2} = \{(x_1, \ldots, x_L) \mid x_{j_1} = i x_{j_2}\}$$

for $j_1, j_2 \in \{1, \ldots, L\}$ with $j_1 \neq j_2$ and $i \in \{1, -1\}$.

In the next definition, we introduce key notions of this section.

Throughout this section, we fix $L \geq 1$ and $N \geq 2$.

Definition 4.11. (1) A self-similar structure $(K, S, \{f_s\}_{s \in S})$ is called a *subsystem of L-dimensional hypercubic tiling*, or a *subsystem of cubic tiling* for short, if $K \subseteq C_*$, $S \subseteq \{1, \ldots, N\}^L$ and, for any $s \in S$, f_s is a restriction of a similitude from $\mathbb{R}^L$ to itself satisfying $f_s(C_*) = C_s$, i.e., there exists $\Phi_s \in \mathbb{B}_L$ such that

$$f_s(x) = \frac{1}{N}\Phi_s x + c_s$$

for any $x \in \mathbb{R}^L$. A subsystem of cubic tiling $(K, S, \{f_s\}_{s \in S})$ is called non-degenerate if $K \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \ldots, L\}$ and $i \in \{1, -1\}$.

(2) A continuous map $\varphi: C_* \to C_*$ is called an *N-folding map* if and only if, for any $s \in \{1, \ldots, N\}^L$, there exists $A_s \in \mathbb{B}_L$ such that

$$\varphi(x) = NA_s(x - c_s) \tag{4.13}$$

for any $x \in C_s$. If no confusion may occur, we omit N in the expression of an "N-folding" map and say a "folding map" for simplicity.

(3) Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. We use the framework of Section 4.1 to define $(T, \mathcal{A}, \phi)$ with $r = \frac{1}{N}$ and $j_s = 1$ for any $s \in S$. In this case, $T_n = S^n$ for any $n \geq 1$. Define a graph (T_n, E_n^{ℓ}) by

$$E_n^{\ell} = \{(w, v) \mid w, v \in T_n, w \neq v, f_w(C_*) \cap f_v(C_*) = f_w(B_{j,i}) \\ \text{for some } j \in \{1, \ldots, L\} \text{ and } i \in \{1, -1\}\}.$$

The subsystem of cubic tiling $\mathcal{L}$ is said to be *strongly connected* if (T_n, E_n^{ℓ}) is connected for any $n \geq 1$.

(4) Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. $\mathcal{L}$ is called *locally symmetric* if $K_w \cup K_v$ is invariant under the reflection in the hyperplane including $f_w(C_*) \cap f_v(C_*)$ for any $n \geq 1$ and $(w, v) \in E_n^{\ell}$.

Remark. Let $\mathcal{L}$ be a subsystem of cubic tiling which is non-degenerate and locally symmetric. Then $E_n^{\ell} \subseteq E_n^*$ by the following arguments. Assume that $(w, v) \in E_n^{\ell}$. Set

$$\ell_{w,v} = f_w(C_*) \cap f_v(C_*). \tag{4.14}$$

By non-degeneracy, $K_w \cap \ell_{w,v} \neq \emptyset$ and by local symmetry, $K_w \cap \ell_{w,v} = K_v \cap \ell_{w,v} \neq \emptyset$. Hence $(w, v) \in E_n^*$. Note that even if $(w, v) \in T_n$ and $f_w(C_*) \cap f_v(C_*) \neq \emptyset$, it may happen that $K_w \cap K_v = \emptyset$.

Remark. Let $\mathcal{L}$ be a subsystem of cubic tiling which is non-degenerate, locally symmetric, and strongly connected. As in the case of the unit square in Example 2.32, define

$$\mathcal{J}_{\ell} = \Big\{\{w, v\} \;\Big|\; (w, v) \in \bigcup_{n\geq 0} E_n^{\ell}\Big\}. \tag{4.15}$$

For explicit examples in the next section except for the chipped Sierpiński carpet, $\mathcal{J}_{\ell}$ is a covering system and is a good substitute for $\mathcal{J}_*$ in the case $p < \dim_{AR}(K, d)$.

By properties of cubic tiling, it is easy to see that Assumption 2.15 holds. In summary, we have the next proposition. Recall that the edges of T_n are given not by E_n^{ℓ} but by E_n^* as it has always been in the previous sections.

Proposition 4.12. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Then the family $\{K_w\}_{w\in T}$ is a partition of K parametrized by the tree $(T, \mathcal{A}, \phi)$. Let d_* be the restriction of the Euclidean metric on K and let μ be the self-similar measure satisfying $\mu(K_w) = (\#(S))^{-|w|}$ for any $w \in T$. Then Assumption 2.15 is satisfied with $d = d_*$, $r = \frac{1}{N}$, $M_* = 1$, $M_0 = 1$, $N_* = \#(S)$ and $L_* \leq 3^L - 1$. In this case, μ is α_H-Ahlfors regular with respect to d_*, where $\alpha_H = \frac{\log \#(S)}{\log N}$.*

The exponent α_H coincides with the Hausdorff dimension of (K, d_*). Note that $\#(S) \le N^L$. Since $\#(S) = N^L$ implies $K = C_*$, we see that $\alpha_H < L$ unless $K = C_*$.

The following theorems are the main results of this section.

Theorem 4.13. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric, and strongly connected. Moreover, suppose that the following condition* (SDR) *is satisfied:*

(SDR) *For any $j_1, j_2 \in \{1, \dots, L\}$ with $j_1 \ne j_2$, there exists $i \in \{1, -1\}$ such that $R^i_{j_1,j_2} \in \mathcal{G}_{(K,T)}$.*

Then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$.

The name (SDR) represents "symmetric with respect to diagonal reflections" as $R^i_{j_1,j_2}$ is the reflection in the diagonal hyperplane $\mathcal{H}^i_{j_1,j_2}$. For generalized Sierpiński carpets, the Menger curve and the hypercube, it follows that $\mathcal{G}_{(K,T)} = \mathbb{B}_L$ and (SDR) is satisfied. However, $\mathcal{G}_{(K,T)}$ does not necessarily coincide with $\mathbb{B}_L$ to satisfy (SDR). For example, the group generated by $\{R^1_{j_1,j_2} \mid j_1, j_2 \in \{1, \dots, L\}, j_1 \ne j_2\}$ is (isomorphic to) the symmetric group of order L, $\mathcal{S}_L$, which is a proper subgroup of $\mathbb{B}_L$, and if $\mathcal{S}_L \subseteq \mathcal{G}_{(K,T)}$, then (SDR) is satisfied. See Example 4.30.

In the case $L = 2$, the advantage of being planar gives another two classes having conductive homogeneity.

Theorem 4.14. *Let $L = 2$ and let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of 2-dimensional cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric, and strongly connected. Moreover, assume one of the following two conditions* (RS) *or* (NS)*:*

(RS) $\Theta_{\frac{\pi}{2}} \in \mathcal{G}_{(K,T)}$*, where $\Theta_{\frac{\pi}{2}}$ is the rotation by $\frac{\pi}{2}$ around* $(0, 0)$.

(NS) *For each $i, j \in \{1, \dots, N-1\}$, there exist $i_1, j_1 \in \{1, \dots, N\}$ such that*

$$\{(i, j_1), (i+1, j_1), (i_1, j), (i_1, j+1)\} \cap S = \emptyset.$$

Then K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$.

The expressions (RS) and (NS) represent "rotational symmetry" and "no symmetry", respectively.

At a glance at definitions, it may look difficult to verify the conditions like "non-degenerate", "strongly continuous", and "locally symmetric". In the course of the discussion, however, we will show useful criteria concerning only the first iteration $\{f_s(C_*)\}_{s\in S}$ to check those conditions.

Proofs of the above theorems will be given later in this section after necessary preparations. The main idea of the proof is to construct a family of paths required (c) of Theorem 4.8 by using local symmetry and an additional geometric condition (SDR), (RS), or (NS). Such an idea was used in [11] and can be traced back to the "knight move" argument by Barlow–Bass [1]. In those previous works, however, the full $\mathbb{B}_L$-

symmetry of the space was required but we find that weaker (or even no) symmetry is good enough under the presence of local symmetry.

Now we start to study the conditions "non-degenerate", "strong continuous", and "locally symmetric". First, we study the nature of folding maps, which turns out to be closely related to the local symmetry.

Lemma 4.15. *Let $\varphi\colon C_* \to C_*$ be a folding map characterized as (4.13). Then for any $s, t \in \{1, \dots, N\}^L$,*

$$A_s = A_t R_j \quad \text{if } C_s \cap C_t = \frac{1}{N} B_{j,i} + c_s \text{ for some } i \in \{1, -1\}.$$

Proof. Assume that $C_s \cap C_t = \frac{1}{N} B_{j,i} + c_s$. Then $C_s \cap C_t = \frac{1}{N} B_{j,-i} + c_t$ as well and $x - c_t = R_j(x - c_s)$ for any $x \in C_s \cap C_t$. On the other hand, as φ is a folding map, we see that

$$NA_s(x - c_s) = NA_t(x - c_t)$$

for any $x \in C_s \cap C_t$. Hence $A_s(x - c_s) = A_t R_j(x - c_s)$ for any $x \in C_s \cap C_t$. This immediately implies $A_s = A_t R_j$. ∎

Note that $R_{j_1} R_{j_2} = R_{j_2} R_{j_1}$ for any $j_1, j_2 \in \{1, \dots, L\}$. So, by the above lemma, we can determine all the folding maps as follows.

Lemma 4.16. *Fix $s^* = (s_1^*, \dots, s_L^*) \in \{1, \dots, N\}^L$. For $A \in \mathbb{B}_L$, define $\varphi_{s^*,A}\colon C_* \to C_*$ by*

$$\varphi_{s^*,A}(x) = NA \prod_{j=1}^{L} (R_j)^{|s_j^* - s_j|}(x - c^N_{(i,j)})$$

for any $x \in C_{(s_1,\dots,s_L)}$. Then $\varphi_{s_0,A}$ is a folding map. Moreover, $\{\varphi_{s^*,A} \mid A \in \mathbb{B}_L\}$ is the totality of folding maps for any $s^* \in \{1, \dots, N\}^L$.

Examples of folding maps in the case of $L = 2$ are given in Figure 4.1. In each example, $s^* = (1, 1)$ and $A = I$. The element of $\mathbb{B}_2$ in each square indicates the corresponding $A(R_1)^{|s_1 - s_1^*|}(R_2)^{|s_2 - s_2^*|}$.

Notation. Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Set

$$K^{(m)} = \bigcup_{w \in T_m} f_w(C_*).$$

Due to the next lemma, one can easily determine the non-degeneracy of K by examining $K^{(1)}$.

Lemma 4.17. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Then $\mathcal{L}$ is non-degenerate if and only if $K^{(1)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$.*

I	R_1	I
R_2	$-I$	R_2
I	R_1	I

$N = 3$

R_2	$-I$	R_2	$-I$
I	R_1	I	R_1
R_2	$-I$	R_2	$-I$
I	R_1	I	R_1

$N = 4$

Figure 4.1. Folding maps.

Proof. Since $K \subseteq K^{(1)}$, the "only if" part is obvious. Assume that $K^{(1)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. We are going to show that $K^{(k)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$, $i \in \{1, -1\}$, and $k \in \{1, \dots, n\}$ by induction on n. Assume that the claim holds for n. Let $w \in T_n$ satisfying $f_w(C_*) \cap B_{j,i} \neq \emptyset$. Since

$$(f_w)^{-1}(f_w(C_*) \cap B_{j,i}) = B_{j_1,i_1}$$

for some $j_1 \in \{1, \dots, L\}$ and $i_1 \in \{1, -1\}$, there exists $s \in T_1$ such that

$$f_s(C_*) \cap (f_w)^{-1}(f_w(C_*) \cap B_{j,i}) \neq \emptyset.$$

This implies that $f_{ws}(C_*) \cap B_{j,i} \neq \emptyset$. Thus we have shown the desired statement for $n + 1$. Now by induction,

$$K^{(k)} \cap B_{j,i} \neq \emptyset$$

for any $j \in \{1, \dots, L\}$, $i \in \{1, -1\}$. Since $K^{(n)}$ is monotonically decreasing and $K = \bigcap_{n \geq 1} K^{(n)}$, it follows that $K \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. ■

The locally symmetric property can also be determined by the first step of the iteration as follows.

