Appendices

A Basic inequalities

The next two lemmas can be deduced from the Hölder inequality.

Lemma A.1. *For $p \in (0, \infty)$,*

$$\left|\sum_{i=1}^{n} a_i\right|^p \leq \max\{1, n^{p-1}\} \sum_{i=1}^{n} |a_i|^p$$

for any $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{R}$.

Lemma A.2. *Let $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$,*

$$\left(\sum_{i=1}^{n} |a_i|^q\right)^{\frac{1}{q}} \leq \max\left\{1, n^{\frac{p-2}{p}}\right\}\left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}}.$$

The following fact implies the comparison of two types of Poincaré constants, $\lambda_{p,m}$ and $\widetilde{\lambda}_{p,m}$, as in (5.4).

Theorem A.3 ([9, Lemma 4.17]). *Let μ be a finite measure on a set X. Then for any $f \in L^p(X, \mu)$ and $c \in \mathbb{R}$,*

$$\|f - c\|_{p,\mu} \geq \frac{1}{2}\|f - (f)_\mu\|_{p,\mu},$$

where $\|\cdot\|_{p,\mu}$ is the L^p-norm with respect to μ and $(f)_\mu = \mu(X)^{-1} \int_X f d\mu$.

The following lemma is a discrete version of the above theorem.

Corollary A.4. *Let $(\mu_i)_{i=1,\dots,n} \in (0,1)^n$ with $\sum_{i=1}^{n} \mu_i = 1$. Then*

$$\sum_{i=1}^{n} |x - a_i|^p \mu_i \geq \left(\frac{1}{2}\right)^p \sum_{i=1}^{n} \left|\sum_{j=1}^{n} \mu_j a_j - a_i\right|^p \mu_i$$

for any $x, a_1, \dots, a_n \in \mathbb{R}$.

B Basic facts on p-energy

Let $G = (V, E)$ be a finite graph. For $A \subseteq V$, set $E_A = \{(x, y) \mid x, y \in A, (w, y) \in E\}$ and $G_A = (A, E_A)$.

Definition B.1. Let $\mu\colon V \to (0,\infty)$ and let $A \subseteq V$. Define $\mathrm{supp}(\mu) = \{x \mid x \in V, \mu(x) > 0\}$. Let $p > 0$. For $u \in \ell(V)$, define

$$\mathcal{E}_p^G(u) = \frac{1}{2}\sum_{(x,y)\in E} |u(x)-u(y)|^p,$$

$$\|u\|_{p,\mu} = \Big(\sum_{x\in V} |u(x)|^p \mu(x)\Big)^{\frac{1}{p}},$$

$$(u)_\mu = \frac{1}{\sum_{y\in V}\mu(y)} \sum_{x\in V} \mu(x)u(x)$$

and

$$\lambda_{p,\mu}^G = \sup_{u\in\ell(V),u\neq 0} \frac{(\min_{c\in\mathbb{R}} \|u - c\chi_V\|_{p,\mu})^p}{\mathcal{E}_p^G(u)},$$

where $\chi_V \in \ell(V)$ is the characteristic function of the set V.

For $A \subseteq U$, set $\mathcal{E}_p^A = \mathcal{E}_p^{G_A}$ and $\lambda_{p,\mu}^A = \lambda_{p,\mu|_A}^{G_A}$.

Lemma B.2. *Define*

$$\widetilde{\lambda}_{p,\mu}^G = \sup_{u\in\ell(V),u\neq 0} \frac{(\|u - (u)_\mu\chi_V\|_{p,\mu})^p}{\mathcal{E}_p^G(u)}.$$

Then

$$\Big(\frac{1}{2}\Big)^p \widetilde{\lambda}_{p,\mu}^G \le \lambda_{p,\mu}^G \le \widetilde{\lambda}_{p,\mu}^G.$$

Proof. By Corollary A.4,

$$\begin{aligned}\sum_{x\in V} |u(x)-(u)_\mu|^p\mu(x) &\ge \min_{c\in\mathbb{R}} \sum_{x\in V} |u(x)-c|^p\mu(x)\\ &\ge \Big(\frac{1}{2}\Big)^p \sum_{x\in V} |u(x)-(u)_\mu|^p\mu(x).\end{aligned}$$ ∎

