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Finite groups of birational transformations

Yuri Prokhorov

Abstract. We survey new results on finite groups of birational transformations of algebraic varieties.

1. Introduction

We work over a field $\Bbbk$ of characteristic 0. Typically, unless otherwise mentioned, we assume that $\Bbbk$ is algebraically closed. The *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ of rank n is the group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, \ldots, x_n)$ of rational functions in n independent variables. Equivalently, $\mathrm{Cr}_n(\Bbbk)$ can be viewed as the group of birational transformations of the projective space $\mathbb{P}^n$. It is easy to show that for $n = 1$, the group $\mathrm{Cr}_n(\Bbbk)$ consists of linear projective transformations:

$$\mathrm{Cr}_1(\Bbbk) = \mathrm{PGL}_2(\Bbbk).$$

On the other hand, for $n \geq 2$, the group $\mathrm{Cr}_n(\Bbbk)$ has an extremely complicated structure. In particular, it contains linear algebraic subgroups of arbitrary dimension and has a lot of normal non-algebraic subgroups [18,24]. We refer to [3,22,23,38,48,95] for surveys, historical résumés, and introductions to the subject.

Examples. (i) Any matrix $A = \|a_{i,j}\| \in \mathrm{GL}_n(\mathbb{Z})$ defines an element $\varphi_A \in \mathrm{Cr}_n(\Bbbk)$ via the following action on $\Bbbk(x_1, \ldots, x_n)$:

$$\varphi_A : x_i \mapsto x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}.$$

Such Cremona transformations are called *monomial*. For $n = 2$ and $A = -\mathrm{id}$, the transformation φ_A is known as the *standard quadratic involution*

$$(x_1, x_2) \mapsto (x_1^{-1}, x_2^{-1}).$$

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(ii) Let S be an algebraic variety admitting a generically finite rational map

$$\pi : S \dashrightarrow \mathbb{P}^{n-1}$$

of degree 2. In an affine piece and suitable coordinates, S can be given by the equation $y^2 = f(x_1, \dots, x_{n-1})$. One can associate with (S, π) an involution $\tau \in \mathrm{Cr}_n(\Bbbk)$ acting on $\Bbbk(x_1, \dots, x_{n-1}, y)$ via

$$\tau : (x_1, \dots, x_{n-1}, y) \mapsto \big(x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}) \cdot y^{-1}\big).$$

If $n = 2$ and S is a hyperelliptic curve, then τ is known as the *de Jonquières involution*.

The study of the Cremona group has a very long history. Basically, it was started in earlier works of A. Cayley and L. Cremona, and since then, this group has been the object of many studies. In these notes, we concentrate on the following particular problem.

Problem 1.1. Describe the structure of finite subgroups of $\mathrm{Cr}_n(\Bbbk)$.

Note, however, that the projective space is not an exceptional variety from the algebro-geometric point of view. So one can ask a similar question replacing $\mathrm{Cr}_n(\Bbbk)$ with the group of birational transformations $\mathrm{Bir}(X)$ of an arbitrary algebraic variety X. Hence it is natural to pose the following problem.

Problem 1.2. Describe the structure of finite subgroups of $\mathrm{Bir}(X)$, where X is an algebraic variety.

We deal with the most recent results related to these problems. Definitely, our survey is not exhaustive.

2. Equivariant minimal model program

In this section, we collect basic facts on the so-called G-minimal model program (abbreviated as G-MMP). This program is the main tool in the study of finite groups of birational transformations. For a detailed exposition, we refer to [89].

Let G be a *finite* group. Following Yu. Manin [68], we say that an algebraic variety X is a *G-variety* if it is equipped with a regular faithful action $G \curvearrowright X$, i.e., if there exists an injective homomorphism $\alpha : G \hookrightarrow \mathrm{Aut}(X)$. A morphism (resp. rational map) $f : X \to Y$ of G-varieties is a *G-morphism* (resp. *G-map*) if there exists a group automorphism $\varphi : G \to G$ such that, for any $g \in G$,

$$f \circ \alpha(g) = \beta\big(\varphi(g)\big) \circ f,$$

where $\alpha : G \hookrightarrow \mathrm{Aut}(X)$ and $\beta : G \hookrightarrow \mathrm{Aut}(Y)$ are the embeddings corresponding to the actions $G \curvearrowright X$ and $G \curvearrowright Y$, respectively.

For any G-variety X, the action $G \curvearrowright X$ induces an embedding $G \hookrightarrow \mathrm{Aut}_{\Bbbk}(\Bbbk(X))$ to the automorphism group of the field of rational functions $\Bbbk(X)$. Conversely, given any finitely generated extension $\mathbb{K}/\Bbbk$ and any finite subgroup $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$, there exists a G-variety X and an isomorphism $\Bbbk(X) \simeq_{\Bbbk} \mathbb{K}$ inducing $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$. Thus, we have the following fact.