Lemma 4.18. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Then $\mathcal{L}$ is locally symmetric if and only if $K_s \cup K_t$ is invariant under the reflection in $\ell_{s,t}$ for any $(s, t) \in E_1^{\ell}$.*

Proof. The "only if" part is obvious. We show the following statement by induction on $n \geq 1$.

For any $k \in \{1, \dots, n\}$ and $(w, v) \in E_k^{\ell}$, $K_w \cup K_v$ is invariant under the reflection in $\ell_{w,v}$.

The case $n = 1$ is exactly the assumption of the lemma. Suppose that the statement holds for n. Let $(w, v) \in E_{n+1}^{\ell}$. In the case $\pi^n(w) = \pi^n(v)$, let $s = \pi^n(w)$. Then $w = sw'$ and $v = sv'$ for some $w', v' \in T_n$. Since $f_w(C_*) = f_s(f_{w'}(C_*))$ and $f_v(C_*) =$

$f_s(f_{v'}(C_*))$, we see $\ell_{w',v'} \in E_n^\ell$. By induction hypothesis, $K_{w'} \cap K_{v'}$ is invariant under the reflection in $\ell_{w',v'}$. Applying f_s, we see that $K_w \cup K_v$ is invariant under the reflection in $\ell_{w,v}$. In the case $\pi^n(w) \neq \pi^n(v)$, let $s = \pi^n(w)$ and let $t = \pi^n(v)$. Since $\ell_{w,v} \subseteq \ell_{s,t} = f_s(B_{j,i})$ for some $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$, we obtain $(s,t) \in E_1^\ell$. So, $K_s \cup K_t$ is invariant under the reflection in $\ell_{s,t}$. Denoting this reflection by R, we see that R coincides with the reflection in $\ell_{w,v}$. Since $R(f_w(C_*)) = f_v(C_*)$, it follows that $R(K_w) = R(K_s \cap f_w(C_*)) = K_t \cap f_v(C_*) = K_v$. So we have verified the statement for $n + 1$. Thus by induction, we have the desired result. ■

Next, we consider the strong connectedness.

Lemma 4.19. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a locally symmetric subsystem of cubic tiling. If $\mathcal{L}$ is non-degenerate and (T_1, E_1^ℓ) is connected, then $\mathcal{L}$ is strongly connected.*

Proof. By the non-degeneracy, we see that $K^{(n)} \cap B_{j,i} \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$.

We are going to show that (T_k, E_k^ℓ) is connected for any $k \in \{1, \dots, n\}$ by induction on $n \geq 1$. Assume that $w, v \in T_{n+1}$. If $\pi^n(w) = \pi^n(v)$, then there exist $w', v' \in T_n$ such that $w = sw'$ and $v = sv'$, where $s = \pi^n(w)$. Since w' and v' are connected by an E_n^ℓ-path, w and v are connected by an E_{n+1}^ℓ-path. In the case $\pi^n(w) \neq \pi^n(v)$, let $s = \pi^n(w)$ and let $t = \pi^n(v)$. Then $w = sw'$ and $v = tv'$ for some $w', v' \in T_n$. Since (T_1, E_1^ℓ) is connected, there exists an E_1^ℓ-path $(s(0), \dots, s(m))$ such that $s(0) = s$, $s(m) = t$ and $(s(i), s(i+1)) \in E_1^\ell$ for any $i = 0, \dots, m-1$. For each $i = 0, \dots, m-1$, since $\bigcup_{w' \in T_n} f_{w'}(C_*) \cap B_{j,i} \neq \emptyset$ for any $j = \{1, \dots, L\}$ and $i \in \{1, -1\}$, there exists $u(i) \in T_n$ such that $f_{s(i)u(i)}(C_*) \cap \ell_{s(i),s(i+1)} \neq \emptyset$. Since $\mathcal{L}$ is locally symmetric, there exists $v(i) \in T_n$ such that $f_{s(i+1)v(i)}(C_*)$ is the image of $f_{s(i)u(i)}(C_*)$ by the reflection in $\ell_{s(i),s(i+1)}$. Define $v(-1) = w'$ and $u(m) = v'$. Then $w = s(0)v(-1)$ and $v = s(m)u(m)$. Since (T_n, E_n^ℓ) is connected, $v(i-1)$ and $u(i)$ are connected by an E_n^ℓ-path for any $i = 0, \dots, m-1$. Adding $s(i)$ at the top, we obtain an E_{n+1}^ℓ-path between $s(i)v(i-1)$ and $s(i)u(i)$. Combining all these E_{n+1}^ℓ-paths, we obtain an E_{n+1}^ℓ-path between w and v. Thus (T_{n+1}, E_{n+1}^ℓ) is connected. By induction, we see that $\mathcal{L}$ is strongly connected. ■

Lemma 4.20. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Assume that $K \cap \mathrm{int}(C_*) \neq \emptyset$. For any $s \in \{1, \dots, N^m\}^L$, if $K \cap \mathrm{int}(C_s^{L,N^m}) \neq \emptyset$, then there exists $w \in T_m$ such that $f_w(C_*) = C_s^{L,N^m}$.*

Proof. Suppose that $f_w(C_*) \neq C_s^{L,N^m}$ for all $w \in T_m$. Then $f_w(C_*) \cap C_s^{L,N^m}$ is included in the boundary of C_s^{L,N^m} and hence $f_w(C_*) \cap \mathrm{int}(C_s^{L,N^m}) = \emptyset$. So,

$$K^{(m)} \cap \mathrm{int}(C_s^{L,N^m}) = \bigcup_{w \in T_m} \left(f_w(C_*) \cap \mathrm{int}(C_s^{L,N^m}) \right) = \emptyset.$$

Since $K \subseteq K^{(m)}$, it follows that $K \cap \mathrm{int}(C_s^{L,N^m}) = \emptyset$. ■

The following relation between a folding map and a subsystem of cubic tiling will be used to characterize local symmetry.

Lemma 4.21. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling. Assume that $K \cap \mathrm{int}(C_*) \neq \emptyset$. Let φ be a folding map. Then the following four statements are equivalent:*

(a) $\varphi(K) = K$.

(b) $\varphi\circ f_s(K^{(m)}) = K^{(m)}$ *for any $s \in S$ and $m \geq 0$.*

(c) $\varphi\circ f_s(K) = K$ *for any $s \in S$.*

(d) $\varphi(K^{(m+1)}) = K^{(m)}$ *for any $m \geq 0$.*

Proof. (a) $\Rightarrow$ (b): Let $s \in S$. Then $\varphi\circ f_s(K) \subseteq K$. For any $w \in T_m$, there exists $\tau = (\tau_1, \dots, \tau_L) \in \{1, \dots, N^m\}^L$ such that $\varphi\circ f_s(f_w(C_*)) = C_\tau^{L,N^m}$. Now

$$K \supseteq \varphi\circ f_s(f_w(K \cap \mathrm{int}(C_*))) = \varphi\circ f_s\circ f_w(K) \cap \mathrm{int}(C_\tau^{L,N^m}).$$

Since $K \cap \mathrm{int}(C_*) \neq \emptyset$, this implies $K \cap \mathrm{int}(C_\tau^{L,N^m}) \neq \emptyset$. Lemma 4.20 shows that $\varphi\circ f_s(f_w(C_*)) = C_\tau^{L,N^m} \subseteq K^{(m)}$, so that

$$\varphi\circ f_s(K^{(m)}) = \bigcup_{w\in T_m} \varphi\circ f_s(f_w(C_*)) \subseteq K^{(m)}.$$

As $\varphi\circ f_s \in \mathbb{B}_L$ preserves the Lebesgue measure of a set, we see $\varphi\circ f_s(K^{(m)}) = K^{(m)}$.

(b) $\Rightarrow$ (c): Since $\bigcap_{m\geq 0} K^{(m)} = K$,

$$\varphi\circ f_s(K) = \varphi\circ f_s\Big(\bigcap_{m\geq 0} K^{(m)}\Big) = \bigcap_{m\geq 0} K^{(m)} = K.$$

(c) $\Rightarrow$ (a): Since $K = \bigcup_{s\in S} f_s(K)$,

$$\varphi(K) = \varphi\Big(\bigcup_{s\in S} f_s(K)\Big) = K.$$

(b) $\Rightarrow$ (d): Since $\bigcup_{s\in S} f_s(K^{(m)}) = K^{(m+1)}$,

$$\varphi(K^{(m+1)}) = \varphi\Big(\bigcup_{s\in S} f_s(K^{(m)})\Big) = K^{(m)}.$$

(d) $\Rightarrow$ (a): Since $\bigcap_{m\geq 0} K^{(m)} = K$,

$$\varphi(K) = \varphi\Big(\bigcap_{m\geq 0} K^{(m+1)}\Big) = \bigcap_{m\geq 0} K^{(m)} = K. \qquad \blacksquare$$

The next theorem tells that a locally symmetric subsystem of cubic tiling is almost an inverse of a folding map.

Theorem 4.22. *Let $\mathcal{L} = (K, S, \{f_s\}_{s\in S})$ be a subsystem of cubic tiling.*

(1) *If $\mathcal{L}$ is strongly connected and locally symmetric, then there exists a folding map satisfying*

$$\varphi^n \circ f_w(K^{(m)}) = K^{(m)}$$

for any $n \geq 1$, $m \geq 0$ and $w \in T_n$. In particular,

$$\varphi^n(K^{(n+m)}) = K^{(m)}$$

for any $n \geq 1$, $m \geq 0$ and

$$\varphi^n(K) = K$$

for any $n \geq 1$. Furthermore, define $F_s\colon C_* \to C_s$ by $F_s = (\varphi|_{C_s})^{-1}$ for each $s \in S$. Then

$$K = \bigcup_{s\in S} F_s(K)$$

and $(K, S, \{F_s\}_{s\in S})$ is a self-similar structure.

(2) *Suppose that $K \cap \mathrm{int}(C_*) \neq \emptyset$. If there exists a folding map φ such that $\varphi(K) = K$, then $\mathcal{L}$ is locally symmetric.*

Proof. (1) Fix $s \in S$. Recall that there exists $\Phi_s \in \mathbb{B}_L$ such that

$$f_s(x) = \frac{1}{N}\Phi_s x + c_s$$

for any $x \in C_*$. Set $A_s = (\Phi_s)^{-1}$ and define $\varphi = \varphi_{s_0,A_s}$. Since $\varphi \circ f_s = I$, it follows that $\varphi^n \circ (f_s)^n = I$ for any $n \geq 1$. Thus letting

$$s_n = \underbrace{ss\cdots s}_{n\text{-times}},$$

we see that $\varphi^n \circ f_{s_n}(K) = K$. Choose $\tau = (\tau_1, \ldots, \tau_L) \in \{1, \ldots, N^n\}^L$ such that $C_\tau^{L,N^n} = f_{s_n}(C_*)$. Let $w \in T_n$. Choose $\xi = (\xi_1, \ldots, \xi_L) \in \{1, \ldots, N^n\}^L$ such that $C_\xi^{L,N^n} = f_w(C_*)$. Since $\mathcal{L}$ is strongly connected, there exists an E_n^ℓ-path $(w(0), \ldots, w(m))$ between s_n and w. Following this path and applying the reflections in $\ell_{w(i),w(i+1)}$, we see that

$$K_w - c_\xi^{L,N^n} = R(K_{s_n} - c_\tau^{L,N^n}),$$

where $R = \prod_{j=1}^{L}(R_j)^{|\tau_j - \xi_j|}$. Note that φ^n is an N^n-folding map. Hence, for any $\gamma \in \{1, \ldots, N^n\}^L$, there exists $A_\gamma \in \mathbb{B}_L$ such that

$$\varphi^n(x) = N^n A_\gamma(x - c_\gamma^{L,N^n})$$

for any $x \in C_\gamma^{L,N^n}$. Applying Lemma 4.16 to φ^n, we see that

$$\begin{aligned}\varphi^n \circ f_w(K) &= \varphi^n(K_w) = N^n A_\xi(K_w - c_\xi^{L,N^n}) \\ &= N^n A_\tau RR(K_{s_n} - c_\tau^{L,N^n}) = \varphi^n(K_{s_n}) = K.\end{aligned}$$

Hence

$$\varphi^n \circ f_w(K) = K$$

for any $n \geq 1$ and $w \in T_n$. Since $K \subseteq K^{(m)}$, it follows that $\varphi^n \circ f_w(K^{(m)}) \supseteq K$. Note that $\varphi^n \circ f_w(K^{(m)}) = \bigcup_{\gamma \in B} C_\gamma^{L,N^n}$ for some subset $B \subseteq \{1, \dots, N^n\}^L$ and $K^{(m)}$ is the minimal of such unions containing K. This shows $\varphi^n \circ f_w(K^{(m)}) \supseteq K^{(m)}$. Since $\varphi^n \circ f_w$ preserves the Lebesgue measure of a set, we conclude that

$$\varphi^n \circ f_w(K^{(m)}) = K^{(m)}.$$

Since $K^{(m+n)} = \bigcup_{w \in T_n} f_w(K^{(m)})$, we obtain $\varphi^n(K^{(n+m)}) = K^{(m)}$. Note that $K = \bigcup_{w \in T_n} f_w(K)$. Hence $\varphi^n(K) = K$. Moreover, if $\varphi(x) = NA_s(x - c_s)$ for $x \in C_s$, then by Lemma 4.21 (c), we have $K = NA_s(K_s - c_s)$. This implies

$$K_s = \frac{1}{N}(A_s)^{-1}K + c_s.$$

Hence letting $F_s(x) = \frac{1}{N}(A_s)^{-1}x + c_s$, we see $K = \bigcup_{s \in S} F_s(K)$.