Lemma B.3 ([36, Proposition 1.5 (2)]). *Let $p \in [1,\infty)$ and let $\mu\colon V \to (0,\infty)$. Assume that $A \subseteq B \subseteq V$. Then for any $u \in \ell(B)$,*

$$|(u)_A - (u)_B| \le \frac{1}{\mu(A)^{\frac{1}{p}}}\big(\widetilde{\lambda}_{p,\mu}^B \mathcal{E}_p^B(u)\big)^{\frac{1}{p}}.$$

Proof. By the Hölder inequality,

$$|(u)_A - (u)_B| \le \frac{1}{\mu(A)} \int_B \chi_A |u-(u)_B| d\mu \le \frac{1}{\mu(A)^{\frac{1}{p}}}\Big(\int_B |u-(u)_B|^p d\mu\Big)^{\frac{1}{p}}.$$ ∎

C Useful facts on combinatorial modulus

In this appendix, we have useful facts on combinatorial modulus. In particular, the last lemma, Lemma C.4, is a result on the comparison of moduli in two different graphs. This lemma plays a key role on several occasions in this paper.

Let V be a countable set and let $\mathcal{P}(V)$ be the power set of V. For $\rho\colon V \to [0,\infty)$ and $A \subseteq V$, define

$$L_\rho(A) = \sum_{x \in A} \rho(x).$$

For $\mathcal{U} \subseteq \mathcal{P}(V)$, define

$$\mathcal{A}(\mathcal{U}) = \{\rho \mid \rho\colon V \to [0,\infty),\ L_\rho(A) \geq 1 \text{ for any } A \in \mathcal{U}\}.$$

Moreover, for $\rho\colon V \to [0,\infty)$, define

$$M_p(\rho) = \sum_{x \in V} \rho(x)^p \quad \text{and} \quad \mathrm{Mod}_p(\mathcal{U}) = \inf_{\rho \in \mathcal{A}(\mathcal{U})} M_p(\rho).$$

Note that if $\mathcal{U} = \emptyset$, then $\mathcal{A}(\mathcal{U}) = [0,\infty)^V$ and $\mathrm{Mod}_p(\mathcal{U}) = 0$.

Lemma C.1. *Assume that $\mathcal{U}$ consists of finite sets. Then there exists $\rho_* \in \mathcal{A}(\mathcal{U})$ such that*

$$\mathrm{Mod}_p(\mathcal{U}) = M_p(\rho_*).$$

Proof. Choose $\{\rho_i\}_{i\geq 1} \subseteq \mathcal{A}(\mathcal{U})$ such that $M_p(\rho_i) \to \mathrm{Mod}_p(\mathcal{U})$ as $i \to \infty$. Since V is countable, there exists a subsequence $\{\rho_{n_j}\}_{j\geq 1}$ such that, for any $v \in V$, $\rho_{n_j}(v)$ is convergent as $j \to \infty$. Set $\rho_*(p) = \lim_{j\to\infty} \rho_{n_j}(p)$. For any $A \in \mathcal{U}$, since A is a finite set, it follows that $L_{\rho_*}(A) \geq 1$. Hence $\rho_* \in \mathcal{A}(\mathcal{U})$. For any $\varepsilon > 0$, there exists a finite set X_ε such that $\sum_{v \in X_\varepsilon} \rho_*(v)^p \geq M_p(\rho_*) - \varepsilon$. As

$$\mathrm{Mod}_p(\mathcal{U}) = \lim_{j\to\infty} M_p(\rho_{n_j}) \geq \lim_{j\to\infty} \sum_{v \in X_\varepsilon} \rho_{n_j}(v)^p,$$

we obtain $\mathrm{Mod}_p(\mathcal{U}) \geq M_p(\rho_*) - \varepsilon$ for any $\varepsilon > 0$. Hence $\mathrm{Mod}_p(\mathcal{U}) \geq M_p(\rho_*)$. On the other hand, since $\rho_* \in \mathcal{A}(\mathcal{U})$, we see $M_p(\rho_*) \geq \mathrm{Mod}_p(\mathcal{U})$. Therefore, $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. ∎

