Chapter 6

Miscellanea

6.1 Uniformity of constants

In this section, we study the uniformity of conductance, Poincaré and neighbor disparity constants with respect to the structure of graphs.

Definition 6.1. (1) A pair (V, E) is called a (non-directed) *graph* if and only if V is a countable set and $E \subseteq V \times V$ such that $(u, v) \in V$ if and only if $(v, u) \in V$. For a graph (V, E), V is called the *vertices* and E is called the *edges*.

(2) Let (V, E) and (V', E') be graphs. A bijective map $\iota\colon V \to V'$ is called an *isomorphism* between (V, E) and (V', E') if "$(w, v) \in E$" is equivalent to "$(\iota(w), \iota(v)) \in E'$" for any $u, v \in V$.

(3) Let (V, E) be a graph. For $p > 0$ and $f \in \ell(V)$, define $\mathcal{E}_p^{(V,E)}(f) \in [0, \infty]$ by

$$\mathcal{E}_p^{(V,E)}(f) = \frac{1}{2} \sum_{(u,v)\in E} |f(u) - f(v)|^p.$$

(4) Let (V, E) be a graph and let $A, B \subseteq V$ with $A \cap B = \emptyset$. Define

$$\mathcal{E}_p^{(V,E)}(A, B) = \inf\{\mathcal{E}_p^{(V,E)}(f) \mid f \in \ell(V), f|_A \equiv 1, f|_B \equiv 0\}.$$

In this section, we always identify isomorphic graphs.

First, we study the uniformity of conductance constants.

Definition 6.2. For $L, N \geq 1$, define

$$\begin{aligned}\mathcal{G}_{\mathcal{E}}(L, N) = \{(V, E) \mid\ &(V, E) \text{ is a connected graph, } V = \{\mathbf{t}, \mathbf{b}\} \cup V_*, \text{where}\\ &\text{the union is a disjoint union and } \mathbf{t} \neq \mathbf{b},\ 1 \leq \#(V_*) \leq LN,\\ &\#(\{v \mid v \in E, (w, v) \in E\}) \leq L \text{ for any } w \in V_*\}.\end{aligned}$$

Since $\mathcal{G}_{\mathcal{E}}(L, N)$ is a finite set up to graph isomorphisms, we have the following theorem.

Theorem 6.3. *For any $L, N \geq 1$ and $p > 0$,*

$$0 < \inf_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\}) \leq \sup_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\}) < \infty.$$

Definition 6.4. Define

$$\underline{c}_{\mathcal{E}}(L, N, p) = \inf_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\}, \{\mathbf{b}\})$$

and

$$\overline{c}_{\mathcal{E}}(L,N,p) = \sup_{(V,E)\in\mathcal{G}_{\mathcal{E}}(L,N)} \mathcal{E}_p^{(V,E)}(\{\mathbf{t}\},\{\mathbf{b}\}).$$

Next we consider Poincaré constants.

Definition 6.5. For $L \geq 1$ and $N \geq 2$, define

$$\mathcal{G}(L,N) = \{(V,E) \mid (V,E) \text{ is a connected graph, } 2 \leq \#(V) \leq N,$$
$$\#(\{v \mid v \in V,\ (w,v) \in E,\}) \leq L \text{ for any } w \in V\}.$$

For a connected graph (V,E), define

$$\mathcal{P}(V,E) = \Big\{\mu \mid \mu \in V \to [0,1], \sum_{v\in V}\mu(v) = 1\Big\}.$$

For $\mu \in \mathcal{P}(V,E)$, define

$$(f)_\mu = \sum_{v\in V} f(v)\mu(v),$$

for $f \in \ell(V)$,

$$\widetilde{\lambda}_{p,\mu}^{(V,E)} = \sup_{f\in\ell(V)} \frac{\sum_{v\in V}|f-(f)_\mu|^p\mu(v)}{\mathcal{E}_p^{(V,E)}(f)}$$

for $p > 0$.

Lemma 6.6. *Let (V,E) be a connected finite graph. Then for any $p \geq 1$,*

$$0 < \inf_{\mu\in\mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} \leq \sup_{\mu\in\mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} < \infty.$$

Proof. Write $\mathcal{E}_p = \mathcal{E}_p^{(V,E)}$. For any $p \geq 1$,

$$|(f)_\mu| + \mathcal{E}_p(f)^{\frac{1}{p}}$$

is a norm on $\ell(V)$. Therefore, if

$$\mathcal{F}_\mu = \{f \mid f \in \ell(V), \mathcal{E}_p(f) = 1, (f)_\mu = 0\},$$

then $\mathcal{F}_\mu$ is a compact subset of $\ell(V)$. Fix $\mu_* \in \mathcal{P}(V,E)$ and set $\mathcal{F} = \mathcal{F}_{\mu_*}$. For any $f \in \ell(V)$ with $\mathcal{E}_p(f) \neq 0$, define

$$f_* = \mathcal{E}_p(f)^{-\frac{1}{p}}(f-(f)_{\mu_*}).$$

Then $f_* \in \mathcal{F}$ and

$$\frac{\sum_{v\in V}|f(v)-(f)_\mu|^p\mu(v)}{\mathcal{E}_p(f)} = \sum_{v\in V}|f_*(v)-(f_*)_\mu|^p\mu(v).$$