(2) Suppose that $(s, t) \in E_1^\ell$. Then by Lemma 4.16, there exist $A_s \in \mathbb{B}_L$ and $j \in \{1, \dots, L\}$ such that

$$\varphi(x) = NA_s(x - c_s)$$

for any $x \in C_s$ and

$$\varphi(x) = NA_s R_j(x - c_t)$$

for any $x \in C_t$. Since $\varphi \circ f_s(K) = K$ and $\varphi \circ f_t(K) = K$ by Lemma 4.21, it follows that

$$K_s - c_s = \frac{1}{N}(A_s)^{-1}K \quad \text{and} \quad K_t - c_t = \frac{1}{N}R_j(A_s)^{-1}K.$$

Therefore,

$$R(K_s - c_s) = R\frac{1}{N}(A_s)^{-1}K = K_t - c_t,$$

so that $K_t \cup K_s$ is invariant under the reflection in $\ell_{s,t}$. Thus Lemma 4.18 shows that $\mathcal{L}$ is locally symmetric. ■

By (2) of the above theorem, we immediately have the following sufficient condition for the local symmetry.

Corollary 4.23. *Let $S \subseteq \{1, \dots, N\}^L$. Assume that $B_{j,i} \cap (\bigcup_{s \in S} C_s) \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. Let φ be an N-folding map. Define*

$$f_s = (\varphi|_{C_s})^{-1}$$

for any $s \in S$. Let K be the unique non-empty compact set satisfying

$$K = \bigcup_{s \in S} f_s(K).$$

Then, $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ is non-degenerate and locally symmetric.

Proof. Since $B_{i,i} \cap (\bigcup_{s \in S} C_s) \neq \emptyset$ for any $j \in \{1, \dots, L\}$ and $i \in \{1, -1\}$, Lemma 4.17 shows that $\mathcal{L}$ is non-degenerate and hence $K \cap \mathrm{int}(C_*) \neq \emptyset$. Moreover, it is immediate to see that $\varphi(K) = K$. Now Theorem 4.22 (2) suffices. ∎

Note that by Theorem 4.22 (1), if a subsystem of cubic tiling is locally symmetric and strongly connected, then it is given by a inverse of a folding map described in Corollary 4.23.

Now we are ready to give a proof of Theorem 4.13.

Proof of Theorem 4.13. By Theorem 4.22, we may assume that $\mathcal{L}$ is given by an inverse of a folding map described in Corollary 4.23 without loss of generality. Note that

$$(\varphi^m|_{f_w(C_*)})^{-1} = f_w \tag{4.16}$$

for any $m \geq 1$ and $w \in T_m$. For any $m \geq 1$ and $e = (w, v) \in E_m^\ell$, by (4.16),

$$\varphi^m|_{f_w(C_*) \cap f_v(C_*)} = (f_w)^{-1}|_{f_w(C_*) \cap f_v(C_*)} = (f_v)^{-1}|_{f_w(C_*) \cap f_v(C_*)}.$$

Hence $X(e) = X(e^r)$ and $\varphi_e = I$, where I is the identity map. Now let

$$\mathcal{I} = \left\{(X(e), X(e^r), \varphi_e) \mid e \in \bigcup_{m \geq 1} E_m^\ell\right\},$$

and set $\mathcal{G}_0 = \{I\}$ and $\mathcal{G}_1 = \mathcal{G}_{(K,T)} \cap \mathbb{B}_L$. We are going to make use of Theorem 4.8. By the fact that $\mathcal{L}$ is strongly connected, we have (a) of Theorem 4.8. Since $\varphi_e = I$ for any $e \in \bigcup_{m \geq 1} E_m^\ell$, (b) of Theorem 4.8 is obvious.

Now it only remains to show (c) of Theorem 4.8. Let $w \in T_n$. Suppose that $f_w(C_*) = \prod_{i=1}^L [\alpha_i, \alpha_i + \frac{2}{N^n}]$. Then every path $\mathbf{p} \in \mathcal{C}_{1,m}^{(1)}(w)$ contains a path between hyperplanes

$$\{(x_1, \dots, x_L) \mid x_j = \alpha_j\} \quad \text{and} \quad \left\{(x_1, \dots, x_L) \mid x_j = \alpha_j - \frac{2}{N^n}\right\}$$

or

$$\left\{(x_1, \dots, x_L) \mid x_j = \alpha_j + \frac{2}{N^n}\right\} \quad \text{and} \quad \left\{(x_1, \dots, x_L) \mid x_j = \alpha_j + \frac{4}{N^n}\right\}$$

for some $j \in \{1, \dots, L\}$. This implies that there exists $j_* \in \{1, \dots, L\}$ such that $\varphi^n(K(\mathbf{p})) \cap B_{j_*,i} \neq \emptyset$ for any $i \in \{1, -1\}$. Note that $\varphi^m(K(\mathbf{p})) = K(\psi_n(\mathbf{p}))$. Hence there exists a path $\mathbf{p}_{j_*} \subseteq \psi_n(\mathbf{p})$ between $B_{j_*,-1}$ and $B_{j_*,1}$. By (SDR), for any $j_1 \neq j_*$, there exists $i_* \in \{1, -1\}$ such that $R^{i_*}_{j_*,j_1} \in \mathcal{G}_{(K,T)}$. Set $\mathbf{p}_{j_1} = (R^{i_*}_{j_*,j_1})^*(\mathbf{p}_{j_*})$. Then $K(\mathbf{p}_{j_1}) \cap B_{j_1,i} \neq \emptyset$ for any $i \in \{1, -1\}$. Moreover, $K(\mathbf{p}_{j_*})$ and $K(\mathbf{p}_{j_1})$ intersects at $H^{i_*}_{j_*,j_1}$. Thus set $\mathbf{p}_* = \bigcup_{k=1}^{L} \mathbf{p}_k$. Then $\mathbf{p}_*$ is connected and $K(\mathbf{p}_*) \cap B_{k,i} \cap K \neq \emptyset$ for any $k \in \{1, \dots, L\}$ and $i \in \{1, -1\}$. Moreover, $\mathbf{p}_* \subseteq \bigcup_{g \in \mathcal{G}_{(K,T)} \cap \mathbb{B}_L} g^*(\psi_n(\mathbf{p}))$. Thus we have verified (c) of Theorem 4.8. ∎

Proof of Theorem 4.14. The arguments are the same as in the proof of Theorem 4.13 except the deduction of (c) of Theorem 4.8.

In the case of (RS), to construct $\mathbf{p}_{j_1}$ from $\mathbf{p}_{j_*}$, we use $\Theta_{\frac{\pi}{2}}$ in place of $R^{i_*}_{j_*,j_1}$. Then the advantage of being planar yields $K(\mathbf{p}_{j_*}) \cap K(\mathbf{p}_j) \neq \emptyset$. The rest is the same as in the proof of Theorem 4.8.

Next, assume (NS). Let $w \in T_n$ and let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{C}^{(1)}_{M,m}(w)$ with $M = 4N - 3$. Note that

$$\#\big(\{\pi^m(w(1)), \dots, \pi^m(w(k))\}\big) \geq M.$$

We are going to show that

$$K(\psi_n(\mathbf{p})) \cap B_{j,i} \neq \emptyset \tag{4.17}$$

for any $j \in \{1, 2\}$ and $i \in \{1, -1\}$. Suppose $K(\psi_n(\mathbf{p})) \cap B_{1,1} = \emptyset$. As $\varphi^{-n}(B_{1,1})$ forms vertical lines at intervals of $\frac{2}{N^n}$, we see that $K(\mathbf{p})$ is contained in the interior of a vertical strip $\bigcup_{j=1,\dots,N^n} C^{2,N^n}_{(i_*,j)} \cup C^{2,N^n}_{(i_*+1,j)}$, which is denoted by Z_{i_*}, for some i_*. Let $C_1, \dots, C_l$ be the collection of connected components of

$$\Big(\bigcup_{w \in T_n} f_w(Q)\Big) \cap Z_{i_*}$$

and set

$$D_i = \{v \mid v \in T_n, f_v(C_*) \subseteq C_i\}$$

for $i = 1, \dots, l$. Then by (NS), we see that

$$\#(D_i) \leq 2(2N - 2).$$

Note that $\bigcup_{i=1}^{k} f_{\pi^m(w(i))}(C_*) \subset C_{i_*}$ for some i_*. Hence

$$4N - 4 \geq \#(D_{i_*}) \geq \#(\{\pi^m(w(i)) \mid i = 1, \dots, k\}) \geq M = 4N - 3.$$

This contradiction shows (4.17). Thus setting $\mathcal{U}_p = \psi_n(\mathbf{p})$, we have (c) of Theorem 4.8. ∎

To conclude this section, we present a useful criterion to determine if $g \in \mathbb{B}_L$ is a symmetry of (K, T) or not.

Lemma 4.24. *Let $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ be a subsystem of cubic tiling. Assume that $\mathcal{L}$ is non-degenerate, locally symmetric and strongly connected. Let φ be the folding map satisfying Theorem 4.22 (1). Then for $g \in \mathbb{B}_L$, if there exists a map $g_*: S \to S$ such that, for any $s \in S$, $g(C_s) = C_{g_*(s)}$ and $A_{g_*(s)} g(A_s)^{-1} = g^k$ for some $k \geq 0$, then $g \in \mathcal{G}_{(K,T)}$.*

Recall that $A_s \in \mathbb{B}_2$ is given in Definition 4.11 (2).