Lemma C.2. *Assume that $\mathcal{U}$ consists of finite sets. For $v \in V$, define $\mathcal{U}_v = \{A \mid A \in \mathcal{U}, v \in A\}$. Then*

$$\rho_*(v)^p \leq \mathrm{Mod}_p(\mathcal{U}_v)$$

for any $\rho_* \in \mathcal{A}(\mathcal{U})$ with $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. In particular, if $\mathcal{U}_v = \emptyset$, then

$$\rho_*(v) = 0.$$

Proof. Suppose that $\rho_* \in \mathcal{A}(\mathcal{U})$ and $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. Assume that $\mathcal{U}_v = \emptyset$ and $\rho_*(v) > 0$. Define ρ'_* by

$$\rho'_*(u) = \begin{cases} \rho_*(u) & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

Then $\rho'_* \in \mathcal{A}(\mathcal{U})$ and $M_p(\rho'_*) < M_p(\rho_*)$. This contradicts the fact that $M_p(\rho_*) = \mathrm{Mod}_p(\mathcal{U})$. Thus if $\mathcal{U}_v = \emptyset$, then $\rho_*(v) = 0$. Next assume that $\mathcal{U}_v \neq \emptyset$. Let $\rho_v \in \mathcal{A}(\mathcal{U}_v)$ with $M_p(\rho_v) = \mathrm{Mod}_p(\mathcal{U}_v)$. Note that such a ρ_v does exist by Lemma C.1. Define

$$\tilde{\rho}(u) = \begin{cases} \max\{\rho_*(u), \rho_v(u)\} & \text{if } u \neq v, \\ \rho_v(v) & \text{if } u = v. \end{cases}$$

Let $A \in \mathcal{U}$. If $v \notin A$, then $\tilde{\rho} \geq \rho_*$ on A, so that $\tilde{\rho} \in \mathcal{A}(A)$. If $v \in A$, then $\tilde{\rho} \geq \rho_v$ on A and hence $\tilde{\rho} \in \mathcal{A}(A)$. Thus we see that $\tilde{\rho} \in \mathcal{A}(\mathcal{U})$. Therefore,

$$\begin{aligned} \mathrm{Mod}_p(\mathcal{U}) \leq M_p(\tilde{\rho}) &\leq \sum_{u \neq v} \rho_*(u)^p + \sum_{u \in V} \rho_v(u)^p \\ &= \mathrm{Mod}_p(\mathcal{U}) - \rho_*(v)^p + \mathrm{Mod}_p(\mathcal{U}_v). \end{aligned}$$ ■

Define $\ell_+(V) = \{f \mid f : V \to [0, \infty)\}$.

Lemma C.3. *Let V_1 and V_2 be finite sets. Let $\mathcal{U}_i \subseteq \mathcal{P}(V_i)$ for $i = 1, 2$. If there exist maps $\xi : \mathcal{U}_2 \to \mathcal{U}_1$, $F : \ell_+(V_1) \to \ell_+(V_2)$ and constants $C_1, C_2 > 0$ such that*

$$C_1 L_{F(\rho)}(\gamma) \geq L_\rho(\xi(\gamma)) \quad \text{and} \quad M_p(F(\rho)) \leq C_2 M_p(\rho)$$

for any $\rho \in \ell_+(V_1)$ and $\gamma \in \mathcal{U}_2$, then

$$\mathrm{Mod}_p(\mathcal{U}_2) \leq (C_1)^p C_2 \mathrm{Mod}_p(\mathcal{U}_1)$$

for any $p > 0$.

Proof. Note that $C_1 F(\rho) \in \mathcal{A}(\mathcal{U}_2)$ for any $\rho \in \mathcal{A}(\mathcal{U}_1)$. Hence if $F'(\rho) = C_1 F(\rho)$, then

$$\begin{aligned} \mathrm{Mod}_p(\mathcal{U}_2) = \min_{\rho \in \mathcal{A}(\mathcal{U}_2)} M_p(\rho) &\leq \min_{\rho \in \mathcal{A}(\mathcal{U}_1)} M_p(F'(\rho)) \\ &= \leq (C_1)^P C_2 \min_{\rho \in \mathcal{A}(\mathcal{U}_1)} M_p(\rho) (C_1)^P C_2 \mathrm{Mod}_p(\mathcal{U}_1). \end{aligned}$$ ■