Proposition 2.1. *Let $\mathbb{K}/\Bbbk$ be finitely generated field extension. Then there exists a* 1-1 *correspondence between finite subgroups $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$ considered modulo conjugacy and G-varieties X such that $\Bbbk(X) \simeq_{\Bbbk} \mathbb{K}$ considered modulo G-birational equivalence.*

Recall that a variety X is said to be *rational* if it is birationally equivalent to the projective space $\mathbb{P}^n$ or, equivalently, if the field extension $\Bbbk(X)/\Bbbk$ is purely transcendental.

Corollary. *There exists a* 1-1 *correspondence between finite subgroups $G \subset \mathrm{Cr}_n(\Bbbk)$ considered modulo conjugacy and rational G-varieties X such that $\Bbbk(X) \simeq_{\Bbbk} \mathbb{K}$ considered modulo G-birational equivalence.*

Next, due to the equivariant resolution theorem (see e.g. [1]), it is possible to replace X with a smooth projective model.

Proposition 2.2 (see, e.g., [89, Lemma 14.1.1]). *For any G-variety X, there exists a* smooth projective *G-variety Y that is G-birationally equivalent to X.*

Thus the above considerations allow us to reduce the problem of classification of finite subgroups of $\mathrm{Bir}(X)$ to the study of subgroups in $\mathrm{Aut}(Y)$, where Y is a smooth projective variety. The main difficulty arising here is that this G-variety Y is not unique in its G-birational equivalence class. So, given G-birational equivalence class of algebraic G-varieties, we need to choose some good representative in it. This can be done by means of the G-MMP. The higher-dimensional MMP forces us to consider varieties with certain very mild, so-called terminal singularities.

Definition. A normal variety X has *terminal singularities* if some multiple mK_X of the canonical Weil divisor K_X is Cartier, and for any birational morphism $f : Y \to X$, one can write

$$mK_Y = f^*mK_X + \sum a_i E_i,$$

where E_i are all the exceptional divisors and $a_i > 0$ for all i. The smallest positive m such that mK_X is Cartier is called the *Gorenstein index* of X.

Definition. A G-variety X has *$G\mathbb{Q}$-factorial singularities* if a multiple of any G-invariant Weil divisor on X is Cartier.

It is important to note that terminal singularities lie in codimension ≥ 3. In particular, terminal surface singularities are smooth.

Example ([72, 93]). Let the cyclic group $\boldsymbol{\mu}_r$ act on $\mathbb{A}^4$ diagonally via

$$(x_1, x_2, x_3, x_4) \mapsto (\zeta x_1, \zeta^{-1} x_2, \zeta^a x_3, x_4), \quad \zeta = \zeta_r = \exp(2\pi \mathrm{i}/r), \quad \gcd(a, r) = 1.$$

Then for a polynomial $f(u, v)$, the singularity of the quotient

$$\{x_1 x_2 + f(x_3^r, x_4) = 0\}/\boldsymbol{\mu}_r$$

at 0 is terminal whenever it is isolated.

The aim of the G-MMP is to replace a G-variety with another one, which is "minimal" in some sense. As we mentioned above, running the G-MMP we have to consider singular varieties, and the class of terminal $G\mathbb{Q}$-factorial singularities is the smallest class that is closed under the G-MMP.

Definition (For simplicity, we assume that $\Bbbk$ is uncountable). A variety X is *uniruled* if for a general point $x \in X$, there exists a rational curve $C \subset X$ passing through x. A variety X is *rationally connected* if two general points $x_1, x_2 \in X$ can be connected by a rational curve.

Note that a rationally connected surface is rational, and an uniruled surface is birationally equivalent to $C \times \mathbb{P}^1$, where C is a curve.

Definition. Let Y be a G-variety with only terminal $G\mathbb{Q}$-factorial singularities and let $f : Y \to Z$ be a G-equivariant morphism with connected fibers to a lower-dimensional variety Z, where the action of G on Z is not necessarily faithful. Then f is called *G-Mori fiber space* (abbreviated as G-Mfs) if the anti-canonical class $-K_Y$ is f-ample and $\operatorname{rk} \operatorname{Pic}(Y/Z)^G = 1$. If Z is a point, then $-K_Y$ is ample, and Y is called *$G\mathbb{Q}$-Fano variety*. Two-dimensional $G\mathbb{Q}$-Fano varieties are traditionally called *G-del Pezzo surfaces*.

Definition. A G-variety Y is said to be a *G-minimal model* if it has only terminal $G\mathbb{Q}$-factorial singularities and the canonical class K_Y is numerically effective (nef).