Proof. We are going to show that $g(K^{(n)}) = K^{(n)}$ for any $n \geq 1$ by induction. For $n = 1$, since $g(C_s) = C_{g_*(s)}$, it follows $g(K^{(1)}) = K^{(1)}$. Next assume that

$$g(K^{(n)}) = K^{(n)}.$$

Then by Theorem 4.22, $\varphi \circ f_s(K^{(n)}) = K^{(n)}$, so that $A_s \Phi_s(K^{(n)}) = K^{(n)}$. Hence

$$f_s(K^{(n)}) = \frac{1}{N}(A_s)^{-1}(K^{(n)}) + c_s.$$

Set $t = g_*(s)$. Then

$$\begin{aligned} g(f_s(K^{(n)})) &= \frac{1}{N} g(A_s)^{-1}(K^{(n)}) + c_t = \frac{1}{N}(A_t)^{-1} A_t g(A_s)^{-1}(K^{(n)}) + c_t \\ &= \frac{1}{N}(A_t)^{-1} g^k(K^{(n)}) + c_t = f_t(K^{(n)}). \end{aligned}$$

Since $K^{(n+1)} = \bigcup_{s \in S} f_s(K^{(n)})$, this yields $g(K^{(n+1)}) = K^{(n+1)}$. Thus using induction, we see that $g(K^{(n)}) = K^{(n)}$ for any $n \geq 1$. Since $\bigcap_{n \geq 1} K^{(n)} = K$, we obtain $g(K) = K$. Now, since $g(K^{(n)}) = K^{(n)}$, it follows that, for any $w \in T_n$, there exists $v \in T_n$ such that $g(f_w(C_*)) = f_v(C_*)$. Set $v = g_*(w)$. Then $g_*: T_n \to T_n$. Since $g(f_w(C_*)) = f_{g_*(v)}(C_*)$ and $g(K_w) \subseteq K$, we see that

$$g(K_w) \subseteq g(f_w(C_*)) \cap K = f_{g_*(w)}(C_*) \cap K = K_{g_*(w)}.$$

Using g^{-1} in place of g in the arguments above, we obtain $g^{-1}(K_{g_*(w)}) \subseteq K_w$ as well. Thus we have shown $g(K_w) = K_{g_*(w)}$, so that $g \in \mathcal{G}_{(K,T)}$. ∎

4.4 Examples: subsystems of (hyper)cubic tiling

In this section, we present examples of subsystems of cubic tiling having conductive homogeneity.

We begin with planar examples where $\dim_{AR}(K, d_*) \leq \dim_H(K, d_*) < 2$, so that they are 2-conductively homogeneous and have self-similar local regular Dirichlet forms constructed in Theorem 4.6.

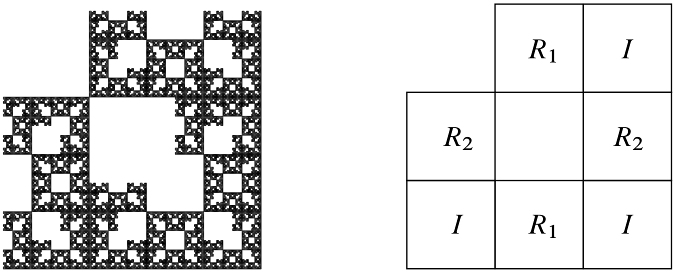


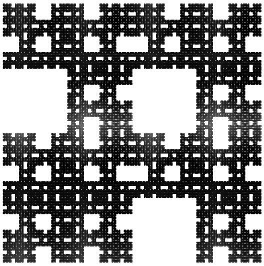
Figure 4.2. Chipped Sierpiński carpet.

Example 4.25 (Chipped Sierpiński carpet). Let $L = 2$ and let $N = 3$. Let S be the set of squares in the right figure of Figure 4.2 where one of R_1, R_2 or I is written. The corresponding f_s is given by

$$f_s(x) = \frac{1}{N}\Phi_s x + c_s^3,$$

where $\Phi_s \in \mathbb{B}_2$ is indicated in Figure 4.2. Note that if the upper-left square belonged to S as well, then K would be the Sierpiński carpet. Lemma 4.17 and Corollary 4.23 show that $\mathcal{L}$ is non-degenerate and locally symmetric, respectively. Then using Lemma 4.19, we see that $\mathcal{L}$ is strongly connected. Finally, Lemma 4.24 shows that $R_{1,2}^{-1} \in \mathcal{G}_{(K,T)}$, so that (SDR) is satisfied. Thus we have confirmed all the assumptions in Theorem 4.13. Note that $K \cap \partial C_*$ has two different ingredients, the line segment, and the Cantor set. The lack of rotational symmetry enables such a phenomenon. Another unique feature is the “countably ramified” property, that is, after removing a certain countable set, every remaining point becomes a connected component. Because of this property, $\mathcal{J}_\ell$ introduced in (4.15) is not a covering system. Furthermore, no matter how we choose a covering system $\mathcal{J} \subseteq \mathcal{J}_*$, we cannot avoid a pair $\{w, v\} \in \mathcal{J}$ where $K_w \cap K_v$ consists of a single point. It is our conjecture that $\dim_{AR}(K, d) = 1$ for the chipped Sierpiński gasket. In this example, since there are enough number of straight lines inside K, (K, d_*) has the chain condition and hence the heat kernel associated with $(\mathcal{E}, \mathcal{W}^2)$ satisfies (3.26) and (3.28).

Example 4.26. Let $L = 2$ and let $N = 4$. As in Example 4.25, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.3. It is easy to see that the corresponding self-similar structure is non-degenerate, locally symmetric, and strongly connected in the same way as Example 4.25. Moreover, Lemma 4.24 shows that $R_{1,2}^{1} \in \mathcal{G}_{(K,T)}$, so that (SDR) is satisfied. Thus we have confirmed all the assumptions of Theorem 4.13. Unlike the chipped Sierpiński carpet, this example is not “countably ramified”. In this example, like the chipped Sierpiński carpet, K contains enough straight lines. This implies that (K, d_*) has the chain condition, so that the heat kernel associated with

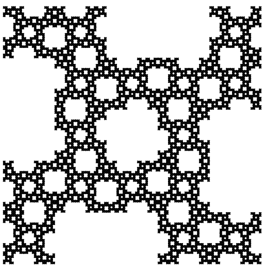


R_2	$-I$	R_2	$-I$
	R_1		R_1
R_2	$-I$	R_2	$-I$
I	R_1		R_1

Figure 4.3. Non-countably ramified example.

$(\mathcal{E}, \mathcal{W}^2)$ satisfies (3.26) and (3.28). In this example, $\mathscr{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.

Example 4.27 (Moulin/Pinwheel). Let $L = 2$ and let $N = 5$. As in the above examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.4. The assumptions of Theorem 4.14 are verified in exactly the same way as before including (RS), i.e., $\Theta_{\frac{\pi}{2}} \in \mathscr{G}_{(K,T)}$. In this example, unlike previous ones, (K, d_*) does not have the chain condition and hence we have (3.26) and (3.27). In this example, $\mathscr{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.



I	R_1			I
	$-I$	R_2	$-I$	R_2
	R_1		R_1	
R_2	$-I$	R_2	$-I$	
I			R_1	I

Figure 4.4. Moulin/Pinwheel.

The next two examples satisfy (NS) and have no $\mathbb{B}_2$-symmetry. Furthermore, $\mathscr{J}_\ell$ given by (4.15) is a covering system with covering numbers $(4, 2)$.

Example 4.28. Let $L = 2$ and let $N = 6$. As in the previous examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.5. In the same manner as before, we verify local symmetry, non-degeneracy and strongly connectedness. By the right figure of Figure 4.5, we verify (NS). We have $\#(S) = 23$, so that $\dim_H(K, d_*) = \frac{\log 23}{\log 6}$.

Example 4.29. Let $L = 2$ and let $N = 7$. As in the previous examples, S and $\{\Phi_s\}_{s \in S}$ are indicated in the right figure of Figure 4.6. In the same manner as before, we verify

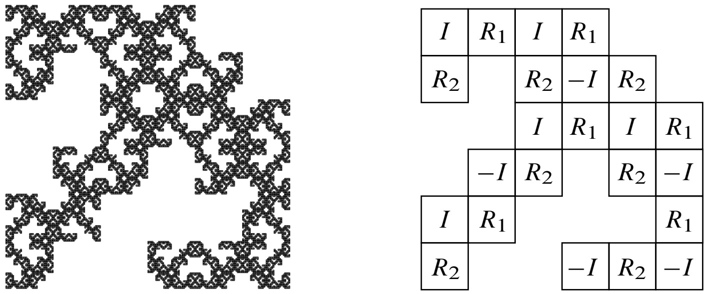


Figure 4.5. Non-symmetric example 1.

local symmetry, non-degeneracy and strongly connectedness. By the right figure of Figure 4.6, we verity (NS). In this example $\#(S) = 30$, so that $\dim_H(K, d_*) = \frac{\log 30}{\log 7}$. Note that

$$\dim_H(K \cap R_{2,1}) = \frac{\log 5}{\log 7} \quad \text{while } \dim_H(K \cap R_{2,-1}) = \frac{\log 4}{\log 7}.$$

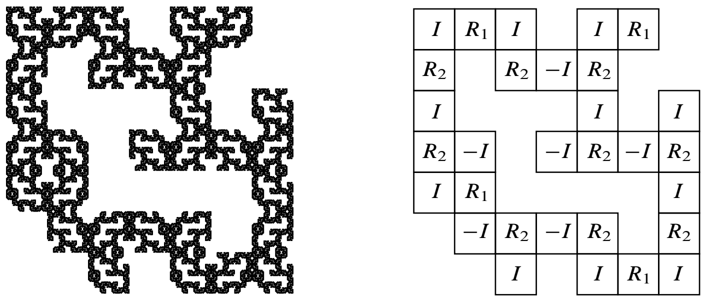


Figure 4.6. Non-symmetric example 2.

In the following examples, we may choose an arbitrary $L \geq 2$.

Example 4.30. Let $S = \{1, \ldots, N\}^L \backslash \{s_*\}$, where $s_* = (1, \ldots, 1)$. Also let $\varphi = \varphi_{s_*, I}$, i.e., φ is a folding map given by

$$\varphi(x) = NA_s(x - c_s)$$

for any $s = (s_1, \ldots, s_L) \in \{1, \ldots, N\}^L$ and $x \in Q_s$, where $A_s = \prod_{j=1}^{L} (R_j)^{|s_i - 1|}$. Note that $(A_s)^{-1} = A_s$. Define

$$f_s(x) = \frac{1}{N} A_s x + c_s$$

and let K be the unique non-empty compact set satisfying

$$K = \bigcup_{s \in S} f_s(K).$$

Then $\mathcal{L} = (K, S, \{f_s\}_{s \in S})$ is a self-similar structure. According to Corollary 4.23, $\mathcal{L}$ is non-degenerate and locally symmetric. Moreover, Lemma 4.19 shows that $\mathcal{L}$ is strongly connected. Additionally, using Lemma 4.24, we see that $\mathcal{G}_{(K,T)}$ is generated by $\{R^1_{j_1,j_2} \mid j_1, j_2 \in \{1, \dots, L\}, j_1 \neq j_2\}$ and it is isomorphic to the symmetric group of order L. Hence by Theorem 4.13, K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Note that $\mathcal{G}_{(K,T)}$ is a proper subgroup of $\mathbb{B}_L$ in this case. In this example, $\mathcal{J}_\ell$ given by (4.15) is a covering system with covering numbers $(2L, L)$.

Example 4.31 (Hypercube). Let $S = \{1, \dots, N\}^L$ and let $f_s(x) = \frac{1}{N}x + c_s$ for any $s \in S$ and $x \in [-1, 1]^L$. Set $K = [-1, 1]^L$. Then $(K, S, \{f_s\}_{s \in S})$ is a self-similar structure. Obviously, $\mathcal{L}$ is non-degenerate, strongly connected and locally symmetric. Moreover, $\mathcal{G}_{(K,T)} = \mathbb{B}_L$. In this case, $\mathcal{J}_\ell$ is a covering system with covering numbers $(2L, L)$. By Theorem 4.13, K is p-conductively homogeneous for any $p > L$. In fact, for any $p > L$, we see that $W^{1,p}(K) = \mathcal{W}^p$ and there exist $c > 0$ such that

$$c\mathcal{E}_p(f) \leq \int_K |\nabla f|^p dx \leq c^{-1}\mathcal{E}_p(f) \tag{4.18}$$

for any $f \in W^{1,p}(K)$, where $\mathcal{E}_p$ is the self-similar p-energy constructed in Section 4.1. The rest of this example is devoted to showing these facts. Let

$$A = \{w(1), w(2), w(3)\} \subseteq T_n.$$

Then $K_{w(1)}$, $K_{w(2)}$ and $K_{w(3)}$ are three consecutive cubes in x_1-direction, i.e.,

$$K_{w(1)} \cap K_{w(2)} = f_{w(1)}(B_{1,1}) = f_{w(2)}(B_{1,-1}),$$
$$K_{w(2)} \cap K_{w(3)} = f_{w(2)}(B_{1,1}) = f_{w(3)}(B_{1,-1}).$$

Let $A_1 = \{w(1)\}$ and let $A_2 = \{w(3)\}$. Then, the function attaining the infimum in the definition of $\mathcal{E}_{p,m}(A_1, A_2, A)$ depends only on the first variable x_1 and is a piecewise linear function in the direction of x_1. Consequently, we see that

$$\mathcal{E}^{\ell}_{p,m}(A_1, A_2, A) \geq 2^{m(L-p)-1}.$$

On the other hand, the comparison of moduli shows

$$\mathcal{M}^{(1)}_{p,m}(A_1, A_2, A) \leq \mathcal{M}^{(1)}_{1,p,m}(w)$$

for any $w \in T$. Therefore, there exists $c_2 > 0$ such that

$$c_2 2^{m(L-p)} \leq \mathcal{E}_{1,p,m}(w, T_{|w|})$$

for any $m \geq 1$ and $w \in T$.