Lemma C.4. *Let V_1 and V_2 be countable sets and let $\mathcal{U}_i \subseteq \mathcal{P}(V_i)$ for $i = 1, 2$. Assume that $H_v \subseteq V_1$ and $\#(H_v) < \infty$ for any $v \in V_2$. Furthermore, assume that, for any $B \in \mathcal{U}_2$, there exists $A \in \mathcal{U}_1$ such that $A \subseteq \bigcup_{v \in B} H_v$. Then*

$$\mathrm{Mod}_p(\mathcal{U}_2) \leq \sup_{v \in V_2} \#(H_v)^p \sup_{u \in V_1} \#(\{v \mid v \in V_2, u \in H_v\}) \mathrm{Mod}_p(\mathcal{U}_1)$$

for any $p > 0$.

Proof. For $\rho: V_1 \to \mathbb{R}$, define

$$F(\rho)(v) = \max_{u \in H_v} \rho(u)$$

for any $v \in V_2$. Then $F: \ell_+(V_1) \to \ell_+(V_2)$ and

$$\begin{aligned} M_p(F(\rho)) &= \sum_{v \in V_2} \max_{u \in H_v} \rho(u)^p \leq \sum_{v \in V_2} \sum_{u \in H_v} \rho(u)^p \\ &\leq \sup_{u \in V_1} \#(\{v \mid v \in V_2, u \in H_v\}) M_p(\rho). \end{aligned}$$

On the other hand, for $B \in \mathcal{U}_2$, choose $\xi(B) \in \mathcal{U}_1$ such that $\xi(B) \subseteq \bigcup_{v \in B} H_v$. Then for any $\rho \in \ell_+(V_1)$ and $B \in \mathcal{U}_2$,

$$\begin{aligned} \sup_{u \in V_2} \#(H_u) L_{F(\rho)}(B) &\geq \sum_{u \in B} \#(H_u) F(\rho)(u) \geq \sum_{u \in B} \sum_{v \in H_u} \rho(v) \\ &= \sum_{v \in \bigcup_{u \in B} H_u} \#(\{u \mid v \in H_u\}) \rho(v) \\ &\geq \sum_{v \in \xi(B)} \rho(v) = L_\rho(\xi(B)). \end{aligned}$$

Hence by Lemma C.3, we have the desired conclusion. ■

D An Arzelà–Ascoli theorem for discontinuous functions

The following lemma is a version of Arzelà–Ascoli theorem showing the existence of a uniformly convergent subsequence of a sequence of functions. The difference between the original version and the current one is that it can handle a sequence of discontinuous functions.

Lemma D.1 (Extension of Arzelà–Ascoli). *Let (X, d_X) be a totally bounded metric space and let (Y, d_Y) be a metric space. Let $u_i: X \to Y$ for any $i \geq 1$. Assume that there exist a monotonically increasing function $\eta: [0, \infty) \to [0, \infty)$ and a sequence $\{\delta_i\}_{i \geq 1} \in [0, \infty)$ such that $\eta(t) \to 0$ as $t \downarrow 0$, $\delta_i \to 0$ as $i \to \infty$ and*

$$d_Y(u_i(x_1), u_i(x_2)) \leq \eta(d_X(x_1, x_2)) + \delta_i \tag{D.1}$$

for any $i \geq 1$ and $x_1, x_2 \in X$. If $\overline{\bigcup_{i \geq 1} u_i(X)}$ is compact, then there exists a subsequence $\{u_{n_j}\}_{j \geq 1}$ such that $\{u_{n_j}\}_{j \geq 1}$ converges uniformly to a continuous function $u: X \to Y$ as $j \to \infty$ satisfying $d_Y(u(x_1), u(x_2)) \leq \eta(d_X(x_1, x_2))$ for any $x_1, x_2 \in X$.

Proof. Since X is totally bounded, there exists a countable subset $A \subseteq X$ which is dense in X and contains a finite τ-net A_τ of X for any $\tau > 0$. Let $K = \overline{\bigcup_{i \geq 1} u_i(X)}$.