It is not difficult to show that the concepts of G-minimal model and G-Mori fiber space are mutually exclusive. Moreover, if $f : Y \to Z$ is a G-Mfs, then its general fiber is rationally connected; hence Y is uniruled. On the other hand, a G-minimal model is never uniruled [70]. The following assertions are usually formulated for varieties without group actions. The corresponding equivariant versions can be easily deduced from non-equivariant ones (see [89]).

Theorem 2.3 ([14]). *Let X be an uniruled G-variety. Then there exists a birational G-map $X \dashrightarrow Y$, where Y has a structure of G-Mfs $f : Y \to Z$.*

Conjecture 2.4. *Let X be a non-uniruled G-variety. Then there exists a birational G-map $X \dashrightarrow Y$, where Y is a G-minimal model.*

The conjecture is known to be true in dimension ≤ 4 (see [73, 99]), as well as in the case where K_X is big [14], and in some other cases. In arbitrary dimension, a weaker notion of quasi-minimal models works quite satisfactory [82].

3. Cremona group of rank 2

The G-MMP for surfaces is much more simple than in higher dimensions. It was developed in the works of Yu. Manin and V. Iskovskikh (see [68]). In the two-dimensional case, the G-MMP works in the category of smooth G-surfaces, and all the birational transformations are contractions of disjoint unions of (-1)-curves. For a G-Mfs $f : Y \to Z$, there are two possibilities:

(i) Z is a point and then Y is a G-del Pezzo surface,

(ii) Z is a curve, any fiber of f is a reduced plane conic and $\operatorname{rk}\operatorname{Pic}(Y)^G = 2$. In this case, f is called *G-conic bundle*.

Thus to study finite subgroups of $\mathrm{Cr}_2(\Bbbk)$, one has to consider the above two classes of G-Mfs's in detail. The classification of del Pezzo surfaces is well known and very short. Hence, to study the case (i) one has to list all finite subgroups $G \subset \operatorname{Aut}(Y)$ satisfying the condition $\operatorname{rk}\operatorname{Pic}(Y)^G = 1$. The full list was obtained by Dolgachev and Iskovskikh [40]. In contrast, the class of conic bundles is huge and consists of an infinite number of families. In this case, a reasonable approach is to find an algorithm of enumerating conic bundles Y/Z together with subgroups $G \subset \operatorname{Aut}(Y/Z)$ satisfying $\operatorname{rk}\operatorname{Pic}(Y)^G = 2$. This also was done by Dolgachev and Iskovskikh [40] (see also [103]). However, even using this algorithm, it is very hard to get a complete list of corresponding groups.

As an example, we present a well-known classical result on the classification of subgroups of order 2 in $\mathrm{Cr}_2(\Bbbk)$. It was obtained by E. Bertini [12] in 1877; however, his arguments were incomplete from a modern point of view. A new rigorous proof was given by L. Bayle and A. Beauville [8].

Theorem 3.1. *Let $G = \{1, \tau\} \subset \mathrm{Cr}_2(\Bbbk)$ be a subgroup of order 2. Then the embedding $G \subset \mathrm{Cr}_2(\Bbbk)$ is induced by one of the following actions on a rational surface X:*

τ		X and τ
1^o	Linear involution	$\mathbb{P}^2$
2^o	de Jonquières involution of genus $g \geq 1$	$X = \{y_1 y_2 = p(x_1, x_2)\} \subset \mathbb{P}(1, 1, g + 1, g + 1)$ p is a homogeneous form of degree $2g + 2$, τ is the deck involution of the projection $X \overset{2:1}{\dashrightarrow} \mathbb{P}(1, 1, g+1)$, $(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1 + y_2)$
3^o	Geiser involution	$X = \{y^2 = p(x_1, x_2, x_3)\} \subset \mathbb{P}(1, 1, 1, 2)$, p is a homogeneous form of degree 4, τ is the deck involution of the projection $X \xrightarrow{2:1} \mathbb{P}(1, 1, 1) = \mathbb{P}^2$
4^o	Bertini involution	$X = \{z^2 = p(x_1, x_2, y)\} \subset \mathbb{P}(1, 1, 2, 3)$, p is a quasihomogeneous form of degree 6, τ is the deck involution of the projection $X \xrightarrow{2:1} \mathbb{P}(1, 1, 2)$

Here $\mathbb{P}(w_1, \dots, w_n)$ *denotes the weighted projective space with corresponding weights.*

In the cases 1^o, 3^o, and 4^o, the variety X is a del Pezzo surface of degree 9, 2, and 1, respectively. In the case 2^o, the projection $X \dashrightarrow \mathbb{P}(1, 1) = \mathbb{P}^1$ becomes a G-conic bundle after blowing up the indeterminacy points.