Hence letting

$$F(\mu, f_*) = \sum_{v \in V} |f_*(v) - (f_*)_\mu|^p \mu(v),$$

we see that

$$\widetilde{\lambda}_{p,\mu}^{(V,E)} = \sup_{f_* \in \mathcal{F}} F(\mu, f_*).$$

Since $\mathcal{P}(V, E) \times \mathcal{F}$ is compact and $F(\mu, f_*)$ is continuous on $\mathcal{P}(V, E) \times \mathcal{F}$, it follows that

$$\begin{aligned} 0 &< \inf_{\mu \in \mathcal{P}(V,E), f_* \in \mathcal{F}} F(\mu, f_*) \le \inf_{\mu \in \mathcal{P}} \widetilde{\lambda}_{p,\mu}^{(V,E)} \le \sup_{\mu \in \mathcal{P}(V,E)} \lambda_{p,\mu}^{(V,E)} \\ &< \sup_{\mu \in \mathcal{P}(V,E), f_* \in \mathcal{F}} F(\mu, f_*) < \infty. \end{aligned}$$

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Since $\mathcal{G}(L, N)$ is a finite set, the above lemma implies the following theorem.

Theorem 6.7. *For $p \ge 1$,*

$$0 < \inf_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} \le \sup_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)} < \infty.$$

Definition 6.8. Define

$$\underline{c}_\lambda(p, L, N) = \inf_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)}$$

and

$$\overline{c}_\lambda(p, L, N) = \sup_{(V,E) \in \mathcal{G}(L,N), \mu \in \mathcal{P}(V,E)} \widetilde{\lambda}_{p,\mu}^{(V,E)}.$$

Finally, we study neighbor disparity constants.

Definition 6.9. Define

$$\begin{aligned} \mathcal{G}_\sigma(L, N_1, N_2) = \big\{ &(V, E_1, \{V_i\}_{i=1}^n, E_2) \mid (V, E_1) \in \mathcal{G}(L, N_1), \\ &(\{1, \dots, n\}, E_2) \in \mathcal{G}(L, N_2), V_i \subseteq V \text{ and } V_i \neq \emptyset \\ &\text{for any } i = 1, \dots, n, V = \textstyle\bigcup_{i=1}^n V_i, V_i \cap V_j = \emptyset \text{ if } i \neq j \big\}. \end{aligned}$$

Let (V, E) be a graph and let $\mu \in \mathcal{P}(V, E)$. For $U \subseteq V$ and $f \in \ell(V)$, define

$$\mu(U) = \sum_{v \in U} \mu(v)$$

and

$$(f)_{U,\mu} = \frac{1}{\mu(U)} \sum_{v \in U} f(v) \mu(v)$$

if $\mu(U) > 0$. For $G = (V, E_1, \{V_i\}_{i=1}^n, E_2) \in \mathcal{G}_\sigma(L, N_1, N_2)$, $\mu \in \mathcal{P}(V, E)$ and $p \geq 1$, define $P_{G,\mu}: \ell(V) \to \ell(\{1, \ldots, n\})$ and $\sigma_{p,\mu}(G)$ by

$$(P_{G,\mu} f)(i) = (f)_{V_i,\mu}$$

for $f \in \ell(V)$ and

$$\sigma_{p,\mu}(G) = \sup_{f \in \ell(V), \mathcal{E}_p^{(V,E)}(f) \neq 0} \frac{\mathcal{E}_p^{(\{1,\ldots,n\},E_2)}(P_{G,\mu} f)}{\mathcal{E}_p^{(V,E)}(f)}.$$

Moreover, define

$$\mathcal{P}(G, \kappa) = \{\mu \mid \mu \in \mathcal{P}(V, E),\ \mu(V_i) \geq \kappa\mu(V_j) \text{ for any } i, j \in \{1, \ldots, n\}\}$$

for $\kappa \in (0, 1]$.