Since K is compact and $\{u_i(x)\}_{i\geq 1} \subseteq K$ is bounded for any $x \in A$, there exists a subsequence $\{u_{m_k}(x)\}_{k\geq 1}$ converging as $k \to \infty$. By the standard diagonal argument, we may find a subsequence $\{u_{n_j}\}_{j\geq 1}$ such that $\{u_{n_j}(x)\}_{j\geq 1}$ converges as $j \to \infty$ for any $x \in A$. Set $v_j = u_{n_j}$ and $\alpha_j = \delta_{n_j}$. Define $v(x) = \lim_{j\to\infty} v(x)$ for any $x \in A$. By (D.1),

$$d_Y(v_j(x_1), v_j(x_2)) \leq \eta(d_X(x_1, x_2)) + \alpha_j$$

for any $x_1, x_2 \in A$. Letting $j \to \infty$, we see that

$$d_Y(v(x_1), v(x_2)) \leq \eta(d_X(x_1, x_2)) \tag{D.2}$$

for any $x_1, x_2 \in A$. Since A is dense in X, v is extended to a continuous function on X satisfying (D.2) for any $x_1, x_2 \in X$. Fix $\varepsilon > 0$. Choose $\tau > 0$ such that $\eta(\tau) < \frac{\varepsilon}{3}$. Since the τ-net A_τ is a finite set, there exists k_0 such that if $k \geq k_0$, then $\alpha_k < \frac{\varepsilon}{3}$ and $d_Y(v(z), v_k(z)) < \varepsilon$ for any $z \in A_\tau$. Let $x \in X$ and choose $z \in A_\tau$ such that $d_X(x, z) < \tau$. If $k \geq k_0$, then

$$\begin{aligned} d_Y(v_k(x), v(x)) &\leq d_Y(v_k(x), v_k(z)) + d_Y(v_k(z), v(z)) + d_Y(v(z), v(x)) \\ &\leq 2\eta(d_X(x, z)) + \alpha_k + d_Y(v_k(z), v(z)) < 2\varepsilon. \end{aligned}$$

Thus $\{v_j\}_{j\geq 1}$ converges uniformly to v as $j \to \infty$. ∎

E Geometric properties of strongly symmetric self-similar sets

In this appendix, we will give proofs of claims on topological and geometric properties of self-similar sets treated in Section 4.6. Namely, we will give proofs of Propositions 4.40 and 4.42. First, we recall the setting of Section 4.6. Let S be a finite subset of $\mathbb{R}^L$ and let $\rho \in (0, 1)$. Let $U_q \in O(L)$ for any $q \in S$. Define $f_q\colon \mathbb{R}^L \to \mathbb{R}^L$ by

$$f_q(x) = \rho U_q(x - q) + q$$

for $x \in \mathbb{R}^L$. Let K be the self-similar set with respect to $\{f_q\}_{q\in S}$, i.e., K is the unique non-empty compact set K satisfying

$$K = \bigcup_{q\in S} f_q(K).$$

The triple $(K, S, \{f_q\}_{q\in S})$ is know to be a self-similar structure defined in Definition 4.1 and the map $\chi\colon S^{\mathbb{N}} \to K$ is given by

$$\{\chi(q_1 q_2 \ldots)\} = \bigcap_{m\geq 0} f_{q_1\ldots q_m}(K)$$

as we have seen in Section 4.1.

Definition E.1. (1) Define $\widetilde{\sigma}\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ by

$$\widetilde{\sigma}(q_1 q_2 \ldots) = q_2 q_3 \ldots \quad \text{for } q_1 q_2 \ldots \in S^{\mathbb{N}}.$$

(2) Define

$$C_K = \bigcup_{i \neq j \in S} K_i \cap K_j, \quad \mathcal{C} = \chi^{-1}(C_K), \quad \mathcal{P} = \bigcup_{k \geq 1} \widetilde{\sigma}^k(\mathcal{C}),$$

and $V_0 = \chi(\mathcal{P})$. The sets $\mathcal{C}$ and $\mathcal{P}$ are called the critical set and the post critical set of $(K, S, \{f_q\}_{q \in S})$, respectively. A self-similar structure $(K, S, \{f_q\}_{q \in S})$ is said to be post critically finite (p.c.f. for short) if $\mathcal{P}$ is a finite set.

By [29, Theorem 1.2.3], we have the following proposition.