Now, for $f: K \to \mathbb{R}$, we define $\widetilde{f}_m: T_m \to T$ by $\widetilde{f}_m(w) = f(f_w(0))$. Then there exists $c > 0$ such that

$$2^{m(p-L)} \mathcal{E}_{p,T_m}(\widetilde{f}_m) \to c \int_K |\nabla f|^p dx \tag{4.19}$$

as $m \to \infty$ for any $f \in C^\infty(K)$. So there exists $c_3 > 0$ such that $\mathcal{E}_{1,p,m}(w, T_{|w|}) \leq c_3 2^{m(L-p)}$ for any $w \in T$. Thus the scaling exponent of σ appearing in (3.17) is 2^{L-p}. Combining this fact and arguments analogous to those in [41, Section 5.3], we have the following Korevaar–Shoen type expression of $\mathcal{W}^p$:

$$\mathcal{W}^p = \left\{ f \;\middle|\; f \in L^p(K, dx), \varlimsup_{r \downarrow 0} \int_K \frac{1}{r^L} \int_{B_{d_*}(x,r)} \frac{|f(x) - f(y)|^p}{r^p} dy dx < \infty \right\}.$$

This expressing enable us to identify $\mathcal{W}^p$ with $W^{1,p}(K)$. By (4.19), we see that (4.18) holds for any $f \in C^\infty(K)$. Since $C^\infty(K)$ is dense in $W^{1,p}(K)$, (4.18) holds for any $f \in \mathcal{W}^p$.

4.5 Rationally ramified Sierpiński crosses

In this section, we present another class of conductively homogeneous spaces called rationally ramified Sierpiński crosses. This example is a planar square-based self-similar set as those in the last section but the sizes of the squares constituting it are not one but two. See Figure 4.7. Consequently, although it has full $\mathbb{B}_2$-symmetry, we should make a little more complicated discussion than that of the previous section to show the conductive homogeneity.

The family of Sierpiński crosses was introduced in [31, Example 1.7.5].

Definition 4.32. Let $r_1, r_2 \in (0, 1)$ satisfying $2r_1 + r_2 = 1$ and $r_1 \geq r_2$. Let $p_1 = (-1, -1)$, $p_2 = (0, -1)$, $p_3 = (1, -1)$, $p_4 = (1, 0)$, $p_5 = (1, 1)$, $p_6 = (0, 1)$, $p_7 = (-1, 1)$ and $p_8 = (-1, 0)$. Set $S = \{1, \ldots, 8\}$. For $s \in S$, define $F_s: C_* \to C_*$ as

$$F_s(x) = \begin{cases} r_1(x - p_s) + p_s & \text{if } s \text{ is odd,} \\ r_2(x - p_s) + p_s & \text{if } s \text{ is even.} \end{cases}$$

The self-similar set K with respect to the family of contractions $\{F_s\}_{s \in S}$ is called the (r_1)-*Sierpiński cross*. Define

$$\ell_{\mathrm{L}} = \{-1\} \times [-1, 1], \quad \ell_{\mathrm{R}} = \{1\} \times [-1, 1],$$
$$\ell_{\mathrm{B}} = [-1, 1] \times \{-1\}, \quad \ell_{\mathrm{T}} = [-1, 1] \times \{1\},$$

where the symbols, L, R, B, and T correspond to left, right, bottom, and top, respectively.

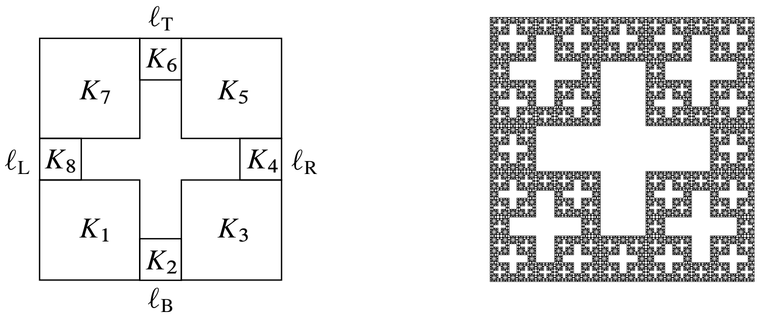


Figure 4.7. The ρ_*-Sierpiński cross: $\rho_* = \sqrt{2} - 1$.

In this section, we will show that if an (r_1)-Sierpiński cross K is rationally ramified, then it is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Roughly speaking an (r_1)-Sierpiński cross is rationally ramified if $\bigcup_{v \in \Gamma_1(w)} K_v$, which represents the local geometry around $w \in T$, has finite types of variety up to the isometries when $w \in T$ varies. See [31] for the exact definition. In fact, in [31, Proposition 1.7.6], it is shown that an (r_1)-Sierpiński cross is rationally ramified if and only if $1 - r_1 = (r_1)^m$ for some $m \geq 2$. For simplicity of arguments, we confine ourselves to the case $m = 2$ hereafter in this section. The generalization to other values of m is a little complicated but the essential idea is the same.

In the case $m = 2$, the value of r_1 equals $\sqrt{2} - 1$. Set $\rho_* = \sqrt{2} - 1$. Our main object of study is now the ρ_*-Sierpiński cross. We take advantage of the framework of Section 4.1 with $r = \rho_*$ and

$$j_s = \begin{cases} 1 & \text{if } s \text{ is odd,} \\ 2 & \text{if } s \text{ is even} \end{cases}$$

to define $(T, \mathcal{A}, \phi)$ and the associated partition of K. In this case, $g(w)$ is the contraction ratio of the map $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ for $w = w_1 \ldots w_m \in S^m$. Note that $g(w) = (\rho_*)^n$ or $(\rho_*)^{n+1}$ for any $(n, w) \in T_n$. For example, $\Lambda^g_{\rho_*} = S$ and

$$\Lambda^g_{(\rho_*)^2} = \{1s, 3s, 5s, 7s \mid s \in S, s\colon \text{even}\} \cup \{1s, 3s, 5s, 7s \mid s \in S, s\colon \text{odd}\} \\ \cup \{2, 4, 6, 8\}.$$

Note that $g(1s) = (\rho_*)^3$ if s is even and $g(1s) = (\rho_*)^2$ if s is odd. Moreover, $\Lambda^g_{\rho_*} \cap \Lambda^g_{(\rho_*)^2} \neq \emptyset$ in this case. Let d_* be the restriction of the Euclidean metric to K. Let $h_{\rho_*}(n, w) = (\rho_*)^n$ for $(n, w) \in T_w$. It is straightforward to see that d_* is 1-adapted to the weight function h_{ρ_*}, i.e., Assumption 2.15 (2B) holds with $M_* = 1$.

For simplicity, to denote an element in T_n, we use w in place of (n, w) hereafter as long as no confusion may occur.

The Hausdorff dimension of (K, d_*) is given by the unique number α_H satisfying

$$4(\rho_*)^{2\alpha_H} + 4(\rho_*)^{\alpha_H} = 1.$$

Consequently, we see that

$$\alpha_H = 1 + \frac{\log 2}{\log(1+\sqrt{2})}.$$

Let μ be the self-similar measure with weight $(\mu_i)_{i \in S}$, where

$$\mu_i = \begin{cases} (\rho_*)^{\alpha_H} & \text{if } i \text{ is odd,} \\ (\rho_*)^{2\alpha_H} & \text{if } i \text{ is even.} \end{cases}$$

Then μ is the normalized α_H-dimensional Hausdorff measure and is α_H-Ahlfors regular with respect to d_*. After those observations, it is easy to see that Assumption 2.15 is satisfied with $M_* = M_0 = 1$, $N_* = 8$. Moreover, we see that $L_* \leq 8$.

The main result of this section is as follows.

Theorem 4.33. *For any $p > 0$, $n, m, k \geq 1$, $w \in T_n$ and $u, v \in T_k$,*

$$\mathcal{M}^{(1)}_{1,p,m}(w) \leq 8(24)^{p+1}\#(T_{k+1})^p \mathcal{M}^{(1)}_{p,m}(u, v, T_k).$$

An immediate consequence of the above theorem is the conductive homogeneity of the Sierpiński cross.

Corollary 4.34. *The ρ_*-Sierpiński cross K is p-conductively homogeneous for any $p > \dim_{AR}(K, d_*)$. Moreover, there exists a self-similar p-energy $\mathcal{E}_p$ on $\mathcal{W}^p$. In particular, there exists a local regular Dirichlet form $(\mathcal{E}, \mathcal{W}^2)$ on $L^2(K, \mu)$ whose associated heat kernel satisfies (3.26) and (3.28).*

Note that due to the two different values of j_s, the self-similarity of the p-energy $\mathcal{E}_p$ is given as

$$\mathcal{E}_p(f) = \sigma \sum_{s:\text{odd}} \mathcal{E}_p(f \circ F_s) + \sigma^2 \sum_{s:\text{even}} \mathcal{E}_p(f \circ F_s)$$

for any $f \in \mathcal{W}^p$.

Proof. By (2.13), it follows that

$$\mathcal{E}_{1,p,m}(w, T_n) \leq c_p \#(T_{k+1})^p \mathcal{E}_{p,m}(u, v, T_k)$$

for any $n, m, k \geq 1$, $w \in T_n$ and $u, v \in T_k$. Moreover, since $p > \dim_{AR}(K, d_*)$, there exist $c > 0$ and $\alpha \in (0, 1)$ such that

$$\mathcal{E}_{1,p,m} \leq c\alpha^m$$

for any $m \geq 1$. Thus we have obtained (3.19) and (3.20), so that K is p-conductively homogeneous by Theorem 3.33. In particular, since $\alpha_H < 2$, K is 2-conductively homogeneous and we have $(\mathcal{E}, \mathcal{W}^2)$. Since (K, d_*) has the chain condition, by Theorem 3.35, we have (3.26) and (3.28). ∎

To show Theorem 4.33, we need to prepare several notions.

Definition 4.35. (1) Set

$$U = \{(2,13), (2,31), (4,35), (4,53), (6,57), (6,75), (8,17), (8,71)\}.$$

For $(i, jk) \in U$, define $R_{i,jk}: K_i \to K_{jk}$ as the reflection in the line segment $K_i \cap K_{jk}$. Moreover, define $R^*_{i,jk}(w)$ for $w \in T(i) \cup T(jk)$ as the unique $v \in T(i) \cup T(jk)$ satisfying $R_{i,jk}(K_w) = K_v$. $R^*_{i,jk}$ is a map from $T(i) \cup T(jk)$ to itself.

(2) For $g \in \mathbb{B}_2$, define $g^*: T \to T$ by

$$g^*(w) = v,$$

where v is the unique $v \in T$ with $g(K_w) = K_v$. Note that $g^*|_{T_n}: T_n \to T_n$.

(3) For $w \in T$, if $w \notin T(2) \cup T(4) \cup T(6) \cup T(8)$, then define

$$\mathcal{H}_w = \{g^*(v) \mid g \in \mathbb{B}_2\}.$$

Otherwise, if $w \in T(i)$ for $i = 2, 4, 6, 8$, then define

$$\mathcal{H}_w = \{g^*(v) \mid g \in \mathbb{B}_2\} \cup \{g_*(R^*_{i,jk}(v)) \mid g \in \mathbb{B}_2, (i, jk) \in U\}.$$

Note that $\#(\mathcal{H}_w) \leq 24$ for any $w \in T_n$.

By the construction of T_n, we see that $g(w) = (\rho_*)^n$ or $g(w) = (\rho_*)^{n+1}$ for any $w \in T_n^n$. In fact, we immediately obtain the following lemma.

