Chapter 5

Knight move implies conductive homogeneity

5.1 Conductance and Poincaré constants

From this section, we start preparations for a proof of Theorem 3.33. To begin with, we will introduce Poincaré constants and study a relationship between Poincaré and conductance constants in this section.

The next lemma concerns an extension of functions on T_n to those on T_{n+m} by means of the partition of unity $\{\varphi_w\}_{w\in T_n}$ given in Lemma 2.19.

Lemma 5.1 ([36, Lemma 2.8]). *Let $p \geq 1$, let $A \subseteq T_n$ and let $\{\varphi_w\}_{w\in A}$ be the partition of unity given in Lemma 2.19. Define $\widehat{I}_{A,m}\colon \ell(A) \to \ell(S^m(A))$ by*

$$(\widehat{I}_{A,m} f)(u) = \sum_{w\in A} f(w)\varphi_w(u).$$

Then

$$\mathcal{E}_{p,A}^{n+m}(\widehat{I}_{A,m} f) \leq c_{5.1}\big(\max_{w\in A} \mathcal{E}_{M,p,m}(w,A)\big)\mathcal{E}_{p,A}^{n}(f),$$

where the constant $c_{5.1} = c_{5.1}(p, L_*, M)$ depends only on p, L_* and M.

Proof. Let $(a_k(u,v))_{u,v\in T_k}$ be the adjacency matrix of (T_k, E_k^*). Set $\widetilde{f} = \widehat{I}_{A,m} f$. Then

$$\mathcal{E}_p^{n+m}(\widetilde{f}) = \frac{1}{2}\sum_{w\in A}\sum_{v\in S^m(w)}\sum_{u\in S^m(\Gamma_1^A(w))} a_{n+m}(u,v)|\widetilde{f}(u) - \widetilde{f}(v)|^p. \tag{5.1}$$

Suppose $v \in S^m(w), u \in S^m(\Gamma_1^A(w))$ and $(u,v) \in E_{n+m}^*$. Then $\varphi_{w'}(u) = \varphi_{w'}(v) = 0$ for any $w' \notin \Gamma_{M+1}^A(w)$. Hence

$$\sum_{w'\in\Gamma_{M+1}^A(w)} \varphi_{w'}(u) = \sum_{w'\in\Gamma_{M+1}^A(w)} \varphi_{w'}(v) = 1.$$

Using this, we see

$$\begin{aligned}\widetilde{f}(u) - \widetilde{f}(v) &= \sum_{w'\in\Gamma_{M+1}^A(w)} f(w')(\varphi_{w'}(u) - \varphi_{w'}(v)) \\ &= \sum_{w'\in\Gamma_{M+1}^A(w)} (f(w') - f(w))(\varphi_{w'}(u) - \varphi_{w'}(v)).\end{aligned}$$

Let $q \geq 1$ be the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then by Lemma A.2,

$$
\begin{aligned}
|\widetilde{f}(u) - \widetilde{f}(v)|^p \leq & \sum_{w' \in \Gamma^A_{M+1}(w)} |f(w') - f(w)|^p \\
& \times \Big(\sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^q \Big)^{\frac{p}{q}} \\
\leq & \, C_1 \sum_{w' \in \Gamma^A_{M+1}(w)} |f(w') - f(w)|^p \\
& \times \sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^p,
\end{aligned}
$$

where $C_1 = \max\{1, (L_*)^{(M+1)(p-2)}\}$. If $w \in A$ and $w' \in \Gamma^A_{M+1}(w)$, then there exist $w(0), \ldots, w(M+1) \in A$ such that $w(0) = w$, $w(M+1) = w'$, $(w(j), w(j+1)) \in E^*_n$ for any $j = 0, \ldots, M$. Then

$$
|f(w') - f(w)|^p \leq (M+1)^{p-1} \sum_{j=0}^{M} |f(w(j)) - f(w(j+1))|^p.
$$

Since $\#(\Gamma^A_{M+1}(w)) \leq (L_*)^{M+1}$, it follows that

$$
\sum_{w' \in \Gamma^A_M(w)} |f(w') - f(w)|^p \leq C_2 \sum_{w', w'' \in \Gamma^A_M(w), (w', w'') \in E^*_n} |f(w') - f(w'')|^p,
$$

where $C_2 = (M+1)^{p-1}(L_*)^M$. On the other hand,

$$
\begin{aligned}
& \sum_{v \in S^m(w)} \sum_{u \in S^m(\Gamma^A_1(w))} a_{n+m}(u, v) \sum_{w' \in \Gamma^A_{M+1}(w)} |\varphi_{w'}(u) - \varphi_{w'}(v)|^p \\
& \leq 2 \sum_{w' \in \Gamma^A_{M+1}(w)} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_{w'}, \varphi_{w'}) \leq 2(L_*)^{M+1} \max_{w' \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_{w'}).
\end{aligned}
$$