Proposition E.2. *The map χ is continuous and surjective. Moreover,*

$$\chi(q_1 q_2 \ldots) = f_{q_1}(\chi(\widetilde{\sigma}(q_1 q_2 \ldots))) \tag{E.1}$$

for any $q_1 q_2 \ldots \in S^{\mathbb{N}}$.

In this appendix, we suppose that Assumption 4.39 holds.

The next lemma gives a proof of Proposition 4.40.

Lemma E.3. *Under Assumption 4.39, we have*

(1) *For any $q \in S$, $\chi^{-1}(q) = \overline{q}$, where $\overline{q} = qqq \ldots \in S^{\mathbb{N}}$.*

(2) *$\mathcal{P} = \{\overline{q} \mid q \in U\}$, where U is the set appearing in Assumption 4.39. In particular, the self-similar structure $(K, S, \{f_q\}_{q \in S})$ is post critically finite and $V_0 = U$.*

Proof. (1) Suppose $\chi(\tau_1 \tau_2 \ldots) = q$. Then by (E.1),

$$q = \chi(\tau_1 \tau_2 \ldots) = f_{\tau_1}(\chi(\tau_2 \tau_3 \ldots)) \in K_{\tau_1}.$$

By Assumption 4.39 (1), it follows that $\tau_1 = q$. Since f_q is invertible, we see that $\chi(\tau_2 \tau_3 \ldots) = q$. Using the same argument as above, we see that $\tau_2 = q$ as well. Thus we deduce that $\tau_k = q$ for any $k \in \mathbb{N}$ inductively.

(2) Suppose that $\chi(\tau_1 \tau_2 \ldots) \in f_{\tau_1}(K) \cap f_q(K)$ for some $q \neq \tau_1$. By (E.1), it follows that $\chi(\tau_1 \tau_2 \ldots) = f_{\tau_1}(\chi(\tau_2 \tau_3 \ldots))$. Hence by Assumption 4.39 (2),

$$\chi(\tau_2 \tau_3 \ldots) \in (f_{\tau_1})^{-1}(f_{\tau_1}(K) \cap f_q(K)) \subseteq U.$$

Thus $\tau_2 \tau_3 \ldots = \overline{q'}$ for some $q' \in U$. Therefore, $\mathcal{P} \subseteq U$.

Conversely, again by Assumption 4.39 (2), for any $q \in U$, there exist $p_1, p_2 \in S$ with $p_1 \neq p_2$ such that $\chi(p_1 \overline{q}) \in f_{p_1}(K) \cap f_{p_2}(K)$. This shows that $p_1 \overline{q} \in \mathcal{C}$ and hence $\overline{q} \in \mathcal{P}$. ∎

In the next two lemmas, we are going to show a sufficient condition for Assumption 4.41.

Lemma E.4. *Suppose that Assumption 4.39 holds and that U_q is the identity map for any $q \in V_0$. Let $q = f_{p_1}(q_1) = f_{p_2}(q_2)$ for some $p_1, p_2 \in S$ with $p_1 \neq p_2$ and $q_1, q_2 \in V_0$. Then there exists $\gamma = \gamma(p_1, p_2, q_1, q_2) > 0$ such that*

$$d(\overline{K_{p_1} \backslash K_{p_1(q_1)^{m-1}}}, K_{p_2}) \geq \gamma \rho^m$$

for any $m \geq 1$, where $d(A, B) = \inf_{x \in A, y \in B} |x - y|$ and $(q)^k = \underbrace{q \dots q}_{k\text{-times}} \in T_k$.

In the following proof, we assume that

$$\#(f_{p_1}(K) \cap f_{p_2}(K)) \leq 1$$

to avoid a non-essential complication of arguments. Without this assumption, the lemma is still true with a technical modification of the proof.