The G-MMP was successfully applied for the classification of various classes of finite subgroups in $\mathrm{Cr}_2(\Bbbk)$: groups of prime order [36], p-elementary groups [9], abelian groups [15, 16], and finally, arbitrary groups [40]. Here is another example of classification results.

Theorem 3.2 ([40]). *Let* $G \subset \mathrm{Cr}_2(\mathbb{C})$ *be a finite simple group. Then* G *is isomorphic to one of the following:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(\mathbf{F}_7),$$

where $\mathfrak{A}_n$ *is the alternating group of degree* n *and* $\mathrm{PSL}_n(\mathbf{F}_q)$ *is the projective special linear group over the finite field* $\mathbf{F}_q$.

Moreover, if $G \not\simeq \mathfrak{A}_5$, *then the embedding* $G \subset \mathrm{Cr}_2(\Bbbk)$ *is induced by one of the following actions on a del Pezzo surface* X:

G	$\lvert G\rvert$	X
$\mathfrak{A}_6$	360	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathbf{F}_7)$	168	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathbf{F}_7)$	168	$\{y^2 = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1\} \subset \mathbb{P}(1, 1, 1, 2)$

A complete classification of embeddings $\mathfrak{A}_5 \hookrightarrow \mathrm{Cr}_2(\Bbbk)$ can be found in [31].

4. Cremona group of rank 3

The MMP in dimension 3 is more complicated than the two-dimensional one, but still, it is developed very well. In particular, terminal threefold singularities are classified up to analytic equivalence [72,93]. The structure of all intermediate steps of the MMP and Mfs's is also studied relatively well (see [89] for a survey).

For a three-dimensional G-Mori fiber space $f : Y \to Z$, there are three possibilities:

(i) Z is a point, then Y is a (possibly singular) $G\mathbb{Q}$-Fano threefold,

(ii) Z is a curve, then f is called a $G\mathbb{Q}$-del Pezzo fibration,

(iii) Z is a surface, then f is a $G\mathbb{Q}$-conic bundle.

A $G\mathbb{Q}$-conic bundle can be birationally transformed into a *standard G-conic bundle*, i.e., $G\mathbb{Q}$-conic bundle such that both X and Z are smooth [6]. For $G\mathbb{Q}$-del Pezzo fibrations, there are only some partial results of this type (see [35,66]). Nevertheless, the main difficulty in the application G-MMP to the classification of finite groups of birational transformations is the lack of a complete classification of Fano threefolds with terminal singularities. At the moment, only some very particular classes of $G\mathbb{Q}$-Fano threefolds are studied (see [4, 5, 52, 79, 80, 88] and references therein). Some roundabout methods work in the case of "large" in some sense (in particular, simple) finite groups.

Theorem 4.1 ([78]). *Let $G \subset \mathrm{Cr}_3(\mathbb{C})$ be a finite simple subgroup. Then G is isomorphic to one of the following:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathfrak{A}_7, \quad \mathrm{PSL}_2(\mathbf{F}_7), \quad \mathrm{PSL}_2(\mathbf{F}_8), \quad \mathrm{PSp}_4(\mathbf{F}_3),$$

where $\mathrm{PSp}_4(\mathbf{F}_3)$ is the projective symplectic group over $\mathbf{F}_3$. All the possibilities occur.

This classification is a consequence of the following more general result.

Theorem 4.2 ([78]). *Let Y be a rationally connected threefold and let $G \subset \mathrm{Bir}(Y)$ be a finite simple group. If G is not embeddable to $\mathrm{Cr}_2(\mathbb{C})$, then Y is G-birationally equivalent to one of the following $G\mathbb{Q}$-Fano threefolds:*

	G	X	Rational?
1^o	$\mathfrak{A}_7$	$X_6' = \{\sigma_{1,7} = \sigma_{2,7} = \sigma_{3,7} = 0\} \subset \mathbb{P}^5 \subset \mathbb{P}^6$	no
2^o	$\mathfrak{A}_7$	$\mathbb{P}^3$	yes
3^o	$\mathrm{PSp}_4(\mathbf{F}_3)$	$\mathbb{P}^3$	yes
4^o	$\mathrm{PSp}_4(\mathbf{F}_3)$	Burkhardt quartic $X_4^{\mathrm{b}} = \{\sigma_{1,6} = \sigma_{4,6} = 0\} \subset \mathbb{P}^4 \subset \mathbb{P}^5$	yes
5^o	$\mathrm{PSL}_2(\mathbf{F}_8)$	Special Fano threefold $X_{12}^{\mathrm{m}} \subset \mathbb{P}^8$ of genus 7	yes
6^o	$\mathrm{PSL}_2(\mathbf{F}_{11})$	Klein cubic $X_3^{\mathrm{k}} = \{x_1x_2^2 + x_2x_3^2 + \cdots x_5x_1^2 = 0\} \subset \mathbb{P}^4$	no
7^o	$\mathrm{PSL}_2(\mathbf{F}_{11})$	Special Fano threefold $X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$ of genus 8	no

Here $\sigma_{d,k} = \sigma_{d,k}(x_1, \dots, x_k)$ is the elementary symmetric polynomial of degree d in k variables.