Lemma 4.36. *Set $T_n^n = \{w \mid w \in T_n, g(w) = (\rho_*)^n\}$ and $T_n^{n+1} = \{w \mid w \in T_n, g(w) = (\rho_*)^{n+1}\}$. Then*

(1) *For any $w \in T_n^n$, $wv \in T_{n+m}$ if and only if $v \in T_m$.*

(2) *For any $w \in T_n^{n+1}$, $wv \in T_{n+m}$ if and only if $v \in T_{m-1}$.*

(3) *$w \in T_{n+1}^{n+1}$ if and only if $w \in T_n^{n+1}$ or $w = \tau j$ for some $\tau \in T_n^n$ and $j \in \{1, 3, 5, 7\}$.*

(4) *$w \in T_{n+1}^{n+2}$ if and only if $w = \tau j$ for some $\tau \in T_n^n$ and $j \in \{2, 4, 6, 8\}$.*

Definition 4.37. (1) Define $\psi^*_{n,m}: S^m(T_n^n) \to T_m$ by

$$\psi^*_{n,m}(wv) = v$$

for $w \in T_n^n$ and $v \in T_m$.

(2) For $w \in T$, define $\mathcal{H}_w^0 \subseteq T$ by

$$\mathcal{H}_w^0 = \begin{cases} \{w, R_{i,jk}^*(w)\} & \text{if } w \in T(jk) \text{ for some } (i, jk) \in U, \\ \{w\} & \text{otherwise.} \end{cases}$$

For $w \in T_{n+1}^{n+1}$ and $u \in T$, define

$$\mathcal{H}_{wu}^n = \begin{cases} \{\tau v \mid v \in \mathcal{H}_{ju}^0\} & \text{if } w = \tau j \text{ for some } \tau \in T_n^n \text{ and } j \in \{1,3,5,7\}, \\ \{wu\} & \text{if } w \in T_n^{n+1}. \end{cases}$$

(3) Define

$$K_{\%} = \bigcup_{s \in S, K_s \cap \ell_{\%} \neq \emptyset} K_s \tag{4.20}$$

for $\% \in \{\mathrm{T}, \mathrm{B}, \mathrm{R}, \mathrm{L}\}$. For example, $K_{\mathrm{B}} = K_1 \cup K_2 \cup K_3$.

Note that if $w \in T_n$, then $\mathcal{H}_w^0 \in T_n$ and that if $w \in T_{n+1}^{n+1}$ and $u \in T_{m-1}$, then $\mathcal{H}_{wu}^n \subseteq T_{n+m}$.

Lemma 4.38. *Assume that there exists a path* $\mathbf{p} = (w(1), \ldots, w(l))$ *of* T_{m-1} *contained in* K_{L} *such that* $K_{w(1)} \cap \ell_{\mathrm{B}} \neq \emptyset$, $K_{w(l)} \cap \ell_T \neq \emptyset$, *and* $\mathbf{p}$ *is* R_2^**-invariant. Set*

$$\mathcal{H}_u^* = \bigcup_{w \in T_{k+1}^{k+1}} \bigcup_{v \in \mathcal{H}_u} \mathcal{H}_{wv}^{k+1}$$

for $u \in T_{m-1}$. *Then for any* $u_1, u_2 \in T_k$, *there exists* $\mathbf{p}_0 \in \mathcal{C}_m^{(1)}(\{u_1\}, \{u_2\}, T_k)$ *such that*

$$\mathbf{p}_0 \subseteq \bigcup_{i=1}^{l} \mathcal{H}_{w(i)}^*. \tag{4.21}$$

Remark. Strictly, $\mathbf{p}_0$ is not a subset but a sequence of points. However, in (4.21), we use $\mathbf{p}_0$ to denote a subset consisting of the points in the sequence. We use such abuse of notations if no confusion may occur.

Proof. Set

$$Y = \mathbf{p} \cup \Theta_{\frac{\pi}{2}}^*(\mathbf{p}) \cup \Theta_{\pi}^*(\mathbf{p}) \cup \Theta_{\frac{3\pi}{2}}^*(\mathbf{p}).$$

Then $Y = g^*(Y)$ for any $g \in \mathbb{B}_2$. Let

$$\mathcal{H}^*(Y) = \bigcup_{w \in T_{k+1}^{k+1}} \bigcup_{v \in Y} \mathcal{H}_{wv}^{k+1}.$$

See Figure 4.8 for an illustration paths ψ and Y along with a part of $\mathcal{H}^*(Y)$. It follows that $K(\mathcal{H}^*(Y))$ is a connected set intersecting K_u for any $u \in T_k$. Therefore, we can choose a path $\mathbf{p}_0$ connecting K_{u_1} and K_{u_2} from $\mathcal{H}^*(Y)$. Since $\mathcal{H}^*(Y) \subseteq \bigcup_{i=1}^{l} \mathcal{H}_{w(i)}^*$, we have the desired statement. ∎

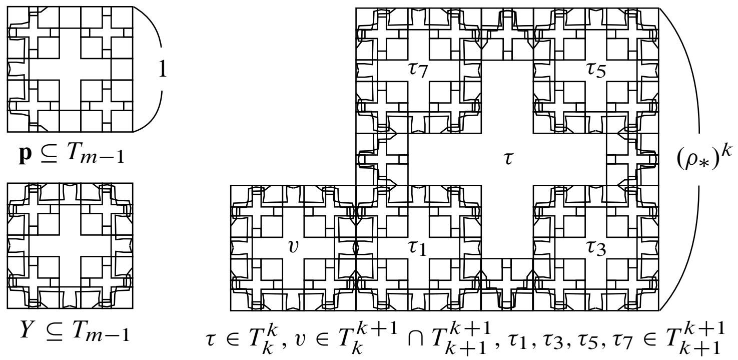


Figure 4.8. Paths ψ and Y, and a part of $\mathcal{H}^*(Y)$.

Proof of Theorem 4.33. Let $w \in T_n$ and let $u_1, u_2 \in T_k$. For any $\mathbf{p} \in \mathcal{C}_{1,m}^{(1)}(w)$, set

$$\mathcal{H}_{m-1}(\mathbf{p}) = \bigcup_{u \in \psi^*_{n+1,m-1}(\mathbf{p} \cap S^{m-1}(T_{n+1}^{n+1}))} \mathcal{H}_u.$$

Then $\mathcal{H}_{m-1}(\mathbf{p}) \subseteq T_{m-1}$ and $g^*(\mathcal{H}_{m-1}(\mathbf{p})) = \mathcal{H}_{m-1}(\mathbf{p})$ for any $g \in \mathbb{B}_2$.

Claim 1. *There exists a path* $\mathbf{p}^*$ *contained in* $\mathcal{H}_{m-1}(\mathbf{p})$ *such that one of the following four statements is true:*

(a) $K(\mathbf{p}^*) \cap \ell_{\mathrm{B}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{T}} \neq \emptyset$,

(b) $K(\mathbf{p}^*) \cap \ell_{\mathrm{T}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{B}} \neq \emptyset$,

(c) $K(\mathbf{p}^*) \cap \ell_{\mathrm{L}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{R}} \neq \emptyset$,

(d) $K(\mathbf{p}^*) \cap \ell_{\mathrm{R}} \neq \emptyset$ *and* $K(\mathbf{p}^*) \cap K_{\mathrm{L}} \neq \emptyset$.

Proof. Let $F_w(C_*) = [a, a+h] \times [b, b+h]$, where $h = (\rho_*)^n$ if $w \in T_n^n$ and $h = (\rho_*)^{n+1}$ if $w \in T_n^{n+1}$. Define

$$A_{w,\gamma} = [a-\gamma, a+h+\gamma] \times [b-\gamma, b+h+\gamma]$$

and $\widetilde{A}_w = K \cap (A_{w,(\rho_*)^{n+1}} \backslash A_{w,(\rho_*)^{n+2}})$. Two typical examples of $\widetilde{A}_w$ is illustrated in Figure 4.9. Since $K_{w(1)} \cap K_w \neq \emptyset$ and $K_{w(l)} \cap A_{w,(\rho_*)^{n+1}} = \emptyset$, a part of $\mathbf{p}$ contained in $\widetilde{A}_w$ connects

$\{(a-(\rho_*)^{n+1}, y) \mid y \in [-1,1]\}$ and $\{(a-(\rho_*)^{n+2}, y) \mid y \in [-1,1]\}$,

$\{(a+h, y+(\rho_*)^{n+2}) \mid y \in [-1,1]\}$ and $\{(a+h+(\rho_*)^{n+1}, y) \mid y \in [-1,1]\}$,

$\{(x, b-(\rho_*)^{n+1}) \mid x \in [-1,1]\}$ and $\{(x, b-(\rho_*)^{n+2}) \mid x \in [-1,1]\}$,

or

$\{(x, b+h+(\rho_*)^{n+2}) \mid x \in [-1,1]\}$ and $\{(x, b+h+(\rho_*)^{n+1}) \mid x \in [-1,1]\}$.

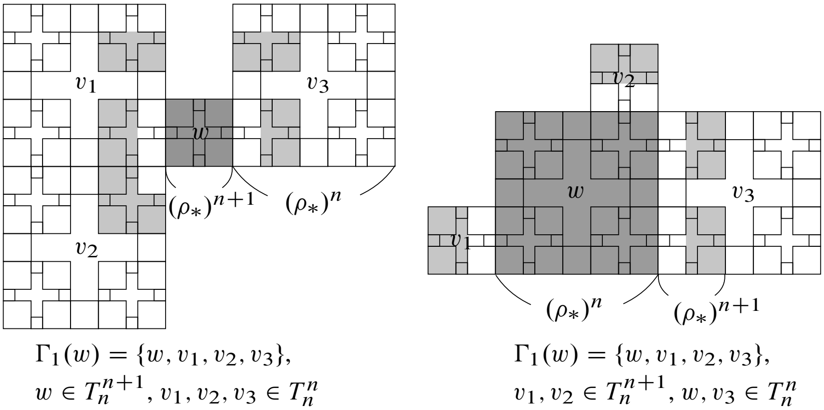


Figure 4.9. Two examples of $\widetilde{A}_w$ (dark grey regions are K_w, light grey regions are $\widetilde{A}_w$).

According to the four possibilities above, we have (a), (b), (c) or (d), where the exact correspondence depends on w. □

Hereafter we assume the first case (a) in Claim 1 in the course of discussion. Other cases may be treated exactly in the same manner. In the following claims, we are going to modify the initial path $\mathbf{p}^*$ step by step. This process of modification is illustrated in Figure 4.10.

Claim 2. *The union* $\mathbf{p}^* \cup R_2^*(\mathbf{p}^*)$ *contains an* R_2*-symmetric path*

$$\mathbf{p}_1 = (v(0), \dots, v(l_1))$$

between ℓ_{B} *and* ℓ_T*, i.e.,* $K_{v(0)} \cap \ell_{\mathrm{B}} \neq \emptyset$*,* $R_2^*(v(i)) = v(l_1 - i)$ *for* $i = 1, \dots, l_1$.

Proof. Let $p_* = (w(1), \dots, w(l))$. By (a), $K(\mathbf{p}^*)$ intersects with the line segment $[-1,1] \times \{0\}$. Set $i_* = \min\{i \mid w(i) \cap [-1,1] \times \{0\} \neq \emptyset\}$. Then connecting $(w(1), \dots, w(i_*))$ and its image by R_2^*, we obtain a desired path. □

Claim 3. *The union* $R_1^*(\mathbf{p}_1) \cup \mathbf{p}_1$ *contains an* R_2*-symmetric path* $\mathbf{p}_2$ *such that*

$$K(\mathbf{p}_2) \subseteq [-1,0] \times [-1,1].$$

Proof. If $\mathbf{p}_1$ or $R_1^*(\mathbf{p}_1)$ is contained in the left half of C_*, then choose $\mathbf{p}_1$ or $R_1^*(\mathbf{p}_1)$ accordingly as our path. Otherwise, applying R_1 to $K(\mathbf{p}_1) \cap [0,1] \times [-1,1]$, we obtain a desired path. □

Claim 4. *Set* $\mathcal{H}_{\mathbf{p}^*} = \bigcup_{u \in \mathbf{p}^*} \mathcal{H}_u$*. Then there exists an* R_2^**-symmetric path* $\mathbf{p}_3 \subseteq \mathcal{H}_{\mathbf{p}^*}$ *contained in* K_{L} *such that* $K(\mathbf{p}_3) \cap \ell_{\mathrm{T}} \neq \emptyset$ *and* $K(\mathbf{p}_3) \cap \ell_{\mathrm{B}} \neq \emptyset$.