Proof. Set $c_m = \inf\{d(K_w, K_v) \mid w, v \in T_m, K_w \cap K_v = \emptyset\}$. Define

$$X_m = \overline{K_{p_1} \backslash K_{p_1(q_1)^{m-1}}} \quad \text{and} \quad Y_m = \overline{K_{p_1 q_1} \backslash K_{p_1(q_1)^{m-1}}}$$

for $m \geq 1$. Then $X_m = Y_m \cup (\bigcup_{q \neq q_1} K_{p_1 q})$ and $K_{p_2} = K_{p_2 q_2} \cup (\bigcup_{q \neq q_2} K_{p_2 q})$. This implies that

$$d(X_m, K_{p_2}) \geq \min\{d(Y_m, K_{p_2 q_2}), c_2\}.$$

On the other hand, letting $f(x) = \rho(x - q) + q$, we see that

$$Y_m \cup K_{p_2 q_2} = f(X_{m-1} \cup K_{p_2}).$$

This yields $d(Y_m, K_{p_2 q_2}) = \rho d(X_{m-1}, K_{p_2})$. Consequently, we have

$$d(X_m, K_{p_2}) \geq \min\{\rho d(X_{m-1}, K_{p_2}), c_2\}.$$

Now inductive argument suffices. ∎

Lemma E.5. *Suppose that Assumption 4.39 holds and that U_q is the identity map for any $q \in V_0$. Then Assumption 4.41 holds.*

Remark. According to the notation in the proof of Lemma E.4, this lemma claims $c_m \geq c\rho^m$ for any $m \geq 1$.

Proof. Suppose that $w, v \in T_m$ and $K_w \cap K_v = \emptyset$. Let $w = w_1 \dots w_m$ and let $v = v_1 \dots v_m$. In the case $w_1 = w_2$,

$$d(K_w, K_v) = \rho d(K_{w_2 \dots w_m}, K_{v_2 \dots v_m}) \geq c_{m-1} \rho.$$

Otherwise, assume that $w_1 \neq v_1$. If $K_{w_1} \cap K_{v_1} = \emptyset$, then $d(K_w, K_v) \geq c_1$. So, the remaining possibility is that $w_1 \neq v_1$ and $K_{w_1} \cap K_{v_1} \neq \emptyset$. In this case, let $q = K_{w_1} \cap K_{v_1}$. Then $q = f_{w_1}(p_{j_1}) = f_{w_2}(p_{j_2})$ for some $j_1, j_2 \in \{1, \ldots, L\}$. By Lemma E.4, it follows that $d(K_w, K_v) \geq \overline{\gamma}\rho^m$, where $\overline{\gamma} = \min\{\gamma(p_1, p_2, q_1, q_2) \mid p_1, p_2 \in S, q_1, q_2 \in V_0, f_{p_2}(q_1) = f_{p_1}(q_2)\}$. Combining all the cases and using induction on m, we see that $c_m \geq \min\{c_1, \overline{\gamma}\}\rho^m$ for any $m \geq 1$. ■

Now we start showing Proposition 4.42, that is, Assumption 2.15 holds under Assumptions 4.39 and 4.41.

Lemma E.6. *Under Assumptions 4.39 and 4.41, Assumption 2.15 (2) holds with $r = \rho$, $M_* = 1$, and $d = d_*$, where d_* is the restriction of the Euclidean metric.*

Proof. (2A) is obvious. Set

$$\Gamma_{1,n}(x) = \bigcup_{\substack{w \in T_n \\ x \in K_w}} \Gamma_1(w)$$

for $x \in K$ and $n \geq 1$. Then for any $v \in T_n \backslash \Gamma_{1,n}(x)$, there exists $w \in T_n$ such that $x \in K_w$ and $K_w \cap K_v = \emptyset$. By Lemma E.5, we see that $d(K_w, x) \geq c\rho^n$ and hence $B_{d_*}(x, cr^n) \cap K_v = \emptyset$. Thus we have

$$B_{d_*}(x, c\rho^n) \subseteq U_1(x : n). \tag{E.2}$$

On the other hand, by (2A), there exists $c' > 0$ such that $\mathrm{diam}(K_w, d_*) \leq c'\rho^{|w|}$ for any $w \in T$. This implies