Below we outline the proof of Theorem 4.2.

Assume that G is not embeddable to $\mathrm{Cr}_2(\Bbbk)$, i.e., it is not isomorphic to any of the groups listed in Theorem 3.2. First, Proposition 2.2 allows us to assume that the action of G is regularized on some smooth projective G-variety X. By running the equivariant MMP, we may assume that X has a structure of a G-Mfs $f : X \to Z$ (because X is rationally connected). Consider the case $\dim Z > 0$. Since G is a simple group, it must act faithfully on the base Z or on the general fiber F. Since the varieties F and Z are rational, this means that G is contained in the plane Cremona group $\mathrm{Cr}_2(\Bbbk)$. The contradiction proves Theorem 4.2 in the case $\dim Z > 0$.

Hence, we may further assume that Z is a point and X is a $G\mathbb{Q}$-Fano threefold. Consider the case where X is not Gorenstein, i.e., the canonical class K_X is not a Cartier divisor. It turns out that this case does not occur. Let $P_1, \dots, P_n \in X$ be all non-Gorenstein points and let $r_1, \dots, r_n$ be the corresponding Gorenstein indices. Arguments based on Bogomolov–Miyaoka inequality (see [55, 57] and [89, §12]) show that

$$\sum \left(r_i - \frac{1}{r_i} \right) < 24.$$

Hence, $n \le 15$. Then using the classification of transitive actions of simple groups [33] and analyzing the action of stabilizers of P_i, one obtains the only possibility:

- $n = 11$, $G \simeq \mathrm{PSL}_2(\mathbf{F}_{11})$, $r_1 = \dots = r_n = 2$.

This case is excluded by a more detailed geometric consideration (see [78, §6]).

Thus, we may assume that K_X is a Cartier divisor. In this case, according to [74], the variety X has a smoothing, that is, there exists a one-parameter flat family $\mathfrak{X}/\mathfrak{B} \ni o$ such that the special fiber $\mathfrak{X}_o$ is isomorphic to X, and a general geometric fiber $\mathfrak{X}_t$ is a smooth Fano threefold. Hence some discrete invariants of X, such as the Picard lattice $\mathrm{Pic}(X)$ and the anticanonical degree $-K_X^3$, are the same as for smooth Fano threefolds, which are completely classified (see [52]). Recall that the Fano index $\iota(X)$ of X is the maximal integer that divides the canonical class K_X in the lattice $\mathrm{Pic}(X)$ [52]. By [80], we have $\operatorname{rk}\mathrm{Pic}(X) \le 4$. Since $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$ and a simple group that is not isomorphic to $\mathfrak{A}_5$ cannot have a nontrivial integer representation of dimension ≤ 4, we have $\operatorname{rk}\mathrm{Pic}(X) = 1$. If $\iota(X) \ge 4$ (resp, $\iota(X) = 3$), then X is isomorphic to the projective space $\mathbb{P}^3$ (resp. a quadric in $\mathbb{P}^4$) [52]. Then from the classification of finite subgroups in $\mathrm{PSL}_4(\Bbbk)$ and $\mathrm{PSL}_5(\Bbbk)$, we get cases 2^o and 3^o. Three-dimensional Fano varieties with $\iota(X) = 2$ are called del Pezzo threefolds. G-Fano threefolds of this type were studied in [79]. As a consequence of these results, we get the case of the group $G = \mathrm{PSL}_2(\mathbf{F}_{11})$ acting on the Klein cubic (case 6^o).

Finally, let $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$. Recall that in this case, the anticanonical degree is written in the form $-K_X^3 = 2\,\mathrm{g}(X) - 2$, where $\mathrm{g}(X) \in \{2, 3, \ldots, 10, 12\}$ [52]. For $\mathrm{g}(X) \le 5$, the variety X has a natural embedding to a (weighted) projective space as a complete intersection [52]. Using this and some facts from representation theory, we obtain for the group G two cases, 1^o and 4^o. The case $\mathrm{g}(X) = 6$ can be excluded using [37, Corollary 3.11]. For $\mathrm{g}(X) \ge 7$, the variety X must be smooth (see [78, Lemma 5.17] and [88]). Further, using some facts about automorphisms of smooth Fano threefolds [63], we obtain for the group G two possibilities, 5^o and 7^o. This completes our sketch of the proof of Theorem 4.2. ∎

A similar technique was applied to the study of finite p-subgroups and quasi-simple subgroups in $\mathrm{Cr}_3(\Bbbk)$ (see [17, 64, 67, 77, 81, 86]).