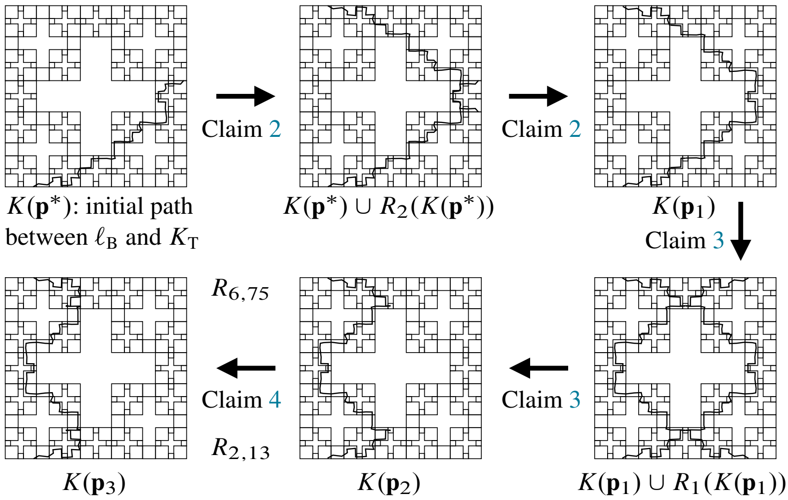


Figure 4.10. Modifications of a path.

Proof. If $K(\mathbf{p}_2) \subseteq K_L$, then we set $\mathbf{p}_2 = \mathbf{p}_3$. Otherwise, use $R^*_{2,13}$ (resp. $R^*_{6,75}$) to reflect the part $K(\mathbf{p}_2) \cap K_2$ (resp. $K(\mathbf{p}_2) \cap K_6$) into K_{13} (resp. K_{75}). Then we obtain a desired path. □

Now we have a path p_3 satisfying all the assumptions of Lemma 4.38. Applying Lemma 4.38 with $\mathbf{p} = \mathbf{p}_3$, we obtain a path $\mathbf{p}_0 \in \mathcal{C}^{(1)}_m(\{u_1\}, \{u_2\}, T_k)$. For $u \in S^m(\Gamma_1(w))$, define

$$H_u = \begin{cases} \bigcup_{v \in \mathcal{H}_{\psi^*_{n+1,m-1}(u)}} \mathcal{H}^*_v & \text{if } u \in S^{m-1}(T^{n+1}_{n+1}), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it follows that $\mathbf{p}_0 \subseteq \bigcup_{v \in \mathbf{p}} H_v$. Since $\#(\mathcal{H}_u) \leq 24$ and $\#(\Gamma_1(w)) \leq 8$,

$$\#(H_w) \leq 48\#(T_{k+1}) \quad \text{and} \quad \#(\{v \mid u \in H_v\}) \leq 24 \cdot 8.$$

So, Lemma C.4 suffices. ■

4.6 Nested fractals

In this section, we show conductive homogeneity of a class of self-similar sets, called strongly symmetric self-similar sets, that are highly symmetric and finitely ramified. This class is a natural extension of nested fractals introduced by Lindstrøm [37], where Brownian motions were constructed on them. In [29, Section 3.8], Lindstrøm's results were extended to strongly symmetric self-similar sets. Typical examples of strongly symmetric self-similar sets are the Sierpiński gasket, the pentakun ("Kun" means "Mr." in Japanese), and the snowflake, whose definitions are given below.

Let $\rho \in (0,1)$ and let S be a finite subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$. For each $q \in S$, let $f_q : \mathbb{R}^L \to \mathbb{R}^L$ be a ρ-similitude whose fixed point is q, i.e., there exists $U_q \in O(L)$ such that

$$f_q(x) = \rho U_q(x - q) + q$$

for any $x \in \mathbb{R}^L$. Let K be the self-similar set with respect to the family of contractions $\{f_q\}_{q \in S}$. Then the triple $(K, S, \{f_q\}_{q \in S})$ is a self-similar structure as is explained in Section 4.1. Throughout this section, we consider a self-similar structure $(K, S, \{f_q\}_{s \in S})$ given in this way.

Assumption 4.39. (1) *If $p, q \in S$ and $p \neq q$, then $p \notin f_q(K)$.*

(2) *There exists $U \subseteq S$ such that*

$$\bigcup_{\substack{q_1, q_2 \in S \\ q_1 \neq q_2}} f_{q_1}^{-1}(f_{q_1}(K) \cap f_{q_2}(K)) = U.$$

(3) *K is connected.*

For purposes of normalization, we assume $\sum_{q \in U} q = 0$ hereafter.

Proposition 4.40. *Under Assumption 4.39, $(K, S, \{f_q\}_{q \in S})$ is a post critically finite self-similar structure with*

$$V_0 = U. \tag{4.22}$$

Moreover, define $\{V_m\}_{m \geq 1}$ inductively by

$$V_{m+1} = \bigcup_{i \in S} f_i(V_m).$$

Then

$$V_m \subseteq V_{m+1} \tag{4.23}$$

for any $m \geq 0$.

The definitions of post critically finite (p.c.f. for short) self-similar structures and V_0 along with the proof of (4.22) is given in Appendix 6.3. Inclusion (4.23) is due to [29, Lemma 1.3.11].

For the self-similar structure $(K, S, \{f_q\}_{q \in S})$, we adopt the framework in Section 4.1 with $r = \rho$ and $j_q = 1$ for any $q \in S$. In this case,

$$T_m = S^m = \{w_1 \dots w_m \mid w_i \in S \text{ for any } i = 1, \dots, m\}.$$

Then we see that

$$V_0 = \bigcup_{e \in E_1^*} X(e),$$

where $X(e)$ is defined in Definition 4.7. Moreover, by [29, Proposition 1.3.5 (2)], it follows that

$$K_w \cap K_v = f_w(V_0) \cap f_v(V_0) \subseteq V_m \tag{4.24}$$

for any $w, v \in T_m$ with $w \neq v$. This implies that

$$V_0 = \bigcup_{(X,Y,\varphi)\in \mathcal{IT}(K,T)} X. \tag{4.25}$$

Let $\alpha_H = -\frac{\log N}{\log \rho}$. Note that $N\rho^{\alpha_H} = 1$. Let μ be the self-similar measure with weight $(\rho^{\alpha_H}, \dots, \rho^{\alpha_H})$. Basic properties of μ are given in Appendix 6.3. Also, let d_* be the restriction of the Euclidean metric to K.

The following assumption is an equivalent condition of Assumption 2.15 (2B) when d is the (restriction of) Euclidean metric. Essentially, the same assumptions have been around from time to time for almost 30 years. See [35, Assumption 2.2] and [38, Assumption (P)]. The assumption is believed to be true for nested fractals but we have no proof so far. In [38], it was shown that this assumption is true if U_q is the same for any $q \in S$. In Appendix 6.3, this assumption is shown to be true if U_q is the identity map for any $q \in V_0$.

Assumption 4.41. *There exists $c > 0$ such that $d(K_w, K_v) \geq c\rho^{|w|}$ for any $n \geq 1$, and $(w, v) \in (T_n \times T_n)\backslash E_n^*$, where $d(A, B) = \inf_{x\in A, y\in B} |x - y|$ for subsets $A, B \subseteq \mathbb{R}^L$.*

Proposition 4.42. *Under Assumptions 4.39 and 4.41, Assumption 2.15 is satisfied with $d = d_*$, $r = \rho$, and $M_* = M_0 = 1$.*

The above proposition is proven in Appendix 6.3.

Definition 4.43. (1) Let $m_* = \#\{|x - y| \mid x, y \in V_0,\ x \neq y\}$, where $|x|$ is the Euclidean length of $x \in \mathbb{R}^L$. Define

$$l_0 = \min\{|x - y| \mid x, y \in V_0,\ x \neq y\}.$$

Moreover, define l_i for $i = 0, 1, \dots, m_* - 1$ inductively by

$$l_{i+1} = \min\{|x - y| \mid x, y \in V_0,\ x \neq y,\ |x - y| > l_i\}.$$

(2) A sequence $(x_i)_{i=1,\dots,k} \subseteq V_m$ is called an *m-walk* if there exists $w(i) \in T_m$ such that $x_i, x_{i+1} \in f_{w(i)}(V_0)$ for any $i = 1, \dots, k - 1$.

(3) A 0-walk $(x_i)_{i=1,\dots,k}$ is called a *strict* 0-*walk* (between x_1 and x_k) if $|x_i - x_{i+1}| = l_0$ for any $i = 1, \dots, k - 1$.

(4) Define

$$\mathcal{G} = \{g \mid g \in O(L), g(V_0) = V_0 \text{ and there exists } g^*\colon T \to T \text{ such that } g(f_w(V_0)) = f_{g^*(w)}(V_0) \text{ for any } w \in T\}.$$

(5) For any $x, y \in \mathbb{R}^L$ with $x \neq y$, define

$$H_{xy} = \{z \mid z \in \mathbb{R}^L,\ |x - z| = |y - z|\}.$$

(H_{xy} is the hyperplane bisecting the line segment xy.) Also let $g_{xy}: \mathbb{R}^L \to \mathbb{R}^L$ be reflection in H_{xy}.

Definition 4.44. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is said to be *strongly symmetric* if Assumption 4.39 is satisfied and there exists a finite subgroup $\mathcal{G}_*$ of $\mathcal{G}$ such that the following properties hold:

(1) For any $x, y \in V_0$ with $x \neq y$, there exists a strict 0-walk between x and y.

(2) If $x, y, z \in V_0$ and $|x - y| = |x - z|$, then there exists $g \in \mathcal{G}_*$ such that $g(x) = x$ and $g(y) = z$.

(3) For any $i = 1, \ldots, m_* - 2$, there exist x, y and $z \in V_0$ such that $|x - y| = l_i$, $|x - z| = l_{i+1}$ and $g_{yz} \in \mathcal{G}_*$.

(4) V_0 is $\mathcal{G}_*$-transitive, i.e., for any $x, y \in V_0$, there exists $g \in \mathcal{G}_*$ such that $g(x) = y$.

Remark. By Definition 4.44 (4), $|q_1| = |q_2|$ for any $q_1, q_2 \in V_0$.

Definition 4.45. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is called a *nested fractal* if Assumption 4.39 holds and $g_{xy} \in \mathcal{G}$ for any $x, y \in V_0$ with $x \neq y$.

By [29, Proposition 3.8.7], we have the following proposition.

Proposition 4.46. *A nested fractal is strongly symmetric.*

We give three examples of strongly symmetric self-similar sets. Note that Assumption 4.41 is satisfied for all three examples because of Lemma E.5. The first two are nested fractals.

Example 4.47 (Pentakun: Figure 4.11). Let $L = 2$ and let $S = \{p_1, \ldots, p_5\}$ be a collection of vertices of a regular pentagon satisfying $\sum_{i=1}^{5} p_i = 0$ and let $\rho = \frac{3-\sqrt{5}}{2}$. Then the associated self-similar set K, called pentakun, is strongly symmetric. (See [29, Example 3.8.11].) In this case $\mathcal{G} = \mathcal{G}_* = D_5$, which is the group of symmetries of a regular pentagon, and $V_0 = \{p_1, \ldots, p_5\}$.

Example 4.48 (Snowflake: Figure 4.12). Let $L = 2$ and let $\{p_1, \ldots, p_6\}$ be a collection of vertices of a regular hexagon satisfying $\sum_{i=1}^{6} p_i = 0$ and let $S = \{p_1, \ldots, p_7, 0\}$. Furthermore, let $\rho = \frac{1}{3}$. Then the associated self-similar set, called snowflake, is strongly symmetric. (See [29, Example 3.8.12].) In this case $\mathcal{G} = \mathcal{G}_* = D_6$, which is the group of symmetries of a regular hexagon and $V_0 = \{p_1, \ldots, p_6\}$.