Hence, by (5.1),

$$
\begin{aligned}
\mathcal{E}^{m+n}_{p, S^m(A)}(\widetilde{f}) \leq & \, C_1 C_2 (L_*)^{M+1} \max_{w \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_w) \\
& \times \sum_{w \in A} \Big(\sum_{w', w'' \in \Gamma^A_{M+1}(w), (w', w'') \in E^*_n} |f(w') - f(w'')|^p \Big) \\
\leq & \, C_1 C_2 (L_*)^{2(M+1)} \max_{w \in A} \mathcal{E}^{n+m}_{p, S^m(A)}(\varphi_w) \mathcal{E}^n_{p, A}(f).
\end{aligned}
$$

So, Lemma 2.19 suffices. ∎

There is another simple way of extension of functions on T_n to those on T_{n+k}.

Lemma 5.2. *Let $p \geq 1$ and let $A \subseteq T_n$. Define $\widetilde{I}_{A,k}: \ell(A) \to \ell(S^k(A))$ by*

$$\widetilde{I}_{A,k} f = \sum_{w \in A} f(w) \chi_{S^k(w)}.$$

Then

$$\mathcal{E}^{n+k}_{p,S^k(A)}(\widetilde{I}_{A,k} f) \leq \max_{w \in A} \#(\partial S^k(w)) \mathcal{E}^n_{p,A}(f).$$

Proof. Let $\widehat{f} = \widetilde{I}_{A,k} f$. Then $\widehat{f}(u) = \widehat{f}(v)$ if $\pi^k(u) = \pi^k(v)$. So if $(u, v) \in E^*_{n+k}$ and $\widehat{f}(u) \neq \widehat{f}(v)$, then $(\pi^k(u), \pi^k(v)) \in E^*_n$. Fix $(w, w') \in E^*_n$. Then

$$\#\{(u, v) \mid (u, v) \in E_{n+k}, \pi^k(u) = w, \pi^k(v) = w'\} \leq \#(\partial S^k(w)).$$

This immediately implies the desired statement. ∎

Combining two previous extensions, we have the following estimate.

Lemma 5.3 ([36, Lemma 2.9]). *Let $p \geq 1$ and let $A \subseteq T_n$. Then, there exists $I_{A,k,m}$: $\ell(A) \to \ell(S^{k+m}(A))$ such that for any $f \in \ell(A)$,*

$$\begin{aligned} \mathcal{E}^{n+k+m}_{p,S^{k+m}(A)}(I_{A,k,m} f) \leq c_{5.3} \max_{w \in A} & \#(\partial S^k(w)) \\ & \times \max_{v \in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A)) \mathcal{E}^n_{p,A}(f), \end{aligned} \tag{5.2}$$

where the constant $c_{5.3} = c_{5.3}(p, L_*, M)$ depends only on p, L_* and M, and

$$(I_{A,k,m} f)(u) = f(w) \tag{5.3}$$

for any $w \in A$ and $u \in S^m(S^k(w) \backslash B_{M,k}(w))$.

Proof. Define $I = \widehat{I}_{S^k(A),m} \circ \widetilde{I}_{A,k}$. Combining Lemmas 5.1 and 5.2, we immediately obtain (5.2). Let $u \in S^{m+k}(A)$. Set $v = \pi^m(u)$ and $w = \pi^k(w)$. If $\Gamma^{S^k(A)}_M(v) \subseteq S^k(w)$, then

$$\begin{aligned} (If)(u) &= \sum_{v' \in S^k(A)} f(\pi^k(v')) \varphi_{v'}(u) = \sum_{v' \in \Gamma^{S^k(A)}_M(v)} f(\pi^k(v')) \varphi_{v'}(u) \\ &= \sum_{v' \in \Gamma^{S^k(A)}_M(v)} f(w) \varphi_{v'}(u) = f(w). \end{aligned}$$

If $v \in S^k(w) \backslash B_{M,k}(w)$, then $\Gamma^{S^k(A)}_M(v) \subseteq \Gamma_M(v) \subseteq S^k(w)$. So the above equality suffices for (5.3). ∎

Next we introduce *p-Poincaré constants*. In fact, there are two kinds of Poincaré constants $\lambda_{p,m}(A)$ and $\widetilde{\lambda}_{p,m}(A)$ but they are almost the same in view of (5.4).