Note that Theorem 4.2 does not describe *embeddings* of groups $\mathfrak{A}_5$, $\mathfrak{A}_6$, and $\mathrm{PSL}_2(\mathbf{F}_7)$ to the space Cremona group. It is obvious that such embeddings exist, but their full classification should be significantly more difficult. There are only some partial results in this direction (see e.g. [26–29, 62]).

5. Jordan property

The methods and results of [40] show that one cannot expect a reasonable classification of all finite subgroups of Cremona groups of higher rank. Thus it is natural to concentrate on the study of general properties of these subgroups. Recall the following two famous results by C. Jordan and H. Minkowski.

Theorem 5.1 ([53]). *There exists a function* $\mathrm{j}(n)$ *such that for any finite subgroup* $G \subset \mathrm{GL}_n(\mathbb{C})$*, there exists a normal abelian subgroup* $A \subset G$ *of index at most* $\mathrm{j}(n)$.

Theorem 5.2 ([69]). *There exists a function* $\mathrm{b}(n)$ *such that for every finite subgroup* $G \subset \mathrm{GL}_n(\mathbb{Q})$*, one has* $|G| \le \mathrm{b}(n)$.

J.-P. Serre [94, 96] asked if these properties hold for Cremona groups. Complete answers to these questions were given in [82, 83] (see below). The following very convenient definitions were suggested by V. L. Popov [75].

Definition.
- A group Γ is *Jordan* if there exists a constant $\mathrm{j}(\Gamma)$ such that any finite subgroup $G \subset \Gamma$ has a normal abelian subgroup A of index $[G : A] \le \mathrm{j}(\Gamma)$.
- A group Γ is *bounded* (or satisfy *bfs* property) if there exists a constant $\mathrm{b}(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$, one has $|G| \le \mathrm{b}(\Gamma)$.

Rationally connected varieties

Theorem 5.3 ([13, 83]). *Let X be a rationally connected variety. Then* $\mathrm{Bir}(X)$ *is Jordan. Moreover,* $\mathrm{Bir}(X)$ *is* uniformly *Jordan; that is, the constant* $\mathrm{j}(\mathrm{Bir}(X))$ *depends only on* $\dim(X)$.

As a consequence, we obtain that the group $\mathrm{Cr}_n(\Bbbk)$ is Jordan.

Originally, Theorem 5.3 was proved modulo the so-called BAB conjecture (in a weak form), which is now settled by C. Birkar:

Theorem 5.4 ([13]). *Fix $d > 0$. The set of all Fano varieties X of dimension at most d with at worst terminal singularities form a bounded family; i.e., they are parameterized by a scheme of finite type.*

It follows from Theorem 5.3 that there is a constant $L = L(n)$ such that for any rationally connected variety X of dimension n and for any prime $p > L(n)$, every finite p-subgroup of $\mathrm{Bir}(X)$ is abelian and generated by at most n elements (see [83]). Recently this result was essentially improved by Jinsong Xu [104]; he showed that $L(n) = n + 1$. The proof is based on a result by O. Haution [47]. Thus we have the following theorem.

Theorem 5.5. *Let X be a rationally connected variety of dimension n and let $G \subset \mathrm{Bir}(X)$ be a finite p-subgroup. If $p > n + 1$, then G is abelian and is generated by at most n elements.*

The results of Theorems 5.3 and 5.5 were applied in the proof of Jordan property of local fundamental groups of log terminal singularities [20, 71].

Varieties over non-closed fields

Theorem 5.6 ([13, 82]). *Let X be a variety over a field $\Bbbk$ of characteristic 0, which is finitely generated over $\mathbb{Q}$. Then the group $\mathrm{Bir}(X)$ is bfs.*

Similar to Theorem 5.3, the proof of this result is based on the BAB conjecture (Theorem 5.4).

In the case $X = \mathbb{P}^2$, an explicit bound was obtained in [94] (see also [41]) in terms of cyclotomic invariants of the field $\Bbbk$. Theorem 5.6 can be reformulated in an algebraic form, which gives the positive answer to a question of J.-P. Serre [96].