$$U_1(x : n) \subseteq B_{d_*}(x, 3c'\rho^n). \tag{E.3}$$

So we have (2B). Choose $x_0 \in K \backslash V_0$ and choose $m_0 \in \mathbb{N}$ such that $2\rho^{m_0} < d(x_0, V_0)$. Let $w \in T_n$ and let $u \in \Gamma_{1,m_0+n}(f_w(x_0))$. Suppose that $u \in T(v)$ for some $v \in T_n$ with $v \neq w$. Since $u \in \Gamma_{1,m_0+n}(f_w(x_0))$, there exists $u_0 \in T_{n+m_0}$ such that $f_w(x_0) \in K_{u_0}$ and $K_{u_0} \cap K_u \neq \emptyset$. Let $y \in K_u$. Since K is connected (and hence arcwise connected by [29, Theorem 1.6.2]), there exists a continuous curve ζ: $[0, 1] \to K_{u_0} \cup K_u$ such that $\zeta(0) = f_w(x_0)$ and $\zeta(1) = y$. Note that $f_w(x_0) \in K_w$ and $y \in K_v$. By (4.24), the curve ζ intersects with $f_w(V_0)$. Therefore, $(K_u \cup K_{u_0}) \cap f_w(V_0) \neq \emptyset$. However, since $\mathrm{diam}(K_u, d_*) = \mathrm{diam}(K_{u_0}, d_*) = \rho^{m_0+n}$, it follows

$$d(f_w(x_0), K_u \cup K_{u_0}) \leq 2\rho^{m_0+n} < d(f_w(x_0), f_w(V_0)),$$

so that $(K_{u_0} \cup K_u) \cap f_w(V_0) = \emptyset$. This contradiction shows that $u \in T(w)$ and hence $U_1(f_w(x_0) : m_0 + n) \subseteq K_w$. By (E.2), we see that

$$B_{d_*}(f_w(x_0), c\rho^{m_0+n}) \subseteq U_1(f_w(x_0) : m_0 + n) \subseteq K_w.$$

This shows (2C). ■

Next set $\alpha_H = -\frac{\log \#(S)}{\log \rho}$. Note that $\rho^{\alpha_H} = \#(S)^{-1}$. Let μ be the self-similar measure on K with weight $(\rho^{\alpha_H}, \dots, \rho^{\alpha_H})$. By [31, Theorem 1.2.7], we see that $\mu(K_w) = \rho^{|w|}$ for any $w \in T$ and consequently $\mu(\{x\}) = 0$ for any $x \in K_w$. These facts show that μ satisfies Assumption 2.12. Moreover, we have the following proposition.

Proposition E.7. *Under Assumptions 4.39 and 4.41, there exist $c_1, c_2 > 0$ such that*

$$c_1 s^{\alpha_H} \le \mu(B_{d_*}(x, s)) \le c_1 s^{\alpha_H} \tag{E.4}$$

for any $s \in [0, 1]$. In particular, μ is α_H-Ahlfors regular with respect to d_* and the Hausdorff dimension of (K, d_*) equals α_H.

Proof. By (E.3), for any $x \in K$ and $n \ge 1$, if $w \in \Gamma_{1,n}(x)$, then

$$(\rho^n)^{\alpha_H} = \mu(K_w) \le \mu(B_{d_*}(x, 3c'\rho^n)). \tag{E.5}$$

On the other hand, by [31, Proposition 1.6.11], there exists $J_* \in \mathbb{N}$ such that

$$\#(\Gamma_{1,n}(x)) \le J_* \tag{E.6}$$

for any $x \in T$ and $n \ge 0$. (Note that $\Lambda^1_{\rho^n,x}$ defined in [31, Definition 1.3.3] equals $\Gamma_{1,n}(x)$.) Therefore by (E.2),

$$\mu(B_{d_*}(x, c\rho^n)) \le \sum_{v \in \Gamma_{1,n}(x)} \mu(K_v) \le J_*(\rho^n)^{\alpha_H}. \tag{E.7}$$

Combining (E.5) and (E.7), we obtain (E.4). ■

The following proposition is immediately deduced from the previous propositions and lemmas. Note that $\Gamma_1(w) \subseteq \Gamma_{1,n}(x)$ for any $w \in T$ and $x \in K_w$. Hence by (E.6), we see that the partition $\{K_w\}_{w \in T}$ is uniformly finite.

Proposition E.8 (Proposition 4.42). *Under Assumptions 4.39 and 4.41, Assumption 2.15 holds with $r = \rho$, $d = d_*$ and $M_* = M_0 = 1$.*

The fact that $M_0 = 1$ is due to the second remark after Assumption 2.6.

F List of definitions and notations

Definitions

Notations