Definition 5.4. Define $\mu(w) = \mu(K_w)$ for $w \in T$. For $A \subseteq T_n$, define $\mu(A) = \sum_{w \in A} \mu(w)$ and $\mu_A \colon A \to [0, \infty)$ by

$$\mu_A(w) = \frac{\mu(w)}{\mu(A)}$$

for $w \in A$. For $f \in \ell(A)$, define

$$(f)_A = \sum_{u \in A} f(u)\mu_A(u)$$

and

$$\|f\|_{p,\mu_A} = \Big(\sum_{u \in A} |f(u)|^p \mu_A(u)\Big)^{\frac{1}{p}}.$$

Moreover, define

$$\lambda_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{\inf_{c \in \mathbb{R}}(\|f - c\chi_{S^m(A)}\|_{p,\mu_{S^m(A)}})^p}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)}$$

and

$$\widetilde{\lambda}_{p,m}(A) = \sup_{f \in \ell(S^m(A))} \frac{(\|f - (f)_{S^m(A)}\|_{p,\mu_{S^m(A)}})^p}{\mathcal{E}^{n+m}_{p,S^m(A)}(f)}.$$

Remark. By Lemma B.2, it follows that

$$\Big(\frac{1}{2}\Big)^p \widetilde{\lambda}_{p,m}(A) \le \lambda_{p,m}(A) \le \widetilde{\lambda}_{p,m}(A). \tag{5.4}$$

Using the previous lemmas, we have a relation between Poincaré and conductance constants as follows.

Lemma 5.5 ([36, Proposition 2.10]). *Let $p \ge 1$ and let $A \subseteq T_n$. For any $m \ge 1$ and $k \ge Mm_0$,*

$$\max_{w \in A} \#(\partial S^k(w)) \max_{v \in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A))\lambda_{p,k+m}(A) \ge c_{5.5}\lambda_{p,0}(A),$$

where the constant $c_{5.5} = c_{5.5}(\gamma, m_0, p, L_*, M)$ depends only on γ, m_0, p, L_* and M.

Proof. Choose $f_0 \in \ell(A)$ such that $\mathcal{E}^n_{p,A}(f_0) = 1$ and

$$\big(\min_{c \in \mathbb{R}} \|f_0 - c\chi_A\|_{p,\mu_A}\big)^p = \lambda_{p,0}(A).$$

Letting $f = I_{A,k,m} f_0$, by Lemma 5.3, we see that

$$\mathcal{E}^{n+k+m}_{p,S^{m+k}(A)}(f) \leq c_{5.3} \max_{w\in A} \#(\partial S^k(w)) \max_{v\in S^k(A)} \mathcal{E}_{M,p,m}(v, S^k(A)). \tag{5.5}$$

On the other hand, by (5.3) and (2.8),

$$\begin{aligned}
&\frac{1}{\mu(A)} \sum_{v\in S^{k+m}(A)} |f(v)-c|^p \mu(v) \\
&\quad = \frac{1}{\mu(A)} \sum_{w\in A} \sum_{v\in S^m(S^k(w))} |f(v)-c|^p \mu(v) \\
&\quad \geq \frac{1}{\mu(A)} \sum_{w\in A} \sum_{v\in S^m(S^k(w)\setminus B_{M,k}(w))} |f_0(w)-c|^p \mu(v) \\
&\quad \geq \gamma^{m_0 M} \frac{1}{\mu(A)} \sum_{w\in A} |f_0(w)-c|^p \mu(w) \geq \gamma^{m_0 M} \lambda_{p,0}(A).
\end{aligned}$$

This and (5.5) yield the desired inequality. ∎

5.2 Relations of constants

In this section, we will establish relations between conductance, neighbor disparity, and Poincaré constants towards a proof of Theorem 3.33. As in the previous section we fix a covering system $\mathcal{J}$ with covering numbers (N_T, N_E) and we write $\sigma_{p,m}$ and $\sigma_{p,m,n}$ in place of $\sigma^{\mathcal{J}}_{p,m}$ and $\sigma^{\mathcal{J}}_{p,m,n}$, respectively.

Definition 5.6. For $w \in T$ and $n \geq 0$, define

$$\xi_n(w) = \max_{v\in S^n(w)} \frac{\mu(v)}{\mu(w)}$$

First, we consider a relation between Poincaré and neighbor disparity constants.