Theorem 5.6a. *Let $\mathbb{K}$ be a finitely generated field over $\mathbb{Q}$. Then the group $\mathrm{Aut}(\mathbb{K})$ is bfs.*

Jordan constants. Define the *Jordan constant* of a group Γ as the number $\mathrm{j}(\Gamma)$ that appears in the definition of Jordan property. The *weak Jordan constant* $\bar{\mathrm{j}}(\Gamma)$ of Γ is the minimal j such that for any finite subgroup $G \subset \Gamma$, there exists an abelian (not necessarily normal) subgroup $A \subset G$ such that $[G : A] \le j$. Easy group-theoretic arguments show that

$$\bar{\mathrm{j}}(\Gamma) \le \mathrm{j}(\Gamma) \le \bar{\mathrm{j}}(\Gamma)^2.$$

The exact value of the Jordan constant is known only for the Cremona group of rank two: $\mathrm{j}(\mathrm{Cr}_2(\Bbbk)) = 7200$ (see [105]). On the other hand, weak Jordan constants are easier to compute. It was proved in [84] that

$$\bar{\mathrm{j}}(\mathrm{Cr}_2) = 288, \quad \bar{\mathrm{j}}(\mathrm{Cr}_3) = 10368.$$

Moreover, the inequality $\bar{\mathrm{j}}(\mathrm{Bir}(X)) \leq 10368$ holds for any rationally connected threefold X.

Jordan property of arbitrary varieties. It turns out that the group of birational transformations of an algebraic variety is not always Jordan. The first example was discovered by Yu. Zarhin.

Example ([106]). Let C be an elliptic curve and let $X = C \times \mathbb{P}^1$. Then the group $\mathrm{Bir}(X)$ is not Jordan.

On the other hand, the exceptions as above are very rare.

Theorem 5.7 (V. L. Popov [75]). *Let X be an algebraic surface. The group* $\mathrm{Bir}(X)$ *is not Jordan if and only if X is birationally equivalent to $\mathbb{P}^1 \times C$, where C is an elliptic curve.*

The proof of this theorem given in [75] essentially uses a result of I. Dolgachev, which in turn is based on the classification of algebraic surfaces. Later, Theorem 5.7 was generalized to higher dimensions with classification independent proofs.

Theorem 5.8 ([82]). *Let X be an algebraic variety. Then the following assertions hold.*

(i) *If X either is non-uniruled or has irregularity $q(X) = 0$, then* $\mathrm{Bir}(X)$ *is Jordan.*

(ii) *If X is non-uniruled and $q(X) = 0$, then* $\mathrm{Bir}(X)$ *is bfs.*

Similar to Theorems 5.6 and 5.3, the proof of Theorem 5.8(i) is based on the boundedness of terminal Fano varieties (Theorem 5.4).

In dimension three, there is the following much more precise result.

Theorem 5.9 ([85]). *Let X be a three-dimensional algebraic variety. Then* $\mathrm{Bir}(X)$ *is not Jordan if and only if either*

(i) *X is birationally equivalent to $C \times \mathbb{P}^2$, where C is an elliptic curve, or*

(ii) *X is birationally equivalent to $S \times \mathbb{P}^1$, where S is one of the following:*

- *a surface of Kodaira dimension $\varkappa(S) = 1$ such that the Jacobian fibration of the pluricanonical map $\phi\colon S \to B$ is locally trivial;*
- *S is either an abelian or bielliptic surface (and $\varkappa(S) = 0$).*

Below we explain the main idea of the proof of the necessity. So we assume that $\mathrm{Bir}(X)$ is not Jordan. By Theorems 5.3 and 5.8, the variety X is uniruled, but it is not rationally connected. Hence there exists a map $X \dashrightarrow Z$ with rationally connected fibers (so-called maximal rationally connected fibration) such that Z is not uniruled and $\dim(Z) = 1$ or 2 (see [56]). We have a natural exact sequence

$$1 \to \mathrm{Bir}(X_\eta) \to \mathrm{Bir}(X) \to \mathrm{Bir}(Z),$$

where X_η is the generic scheme-theoretic fiber. Since X_η is rationally connected and Z is not uniruled, the groups $\mathrm{Bir}(X_\eta)$ and $\mathrm{Bir}(Z)$ must be Jordan. Then group-theoretic arguments show that both groups $\mathrm{Bir}(X_\eta)$ and $\mathrm{Bir}(Z)$ are not bfs (see, e.g., [82, Lemma 2.8]). In the case where Z is a curve, this implies that Z is elliptic, and applying the following fact with $\mathbb{K} = \Bbbk(Z)$ and $S := X_\eta$, we obtain that X is birationally equivalent to $Z \times \mathbb{P}^2$.

Proposition 5.10 ([85]). *Let $\mathbb{K}$ be a field containing all roots of* 1 *and let S be a surface over $\mathbb{K}$ such that S is not $\mathbb{K}$-rational, S is $\bar{\mathbb{K}}$-rational, and $S(\mathbb{K}) \neq \varnothing$. Then the group* $\mathrm{Bir}(S)$ *is bfs.*

Note that the condition of the existence of a $\mathbb{K}$-point on S in the above statement is important. The groups of (birational) automorphisms of geometrically rational surfaces without rational points were studied in the series of papers [100–102].