Theorem 6.10. *For any $p \geq 1$, $L, N_1, N_2 \geq 1$ and $\kappa \in (0, 1]$,*

$$\begin{aligned} 0 &< \inf\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\} \\ &\leq \sup\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\} < \infty. \end{aligned}$$

Proof. First fix

$$G = (V, E_1, \{V_i\}_{i=1}^n, E_2) \in \mathcal{G}_\sigma(L, N_1, N_2)$$

and fix

$$\mu_* \in \mathcal{P}(G, \kappa).$$

Define $\mathcal{F}$ as in the proof of Lemma 6.6. For any $f \in \ell(V)$, setting

$$f_* = \mathcal{E}_p(f)^{-\frac{1}{p}} \times (f - (f)_{\mu_*}),$$

we see that $f_* \in \mathcal{F}$ and

$$\frac{|(f)_{V_1,\mu} - (f)_{V_2,\mu}|^p}{\mathcal{E}_p(f)} = |(f_*)_{V_1,\mu} - (f_*)_{V_2,\mu}|^p$$

for any $\mu \in \mathcal{P}(G, \kappa)$. Let $F: \mathcal{F} \times \mathcal{P}(G, \kappa) \to \mathbb{R}$ by

$$F(f, \mu) = |(f)_{V_1,\mu} - (f)_{V_2,\mu}|.$$

Since F is continuous and $\mathcal{F} \times \mathcal{P}(G, \kappa)$ is compact,

$$\begin{aligned} 0 &< \inf_{\mu \in \mathcal{P}(G,\kappa), f \in \mathcal{F}} F(f, \mu) \leq \inf_{\mu \in \mathcal{P}(G,\kappa)} \sigma_{p,\mu}(G) \leq \sup_{\mu \in \mathcal{P}(G,\kappa), f \in \mathcal{F}} F(f, \mu) \\ &= \sup_{\mu \in \mathcal{P}(G,\kappa)} \sigma_{p,\mu}(G) < \infty. \end{aligned}$$

Now the desired statement follows by the fact that $\mathcal{G}_\sigma(L, N)$ is a finite set up to graph isomorphisms. ∎

Definition 6.11. Define

$$\underline{c}_\sigma(L, N_1, N_2, \kappa) = \inf\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\},$$
$$\overline{c}_\sigma(L, N_1, N_2, \kappa) = \sup\{\sigma_{p,\mu}(G) \mid G \in \mathcal{G}_\sigma(L, N_1, N_2), \mu \in \mathcal{P}(G, \kappa)\}.$$

6.2 Modification of the structure of a graph

In the original work of Kusuoka–Zhou [36], they used a subgraph of (T_n, E_n^*) to define their version of $\mathcal{E}_2^m$ in the case of the Sierpiński carpet. Namely, in our terminology, their subgraph is

$$E_n^1 = \{(u, v) \mid (u, v) \in E_1^*, \dim_H(K_v \cap K_u) = 1\}$$

and their energy is

$$\mathcal{E}_p^{1,n}(f) = \frac{1}{2} \sum_{(u,v) \in E_n^1} |f(u) - f(v)|^p$$

for $f \in \ell(T_n)$. (They only consider the case $p = 2$.) Our theory in this paper works well if we replace our energy $\mathcal{E}_p^n$ with Kusuoka–Zhou's energy $\mathcal{E}_p^{1,n}$ because they are uniformly equivalent, i.e., there exist $c_1, c_2 > 0$ such that

$$c_2 \mathcal{E}_p^n(f) \leq \mathcal{E}_p^{1,n}(f) \leq c_2 \mathcal{E}_p^n(f)$$

for any $n \geq 1$ and $f \in \ell(T_n)$. More generally, if we replace our graph (T_n, E_n^*) with a subgraph (T_n, E_n) satisfying conditions (A) and (B) below, all the results in this paper remain true except for changes in the constants.

(A) $G_n = (T_n, E_n)$ is a connected graph for each n having the following properties:
   - (i) If $(w, v) \in E_n$, then $K_w \cap K_v \neq \emptyset$.
   - (ii) If $(w, v) \in E_n$ for $n \geq 1$, then $\pi(w) = \pi(v)$ or $(\pi(w), \pi(v)) \in E_{n-1}$.
   - (iii) If $(w, v) \in E_n$ for $n \geq 1$, then there exist $w_1 \in S(w)$ and $v_1 \in S(v)$ such that $(w_1, w_2) \in E_{n+1}$.
   - (iv) For any $n \geq 0$ and $w, v \in T_n$ with $K_w \cap K_v \neq \emptyset$, there exist $w(0), \ldots, w(k) \in \Gamma_1(w)$ satisfying $w(0) = w, w(k) = v$ and $(w(i), w(i+1)) \in E_n$ for any $i = 0, \ldots, k-1$.

(B) For any $w \in T$, the graphs $(S^n(w), E_{n+|w|}^{S^n(w)})$ associated with the partition $T(w)$ of K_w satisfies the counterparts of conditions (i), (ii), (iii) and (iv) of (A).

Naturally, the graph (T_n, E_n^*) satisfies (A) and (B).

6.3 Open problems

In the final section, we gather some of open problems and future directions of our research.