Lemma 5.7 ([36, Proposition 2.13 (1)]). *Let $p \geq 1$. For any $w \in T$ and $n, m \geq 1$,*

$$\widetilde{\lambda}_{p,n+m}(w) \leq 2^{p-1}\big(\xi_n(w) \max_{v\in S^n(w)} \widetilde{\lambda}_{p,m}(v) + L_* c_{2.27} \widetilde{\lambda}_{p,n}(w)\sigma_{p,m,n+|w|}\big).$$

Proof. By Theorem A.3, for any $f \in \ell(S^{n+m}(w))$,

$$\begin{aligned}
&\frac{1}{\mu(w)} \sum_{u\in S^{n+m}(w)} |f(u)-(f)_{S^{n+m}(w)}|^p \mu(v) \\
&\quad \leq \frac{C_p}{\mu(w)} \sum_{v\in S^n(w)} \sum_{u\in S^m(v)} (|f(u)-(f)_{S^m(v)}|^p \\
&\qquad\qquad + |(f)_{S^m(v)} - (f)_{S^{n+m}(w)}|^p)\mu(u),
\end{aligned}$$

where $C_p = 2^{p-1}$ for $p \neq 2$ and $C_2 = 1$. Examining the first half of the above inequality, we obtain

$$\begin{aligned}\frac{1}{\mu(w)} \sum_{v \in S^n(w)} \sum_{u \in S^m(v)} |f(u) - (f)_{S^m(v)}|^p \mu(u) \\ \leq \sum_{v \in S^n(w)} \frac{\mu(v)}{\mu(w)} \widetilde{\lambda}_{p,m}(v) \mathcal{E}^{|w|+n+m}_{p,S^m(v)}(f) \leq \xi_n(w) \\ \times \max_{v \in S^n(w)} \widetilde{\lambda}_{p,m}(v) \mathcal{E}^{|w|+n+m}_{p,S^{n+m}(w)}(f).\end{aligned}$$

For the other half, by Lemma 2.27,

$$\begin{aligned}&\frac{1}{\mu(w)} \sum_{v \in S^n(w)} \sum_{u \in S^m(v)} |(f)_{S^m(v)} - (f)_{S^{n+m}(w)}|^p \mu(u) \\ &= \sum_{v \in S^n(w)} \frac{\mu(v)}{\mu(w)} |(P_{n+|w|,m} f)(v) - (P_{n+|w|,m} f)_{S^n(w)}|^p \\ &\leq \widetilde{\lambda}_{p,n}(w) \mathcal{E}^{|w|+n}_{p,S^n(w)}(P_{n+|w|,m} f) \\ &\leq L_* \widetilde{\lambda}_{p,n}(w) c_{2.27} \sigma_{p,m,n+|w|} \mathcal{E}^{n+m+|w|}_{p,S^{n+m}(w)}(f).\end{aligned}$$

Combining all, we see

$$\begin{aligned}\widetilde{\lambda}_{p,n+m}(w) \leq C_p \Big(\xi_n(w) \max_{v \in S^n(w)} \widetilde{\lambda}_{p,m}(v) \\ + L_* c_{2.27} \widetilde{\lambda}_{p,n}(w) \sigma_{p,m,n+|w|}(v, v') \Big).\end{aligned}$$

■

Definition 5.8. Define

$$\overline{\lambda}_{p,m} = \sup_{w \in T} \widetilde{\lambda}_{p,m}(w).$$

By Theorem 6.7, $\overline{\lambda}_{p,m}$ is finite for any $m \geq 1$.

Making use of Lemma 5.7, we have the following inequality.

Lemma 5.9. *Define*

$$\xi_n = \sup_{w \in T} \xi_n(w).$$

Then

$$\overline{\lambda}_{p,n+m} \leq 2^{p-1} \big(\xi_n \overline{\lambda}_{p,m} + L_* c_{2.27} \overline{\lambda}_{p,n} \sigma_{p,m} \big) \tag{5.6}$$

for any $n, m \geq 1$.

Remark. By Lemma 2.13, μ is exponential, so that there exist $\xi \in (0, 1)$ and $c > 0$ such that

$$\xi_n \leq c \xi^n$$

for any $n \geq 1$.

Next, we examine the relationship between the conductance and Poincaré constants.

Lemma 5.10. *For any $w \in T$, $l, m \geq 1$ and $k \geq m_0 M_0$,*

$$\overline{D}_k \mathcal{E}_{M_*,p,m,|w|+k+l} \widetilde{\lambda}_{p,k+m+l}(w) \geq c_{5.10} \widetilde{\lambda}_{p,l}(w), \tag{5.7}$$

where $\overline{D}_k = \max_{v \in T \setminus \{\phi\}} \#(\partial S^k(v))$ and the constant $c_{5.10} = 2^{-p} c_{5.5}$ depends only on γ, m_0, p, L_* and M_0. In particular,

$$\overline{D}_k \mathcal{E}_{M_*,p,m} \overline{\lambda}_{p,k+m+l} \geq c_{5.10} \overline{\lambda}_{p,l} \tag{5.8}$$