The last example is not a nested fractal.

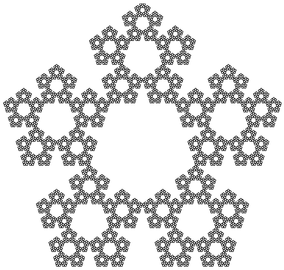


Figure 4.11. Pentakun.

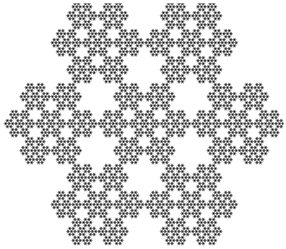


Figure 4.12. Snowflake.

Example 4.49. Let $L = 3$ and let

$$S = \{-1, 0, 1\}^3 \cup \left\{-\frac{1}{2}, \frac{1}{2}\right\}^3, \quad U = \{1, -1\}^3,$$

and $\rho = \frac{1}{5}$. Note that U is the collection of vertices of the cube $[-1, 1]^3$ and

$$f_q([-1, 1]^3) = \left[\frac{4q_1 - 1}{5}, \frac{4q_1 + 1}{5}\right] \times \left[\frac{4q_2 - 1}{5}, \frac{4q_2 + 1}{5}\right] \times \left[\frac{4q_3 - 1}{5}, \frac{4q_3 + 1}{5}\right]$$

for any $q = (q_1, q_2, q_3) \in S$. It is straightforward to see that the associated self-similar set is strongly symmetric with $V_0 = U$ and $\mathcal{G} = \mathcal{G}_* = \mathbb{B}_3$. This self-similar set is not a nested fractal because $g_{xy} \notin \mathcal{G}$ if $x = (-1, -1, -1)$ and $y = (1, 1, 1)$.

Using Theorem 4.8, we have the following theorem.

Theorem 4.50. *Suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric and that Assumption 4.41 holds. Then (K, d_*) is p-conductively homogeneous for any $p >$ $\dim_{AR}(K, d_*)$.*

As for $\dim_{AR}(K, d_*)$, it was shown in [44] that $\dim_{AR}(K, d_*) = 1$ if (K, d_*) is the Sierpiński gasket. In general, we have the following fact.

Proposition 4.51. *Suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric and that Assumption 4.41 holds. Then $\dim_{AR}(K, d_*) < 2$.*

Proof. For $m \geq 0$, define $\widetilde{E}_m = \{(f_w(x), f_w(y)) \mid w \in T_m, x, y \in V_0, x \neq y\}$. Then the sequence $\{(V_m, \widetilde{E}_m)\}_{m \geq 0}$ is a proper system of horizontal networks in the sense of [34, Definition 4.6.5]. Define

$$\begin{aligned}\mathcal{L}_s(V_0) = \{(D_{xy})_{x,y \in V_0} \mid\ & \text{there exists } (D_0, \dots, D_{m_*-1}) \in [0, \infty)^{m_*} \\ & \text{such that } D_0 = 1,\ D_{xy} = D_i \text{ if } |x - y| = l_i, \\ & \text{and } \textstyle\sum_{y \in V_0} D_{xy} = 0 \text{ for any } x \in V_0\}.\end{aligned}$$

In particular, let $D^1 \in \mathcal{L}_s(V_0)$ satisfy $(D^1)_{xy} = 1$ for any $x, y \in V_0$ with $x \neq y$. For $D = (D_{xy})_{x,y \in V_0} \in \mathcal{L}_s(V_0)$, define

$$\mathcal{E}^D_{2,m}(h) = \frac{1}{2} \sum_{w \in T_m, x, y \in V_0} D_{xy}(h(f_w(x)) - h(f_w(y)))^2$$

for $h \in \ell(V_m)$ and

$$\mathcal{E}^D_{2,m,w} = \inf\{\mathcal{E}^D_{2,n+m}(h) \mid h \in \ell(V_{n+m}),\ h|_{V_{n+m} \cap K_w} = 1,\ h|_{V_{n+m} \cap (\cup_{v \notin \Gamma_1(w)} K_v)} = 0\}$$

for any $w \in T_n$. Then by [29, Theorem 3.8.10 and Corollary 3.1.9], there exist $D_* \in \mathcal{L}_s(V_0)$ and $\sigma > 1$ such that $(D_*, (\sigma^{-1}, \ldots, \sigma^{-1}))$ is a harmonic structure, that is, for any $h \in \ell(V_m)$,

$$\sigma^m \mathcal{E}^{D_*}_{2,m}(h) = \min\{\sigma^{m+1} \mathcal{E}^{D_*}_{2,m+1}(g) \mid g \in \ell(V_{m+1}), g|_{V_m} = h\}.$$

This implies that there exist $c_1, c_2 > 0$ and $k \geq 1$ such that

$$c_1 \sigma^{-m} \leq \sup_{w \in T \setminus T_k} \mathcal{E}^{D_*}_{2,m,w} \leq c_2 \sigma^{-m}.$$

On the other hand, there exist $c_3, c_4 > 0$ such that

$$c_3 \mathcal{E}^{D_*}_{2,m}(h) \leq \mathcal{E}^{D^1}_{2,m}(h) \leq c_4 \mathcal{E}^{D_*}_{2,m}(h)$$

for any $m \geq 0$ and $h \in \ell(V_m)$. Thus we see that $\sup_{w \in T} \mathcal{E}^{D_1}_{2,m,w} \leq C\sigma^{-m}$ for any $m \geq 0$. Therefore, by [34, Theorems 4.6.9 and 4.9.1], $\dim_{AR}(K, d_*) < 2$. ■

The rest of this section is devoted to proving Theorem 4.50. We suppose that $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric hereafter in this section. We have the following theorem by [29, Proposition 3.8.19],

Lemma 4.52. *If $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric, then $g(K_w) = K_{g^*(w)}$ for any $g \in \mathcal{G}$ and $w \in T$. In particular, $\mathcal{G} \subseteq \mathcal{G}_{(K,T)}$.*

Lemma 4.53. *If $(K, S, \{f_i\}_{i \in S})$ is strongly symmetric, $x_1, x_2, y_1, y_2 \in V_0$ and $|x_1 - x_2| = |y_1 - y_2|$, then there exists $g \in \mathcal{G}_*$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.*

Proof. According to Definition 4.44 (4), there exists $g_1 \in \mathcal{G}_*$ such that $g_1(x_1) = y_1$. Let $g_1(x_2) = z$. Then $|y_1 - y_2| = |y_1 - z|$. Hence by Definition 4.44 (2), there exists $g_2 \in \mathcal{G}_*$ such that $g_2(y_1) = y_1$ and $g_2(z) = y_2$. Thus letting $g = g_2 \circ g_1$, we see that $g(x_1) = g_2(y_1) = y_1$ and $g(x_2) = g_2(z) = y_2$. ■

Definition 4.54. A path $(w(1), \ldots, w(k))$ of (T_m, E^*_m) is said to connect $x \in K$ and $y \in K$ if $x \in K_{w(1)}$ and $y \in K_{w(k)}$.

Lemma 4.55. *Let* $\mathbf{p}$ *be a path of* (T_m, E_m^*) *connecting* $x_1 \in V_0$ *and* $x_2 \in V_0$*. Suppose* $|x_1 - x_2| = l_i$ *for some* $i = 1, \ldots, m_* - 1$*. Then there exist a path* $\mathbf{p}_1$ *of* (T_m, E_m^*)*,* $x \in V_0$ *and* $y \in V_0$ *such that* $\mathbf{p}_1$ *connects* x *and* y*,* $\mathbf{p}_1 \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$ *and* $|x - y| = l_{i-1}$*.*

Notation. For a path $\mathbf{p} = (w(1), \ldots, w(k))$ and $g \in \mathcal{G}$, set

$$g^*(\mathbf{p}) = (g^*(w(1)), \ldots, g^*(w(k))).$$

Remark. As was done before, we regard $\mathbf{p}_1$ and $g^*(\mathbf{p})$ as subsets of T_m in the above lemma. We are going to keep doing such an abuse of notation as long as no confusion may occur.

Proof. By Definition 4.44 (2), there exist $x, y, z \in V_0$ such that $|x - y| = l_{i-1}$, $|x - z| = l_i$ and $g_{yz} \in \mathcal{G}_*$. Also, Lemma 4.53 shows that there exists $h \in \mathcal{G}_*$ such that $h(x_1) = x$ and $h(x_2) = z$. Since $|x - y| < |x - z|$, x and z belong to different sides of H_{yz}. Hence the path $h^*(\mathbf{p})$ intersects with H_{yz}. Therefore, $h^*(\mathbf{p})$ and $(g_{yz})^* \circ h^*(\mathbf{p})$ has an intersection in H_{yz}. Since $(g_{yz})^* \circ h^*(\mathbf{p})$ connects $g_{yz}(x)$ and $y = g_{yz}(z)$, we can extract a path $\mathbf{p}_1$ from $h^*(\mathbf{p}) \cup (g_{yz})^* \circ h^*(\mathbf{p})$ connecting x and y, and included in $\bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$. Since $|x - y| = l_{i-1}$, $\mathbf{p}_1$ is a desired path. ■

Lemma 4.56. *Let* $\mathbf{p}$ *be a path of* (T_m, E_m^*) *connecting two distinct points in* V_0*. Then for any* $x, y \in V_0$*, there exists a path* $\mathbf{p}'$ *of* (T_m, E_m^*) *connecting* x *and* y *such that* $\mathbf{p}' \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$*.*

Proof. Inductive use of Lemma 4.55 shows that there exists a path $\mathbf{p}_0$ of (T_m, E_m^*) connecting two distinct points z_1 and z_2 in V_0 such that $|z_1 - z_2| = l_0$ and $\mathbf{p}_0 \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\mathbf{p})$. By Definition 4.44 (1), there exists a strict 0-walk $(x_1, \ldots, x_{j_0})$ satisfying $x_1 = x$ and $x_{j_0} = y$. By Lemma 4.53, for any $j = 1, \ldots, j_0 - 1$, there exists $g_j \in \mathcal{G}_*$ such that $g_j(z_1) = x_j$ and $g_j(z_2) = x_{j+1}$. Concatenating $(g_1)^*(\mathbf{p}_0), \ldots, (g_{j_0-2})^*(\mathbf{p}_0)$ and $(g_{j_0-1})^*(\mathbf{p}_0)$, we obtain a desired path connecting x and y. ■

Proof of Theorem 4.50. We are going to use Theorem 4.8. Let $\mathcal{I} = \mathcal{I}\mathcal{T}(K, T)$ and let $\mathcal{G}_0 = \mathcal{G}_1 = \mathcal{G}_*$. By (4.25) and the fact that $\mathcal{I} = \mathcal{I}\mathcal{T}(K, T)$, we see that $E_m^{\mathcal{I}} = E_m^*$. Hence (a) of Theorem 4.8 is satisfied, and (b) is also satisfied due to the fact that $\mathcal{G}_*$ is transitive on V_0.

Let $w \in T_n$, let $u, v \in T_k$ and let $\mathbf{p} \in \mathcal{C}_{1,m}^{(1)}(w)$. Then $\mathbf{p}$ contains a path connecting two distinct points in $\bigcup_{w' \in T_n} f_{w'}(V_0)$. Thus $\psi_n(\mathbf{p})$ contains a path between two distinct points in V_0. By Lemma 4.56, for any $x, y \in V_0$, there exists a path $\mathbf{p}_{xy} \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\psi_n(\mathbf{p}))$ connecting x and y. Set $\mathcal{U}_{\mathbf{p}} = \bigcup_{x,y \in V_0} \mathbf{p}_{xy}$. Then since $K(\mathcal{U}_{\mathbf{p}}) \supseteq V_0$, it follows that $g(K(\mathcal{U}_{\varphi})) \supseteq V_0$ for any $g \in \mathcal{G}_*$. Moreover, $K(\mathcal{U}_{\mathbf{p}})$ is connected and $\mathcal{U}_{\mathbf{p}} \subseteq \bigcup_{g \in \mathcal{G}_*} g^*(\psi_n(\mathbf{p}))$. Thus we have verified (c) of Theorem 4.8. Now, Theorem 4.8 suffices. ■