1. *Regularity of $\mathcal{W}^p$ for $p \in [1, \dim_{AR}(K,d)]$*: As we have already mentioned, it is not known whether or not $C(K) \cap W^p$ is dense in $\mathcal{W}^p$ for $p \in [1, \dim_{AR}(K,d)]$. The first step should be to establish an elliptic Harnack principle for p-harmonic functions on approximating graphs and/or the limiting object $(\mathcal{W}^p, \widehat{\mathcal{E}}_p(\cdot) + \|\cdot\|_{p,\mu})$. Even in the case of $p = 2$, this problem is open except for the case of generalized Sierpiński carpets. The conjecture

$$\mathcal{W}^p \subseteq C(K) \quad \text{if and only if} \quad p > \dim_{ARC}(K,d)$$

in the introduction is closely related to this problem as well.

2. *Construction of p-form and p-Laplacian*: In this paper, we have constructed a p-energy $\widehat{\mathcal{E}}_p(f)$ but not a p-form $\widehat{\mathcal{E}}_p(f,g)$. Let

$$\Phi_p(t) = \begin{cases} |t|^{p-2}t & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

On a graph $G = (V, E)$, if we define

$$\mathcal{E}_p(f,g) = -\sum_{x \in V} (\Delta_p f)(x) g(x)$$

for $f, g \in \ell(V)$, where Δ_p is the p-Laplacian defined by

$$(\Delta_p f)(x) = \sum_{y \in V, (x,y) \in E} \Phi_p(f(y) - f(x)),$$

then it follows that

$$\mathcal{E}_p(f) = \frac{1}{2} \sum_{(x,y) \in E} |f(x) - f(y)|^p = \mathcal{E}_p(f,f).$$

As a natural counterpart, we expect to have a p-form $\widehat{\mathcal{E}}_p(f,g)$ which is linear in g, satisfies

$$\widehat{\mathcal{E}}_p(f) = \widehat{\mathcal{E}}_p(f,f)$$

for any $f \in \mathcal{W}^p$, and has an expression such as

$$\mathcal{E}_p(f,g) = -\int_K (\Delta_p f)(x) g(x) \mu(dx).$$

3. *Existence of p-energy measure*: In the case $p = 2$, there is the notion of energy measures associated with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is the form and $\mathcal{F}$ is the domain. Roughly speaking, the energy measure μ_f associated with $f \in \mathcal{F}$ is a positive Radon measure satisfying

$$\int_X u(x)\, d\mu_f(dx) = 2\mathcal{E}(uf, f) - \mathcal{E}(f^2, u)$$

for any $u \in \mathcal{F} \cap C_0(X)$. See [19] for details. So, what is a counterpart of this in the case of $\widehat{\mathcal{E}}^p$? Is there any natural measure μ_f for $f \in \mathcal{W}^p$ such that

$$\int_K d\mu_f(dx) = \widehat{\mathcal{E}}_p(f)?$$

For $\mathbb{R}^n$, the answer is yes and

$$\mu_f = |\nabla f|^p dx.$$

For the planar Sierpiński carpet, this problem has already been studied in [41]. However, we know almost nothing beyond those examples.

4. *Fractional Korevaar–Shoen type expression*: As we have already mentioned, a fractional Korevaar–Shoen type expression of $\mathcal{W}^p$ has already shown in [41] in the case of the planar Sierpiński carpet. Namely, we have

$$\mathcal{W}^p = \left\{ f \;\middle|\; f \in L^p(K, \mu),\ \overline{\lim_{r \downarrow 0}} \int_K \frac{1}{r^{\alpha_H}} \int_{B_{d_*}(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}}\, dx dy < \infty \right\},$$

and it is shown in [41] that $\beta_p > p$ for any $p > 1$. How about other cases? Suppose that Assumption 2.15 holds and μ is α_H-Ahlfors regular with respect to the metric d. Then we expect that

$$\beta_p = \alpha_H + \tau_p$$

and we know

$$\alpha_H + \tau_p \geq p$$

by [34, (4.6.14)]. Now our questions are:

- Do we have a fractional Korevaar–Shoen type expression as above?
- When does $\beta_p > p$ hold? (Apparently, if $K = [-1, 1]^L$, then $\beta_p = p$.)

A related question is: If $\beta_p = p$, then does $\mathcal{W}^p$ coincide with any of the Sobolev type spaces given by approaches using upper gradients?

5. *Without local symmetry*: In Sections 4.3, 4.4, 4.5 and 4.6, we have shown the conductive homogeneity of self-similar sets having local symmetry, which helped us to extend a path from one piece of K_w to neighbors by the reflection in its boundaries.

However, the local symmetry does not seem indispensable for having conductive homogeneity. Intuitively the essence should be the balance of conductances in different directions, for example, the vertical and the horizontal directions for square-based self-similar sets. Unfortunately, we have not had any example without local symmetry yet except for finitely ramified cases.