Proof. Applying Lemma 5.5 with $M = M_0$ and $A = S^l(w)$, we obtain

$$\overline{D}_k \max_{v \in S^{k+l}(w)} \mathcal{E}_{M_0,p,m}(v, S^{k+l}(w)) \lambda_{p,k+m}(S^l(w)) \geq c_{5.5} \lambda_{p,0}(S^l(w)).$$

Lemma 2.18 shows

$$\mathcal{E}_{M_0,p,m}(v, S^{k+l}(w)) \leq \mathcal{E}_{M_*,p,m}(v, T_{|w|+k+l}) \leq \mathcal{E}_{M_*,p,m,|w|+k+l}.$$

Moreover, $\lambda_{p,k+m}(S^l(w)) = \lambda_{p,k+m+l}(w)$ and $\lambda_{p,0}(S^l(w)) = \lambda_{p,l}(w)$ by definition. So letting $c_{5.10} = 2^{-p} c_{5.5}$, we obtain (5.7). ∎

The next theorem is one of the main results of this section.

Theorem 5.11. *Assume that $p > 1$. If either*

$$\lim_{n \to \infty} \xi_n \mathcal{E}_{p,n-m_0 M_0} = 0 \tag{5.9}$$

or

$$\lim_{n \to \infty} \xi_n \overline{D}_{n-1} = 0, \tag{5.10}$$

then there exists $C > 0$ such that

$$\overline{\lambda}_{p,m} \leq C \sigma_{p,m}, \tag{5.11}$$

$$\overline{\lambda}_{p,m+n} \leq C \overline{\lambda}_{p,n} \sigma_{p,m} \tag{5.12}$$

and

$$(\mathcal{E}_{M_*,p,n})^{-1} \overline{\lambda}_{p,m} \leq C \overline{\lambda}_{p,m+n} \tag{5.13}$$

for any $n, m \geq 1$.

Remark. Inequalities (5.12) and (5.13) correspond to [36, (2.4)] and [36, (2.3)], respectively.

Unlike (5.9), (5.10) does not depend on p. So, once (5.10) holds, then we have (5.11), (5.12) and (5.13) for any $p > 1$. See Proposition 5.12 after the proof for more discussion on (5.10).

Proof. For ease of notation, we write $\overline{\lambda}_m = \overline{\lambda}_{p,m}$, $\sigma_m = \sigma_{p,m}$ and $\mathcal{E}_{M_*,p,m} = \mathcal{E}_m$. By (5.8), if $n > k \geq m_0 M_0$, then

$$\overline{D}_k \mathcal{E}_{n-k} \overline{\lambda}_{n+m} \geq c_{5.10} \overline{\lambda}_m. \tag{5.14}$$

This and (5.6) show

$$\overline{\lambda}_{n+m} \leq 2^{p-1}((c_{5.10})^{-1} \overline{D}_k \mathcal{E}_{n-k} \xi_n \overline{\lambda}_{n+m} + L_* c_{2.27} \overline{\lambda}_n \sigma_m). \tag{5.15}$$

Suppose that (5.9) holds. Let $k = m_0 M_0$. Then there exists n_0 such that, for any $n \geq n_0$,

$$2^{p-1}(c_{5.10})^{-1} \overline{D}_{m_0 M_0} \mathcal{E}_{n-m_0 M_0} \xi_n \leq \frac{1}{2}$$

and hence by (5.15),

$$\overline{\lambda}_{n+m} \leq 2^p L_* c_{2.27} \overline{\lambda}_n \sigma_m. \tag{5.16}$$

Next suppose that (5.10) holds. Then there exists n_0 such that, for any $n \geq n_0$,

$$2^{p-1}(c_{5.10})^{-1} \overline{D}_{n-1} \mathcal{E}_1 \xi_n \leq \frac{1}{2},$$

so that we have (5.16) as well. Thus we have seen that if either (5.9) or (5.10) holds, then there exists n_0 such that (5.16) holds for any $n \geq n_0$.

Now, let $n_* = \max\{m_0 M_0 + 1, n_0\}$. Then by (5.14) and (5.16),

$$c_{5.10}(\overline{D}_{m_0 M_0})^{-1}(\mathcal{E}_{p,n_*-m_0 M_0})^{-1} \overline{\lambda}_m \leq \overline{\lambda}_{n_*+m} \leq 2^p L_* c_{2.27} \overline{\lambda}_{n_*} \sigma_m$$

for any $m \geq 1$. This immediately implies (5.11). Using this and (3.18), we have

$$\overline{\lambda}_{m+n} \leq \sigma_{m+n} \leq C \sigma_m \sigma_n.$$

Therefore, for any $m \geq 1$ and $n \in \{1, \dots, n_0\}$,

$$\frac{\overline{\lambda}_{m+n}}{\overline{\lambda}_n \sigma_{p,m}} \leq C \frac{\sigma_n}{\overline{\lambda}_n} \leq C \max_{n=1,\dots,n_0} \frac{\sigma_n}{\overline{\lambda}_n}.$$

So we have verified (5.12) for any $n, m \geq 1$. Letting $k = m_0 M_0$ in (5.8) and using (5.12), we obtain (5.13) as well. ■

The following proposition gives a geometric sufficient condition for (5.10).

Proposition 5.12. *Suppose that Assumption 2.15 holds. Assume that μ is α_H-Ahlfors regular with respect to the metric d. If there exist $\tilde{\alpha} < \alpha_H$ and $c > 0$ such that*

$$\#(\partial S^m(w)) \leq c r^{-m\tilde{\alpha}}$$

for any $w \in T$ and $m \geq 0$, then (5.10) holds.

Under the assumptions of Proposition 5.12, $\alpha_H = \dim_H(K, d)$, which is the Hausdorff dimension of (K, d), while $\dim_H(B_w, d) \le \tilde{\alpha}$ for any $w \in T$. So, roughly speaking, Proposition 5.12 says that if

$$\dim_H(K, d) > \sup_{w \in T} \dim_H(B_w, d),$$

then (5.10) is satisfied. By this proposition, one can verify (5.10) for generalized Sierpiński carpets for example.

Proof. By [34, Theorem 3.1.21], there exist $c_1, c_2 > 0$ such that

$$c_1 r^{\alpha_H |w|} \le \mu(K_w) \le c_2 r^{\alpha_H |w|}$$

for any $w \in T$. Hence $\xi_n \le c r^{\alpha_H n}$, while $\overline{D}_n \le r^{-\tilde{\alpha} n}$. ∎

To conclude this section, we present a lemma providing a control of the difference of a function on T_n through $\mathcal{E}_p^n(f)$ and the Poincaré constant.

Lemma 5.13. *For any $w \in T$, $n \ge m \ge 1$, $f \in \ell(S^n(w))$, and $u, v \in S^n(w)$, if $\pi^{n-m}(u) = \pi^{n-m}(v)$, then*

$$|f(u) - f(v)| \le 2\gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \sum_{i=1}^{n-m} (\overline{\lambda}_{p,i})^{\frac{1}{p}}.$$

Proof. Let $u \in S^n(w)$. Set

$$S_i(u) = S^i(\pi^i(u))$$

for $u \in S^n(w)$ and $i = 0, 1, \ldots, n$. By Lemma B.3 and (2.5), for any $k = 1, \ldots, n$,

$$\begin{aligned}
|f(u) - (f)_{S_k(u)}| &\le \sum_{i=1}^{k} |(f)_{S_{i-1}(u)} - (f)_{S_i(u)}| \\
&\le \sum_{i=1}^{k} \left(\frac{\mu(\pi^i(u))}{\mu(\pi^{i-1}(u))}\right)^{\frac{1}{p}} \left(\widetilde{\lambda}_{s,p,i}(\pi^i(u)) \mathcal{E}_{p,S_i(u)}^{n+w}(f)\right)^{\frac{1}{p}} \\
&\le \gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \sum_{i=1}^{k} \left(\widetilde{\lambda}_{p,i}(\pi^i(u))\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}
|f(u) - f(v)| &\le |f(u) - (f)_{S_{n-m}(u)}| + |(f)_{S_{n-m}(v)} - f(v)| \\
&\le \gamma^{-\frac{1}{p}} \mathcal{E}_{p,S^n(w)}^{n+|w|}(f)^{\frac{1}{p}} \left(\sum_{i=1}^{n-m} \left((\widetilde{\lambda}_{p,i}(\pi^i(v)))^{\frac{1}{p}} + (\widetilde{\lambda}_{p,i}(\pi^i(w)))^{\frac{1}{p}} \right) \right). \quad ∎
\end{aligned}$$

5.3 Proof of Theorem 3.33

Finally, we are going to give a proof of the "if" part of Theorem 3.33. Recall that by (3.19), there exist $c > 0$ and $\alpha \in (0, 1)$ such that

$$\mathcal{E}_{M_*,p,m} \leq c\alpha^m$$

for any $m \geq 0$. Then since $\xi_n \leq 1$, (5.9) is satisfied and hence (5.11), (5.12) and (5.13) turn out to be true.

As in the previous sections, a set $\mathcal{J}$ is a covering system with covering numbers (N_T, N_E). Furthermore, recall that by the definition of covering systems,

$$\sup_{A \in \mathcal{J}} \#(A) < \infty.$$

We denote the above supremum by N_c.

Lemma 5.14. *Set $\rho = \alpha^{\frac{1}{p}}$. There exists $C > 0$ such that for any $w \in T$, $k, m \geq 1$ with $m \geq k$ and $f \in \ell(S^m(w))$, if $u, v \in S^m(w)$ and $\pi^{m-k}(u) = \pi^{m-k}(v)$, then*

$$|f(u) - f(v)| \leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} \mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}}.$$

Proof. By (5.13),

$$\overline{\lambda}_{p,i} \leq C\overline{\lambda}_{p,m}\mathcal{E}_{p,m-i} \leq C\overline{\lambda}_{p,m}\rho^{p(m-i)}. \tag{5.17}$$

Using this and applying Lemma 5.13, we have

$$|f(u) - f(v)| \leq C\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}} \sum_{i=1}^{m-k} (\overline{\lambda}_{p,i})^{\frac{1}{p}} C\mathcal{E}^{|w|+m}_{p,S^m(w)}(f)^{\frac{1}{p}} (\overline{\lambda}_{p,m})^{\frac{1}{p}} \sum_{i=k}^{m-1} \rho^i. \ \blacksquare$$

Lemma 5.15. *Set $\varepsilon = (N_c)^{-\frac{2}{p}}$. There exist $n_* \geq 1$ and $m_* \geq n_*$ such that if $m \geq m_*$, then there exist $w \in T$ and $f \in \ell(S^m(w))$ such that*

$$\min_{u \in S^{m-n_*}(y_1)} f(u) - \max_{u \in S^{m-n_*}(y_2)} f(u) \geq \frac{1}{8}\varepsilon$$

for some $y_1, y_2 \in S^{n_*}(w)$ and

$$\mathcal{E}^{|w|+m}_{p,S^m(w)}(f) \leq \frac{2}{\sigma_{p,m}}.$$

Proof. Choose $A \in \mathcal{J}$ such that $\sigma_{p,m}(A) \geq \frac{1}{2}\sigma_{p,m}$. Suppose that $A \subseteq T_n$ and choose $f \in \ell(S^m(A))$ such that $\mathcal{E}^n_{p,A}(P_{n,m}f) = 1$ and

$$\mathcal{E}^{n+m}_{p,S^m(A)}(f) = \frac{1}{\sigma_{p,m}(A)}. \tag{5.18}$$

Claim 1. *There exists* $c_1 > 0$*, which is independent of* m *and* A*, such that if* $u_1, u_2 \in S^m(A)$ *and* $(u_1, u_2) \in E^*_{n+m}$*, then*

$$|f(u_1) - f(u_2)| \leq c_1 \rho^m. \tag{5.19}$$

Proof. By (5.11), (5.17) and (5.18), we have

$$|f(u_1) - f(u_2)|^p \leq \mathcal{E}^{n+m}_{p,S^m(A)}(f) = \frac{1}{\sigma_{p,m}(A)} \leq \frac{2}{\sigma_{p,m}} \leq \frac{C}{\overline{\lambda}_{p,m}} \leq C\rho^{pm}.$$

This proves the claim. □

Claim 2. *There exists* $c_2 > 0$*, which is independent of* m *and* A*, such that if* $u_1, u_2 \in A$ *and* $\pi^{m-k}(u_1) = \pi^{m-k}(u_2)$ *for some* $k \in \{1, \ldots, m\}$*, then* $|f(u_1) - f(u_2)| \leq c_2 \rho^k$*.*

Proof. It follows that $u_1, u_2 \in S^m(w)$ for some $w \in A$. Using Lemma 5.14, we obtain

$$\begin{aligned} |f(u_1) - f(u_2)| &\leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} \mathcal{E}^{n+m}_{p,S^m(w)}(f)^{\frac{1}{p}} \\ &\leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} \mathcal{E}^{n+m}_{p,S^m(A)}(f)^{\frac{1}{p}} \leq C\rho^k (\overline{\lambda}_{p,m})^{\frac{1}{p}} (\sigma_{p,m})^{-\frac{1}{p}}. \end{aligned}$$

Now (5.11) immediately shows the claim. □

Since $\#(A) \leq N_c$, it follows that $\#(E^*_n(A)) \leq (N_c)^2$. Therefore, the fact that $\mathcal{E}^n_{p,A}(P_{n,m} f) = 1$ shows that there exists $(w_1, w_2) \in E^*_n(A)$ such that

$$|(f)_{S^m(w_1)} - (f)_{S^m(w_2)}|^p \geq (N_c)^{-2} = \varepsilon^p.$$

Exchanging f by $-f$ if necessary, we may assume that

$$(f)_{S^m(w_1)} - (f)_{S^m(w_2)} \geq \varepsilon$$

without loss of generality. Define

$$\begin{aligned} n_* &= \inf\{n \mid n \in \mathbb{N}, \varepsilon \geq 16 c_2 \rho^n\}, \\ m_* &= \max\{n_*, \inf\{m \mid m \in \mathbb{N}, \varepsilon \geq 2 c_1 \rho^m\}\}. \end{aligned}$$

Hereafter, we assume that $m \geq m_*$.

Claim 3. *For* $i = 1$ *or* 2*, there exist* $u_1, u_2 \in S^m(w_i)$ *such that* $u_2 \in \partial S^m(w_i)$ *and*

$$|f(u_1) - f(u_2)| \geq \frac{1}{4}\varepsilon.$$

Proof. Choose $v_{11}, v_{12} \in S^m(w_1)$ and $v_{21}, v_{22} \in S^m(w_2)$ such that

$$f(v_{11}) \geq (f)_{S^m(w_1)}, \quad f(v_{22}) \leq (f)_{S^m(w_2)}, \quad (v_{12}, v_{21}) \in E^*_{|w_1|+m}.$$

Since

$$f(v_{11}) - f(v_{12}) + f(v_{12}) - f(v_{21}) + f(v_{21}) - f(v_{22}) = f(v_{11}) - f(v_{22}) \geq \varepsilon,$$

(5.19) shows that, for either $i = 1$ or 2,

$$|f(v_{i1}) - f(v_{i2})| \geq \frac{1}{2}(\varepsilon - c_1\rho^m) \geq \frac{1}{4}\varepsilon.$$

Letting $u_1 = v_{i1}$ and $u_2 = v_{i2}$, we have the claim. □

Let $w = w_i$ where i is chosen in Claim 3. Exchanging f by $-f$ if necessary, we see that there exists $u_1 \in S^m(w)$ and $u_2 \in \partial S^m(w)$ such that

$$f(u_1) - f(u_2) \geq \frac{1}{4}\varepsilon.$$

Set $y_i = \pi^{m-n_*}(u_i)$ for $i = 1, 2$. Note that $y_i \in S^{n_*}(w)$. By Claim 2,

$$\min_{u \in S^{m-n_*}(y_1)} f(u) - \max_{u \in S^{m-n_*}(y_2)} f(u) \geq \frac{1}{4}\varepsilon - 2c_2\rho^{n_*} \geq \frac{1}{8}\varepsilon.$$ ■

Proof of Theorem 3.33. Let $m \geq m_*$. Then there exist $w \in T$ and $f \in \ell(S^m(w))$ satisfying the conclusions of Lemma 5.15. Set $c_0 = \max_{u \in S^{m-n_*}(y_2)} f(u)$. Define

$$h(v) = \begin{cases} 1 & \text{if } 8(f(v) - c_0) \geq \varepsilon, \\ 8\varepsilon^{-1}(f(v) - c_0) & \text{if } 0 < 8(f(v) - c_0) < \varepsilon, \\ 0 & \text{if } 8(f(v) - c_0) < 0 \end{cases}$$

for any $v \in S^m(w)$. Then $h|_{S^{n_*}(y_1)} \equiv 1$, $h|_{S^{n_*}(y_2)} \equiv 0$ and

$$\mathcal{E}_{p,m-n_*}(y_1, y_2, S^{n_*}(w)) \leq \mathcal{E}^{|w|+m}_{p,S^m(w)}(h) \leq 8^p\varepsilon^{-p}\mathcal{E}^{|w|+m}_{p,S^m(w)}(f) \leq \frac{2^{3p+1}(N_c)^2}{\sigma_{p,m}}.$$

By (3.20),

$$\mathcal{E}_{M_*,p,m-n_*} \leq c(n_*)\mathcal{E}_{p,m-n_*}(y_1, y_2, S^{n_*}(w)) \leq \frac{c(n_*)2^{3p+1}(N_c)^2}{\sigma_{p,m}}.$$

Making use of the sub-multiplicative property of $\mathcal{E}_{M_*,p,n}$, we have

$$\mathcal{E}_{M_*,p,m} \leq C\mathcal{E}_{M_*,p,n_*}\mathcal{E}_{M_*,p,m-n_*}.$$

Finally, the last two inequalities show

$$\mathcal{E}_{M_*,p,m}\sigma_{p,m} \leq C\mathcal{E}_{M_*,p,n_*}c(n_*)2^{3p+1}(N_c)^2$$

for any $m \geq m_*$, where the right-hand side is independent of m. Thus K is p-conductively homogeneous. ■