Now assume that Z is a surface. According to the main result of [7], the threefold X is birationally equivalent to $Z \times \mathbb{P}^1$. By Theorem 5.8 we have $q(Z) > 0$. Thus in the case $\varkappa(Z) = 0$, the surface Z must be either abelian or bielliptic. Since the group $\mathrm{Bir}(Z)$ is not finite in our case, Z cannot be a surface of general type. Consider the case $\varkappa(Z) = 1$. Then the pluricanonical map $\phi : Z \to B$ is a $\mathrm{Bir}(Z)$-equivariant elliptic fibration. Let

$$\mathrm{Jac}(\phi) : E \to B$$

be the corresponding Jacobian fibration. The automorphism group $\mathrm{Aut}(Z_\eta)$ of the generic fiber Z_η over B is embedded to $\mathrm{Bir}(Z)$ as a normal subgroup. Analyzing singular fibers, one can conclude that $\mathrm{Aut}(Z_\eta)$ is of finite index in $\mathrm{Bir}(Z)$. In turn, $\mathrm{Aut}(Z_\eta)$ has a subgroup $\mathrm{Aut}'(Z_\eta)$ of index at most 6 isomorphic to the group of $\Bbbk(B)$-points of E_η. Assume that the fibration $\mathrm{Jac}(\phi)$ is not locally trivial. Then by the functional version of Mordell–Weil theorem, known as Lang–Néron theorem (see, e.g., [32]), the group of $\Bbbk(B)$-points of E_η is finitely generated, and in particular, the torsion subgroup of the group of points of E_η is finite. This implies that $\mathrm{Aut}'(Z_\eta)$ is finite. ■

6. Invariants and rigidity

The most important part of the classification of finite subgroups in $\mathrm{Bir}(X)$ is to distinguish conjugacy classes.

Problem 6.1. Let $G, G' \subset \mathrm{Bir}(X)$ be finite subgroups such that $G \simeq G'$. How can one conclude that G and G' are *not* conjugate?

This is equivalent to the following.

Problem 6.1a. Let X and X' be G-varieties. How can one conclude that X and X' are *not* G-birational?

Below we describe a few approaches to solve the above problems. Note, however, that there are no universal methods.

Fixed point locus. Let X be a smooth projective G-variety. By $\mathrm{Fix}(X, G)$, we denote the set of G-fixed points. It is not difficult to show (see [87]) that $\mathrm{Fix}(X, G)$ has at most one codimension one component that is not uniruled. Denote this component by $\mathrm{F}^{\mathrm{nu}}(X, G)$. This is a natural birational invariant in the category of smooth projective G-varieties.

Proposition 6.2 ([87]). *Let X and X' be smooth projective G-varieties. If X and X' are G-birational, then* $\mathrm{F}^{\mathrm{nu}}(X, G_0)$ *and* $\mathrm{F}^{\mathrm{nu}}(X', G_0)$ *are birational for any subgroup* $G_0 \subset G$.

If $G_0 \subset G$ is a normal subgroup, then the set $\mathrm{F}^{\mathrm{nu}}(X, G_0)$ (if it is not empty) has a structure of (G/G_0)-variety. Clearly, the birational type of this (G/G_0)-variety is also a birational invariant (cf. [16]).

Example. According to Theorem 3.1 for subgroups $G \subset \mathrm{Cr}_2(\Bbbk)$ of order 2, we have one of the following possibilities:

	Involution $\tau \in G$	$\mathrm{F}^{\mathrm{nu}}(X, G)$
1^o	Linear on $\mathbb{P}^2$	$\varnothing$
2^o	de Jonquières of genus $g \geq 1$	Hyperelliptic curve of genus g
3^o	Geiser	Non-hyperelliptic curve of genus 3
4^o	Bertini	Special non-hyperelliptic curve of genus 4

Thus the curve $\mathrm{F}^{\mathrm{nu}}(X, G)$ distinguishes conjugacy classes in this case. The same assertion is true for subgroups of prime order [36], but it fails in general [15].

Cohomological invariants. It is not difficult to see that for a smooth projective G-variety X, the cohomology group

$$H^1\big(G,\ \mathrm{Pic}(X)\big)$$

is a G-birational invariant (see [19]). More generally, we say that G-varieties X and X' are *stably* G-birationally equivalent if for some n and m the products $X \times \mathbb{P}^n$ and $X' \times \mathbb{P}^m$ are G-birationally equivalent, where the action of G on $\mathbb{P}^n$ and $\mathbb{P}^m$ is supposed to be trivial. Then we have the following theorem.

Theorem 6.3 ([19]). *Let X and X' be smooth projective G-varieties. If X and X' are stably G-birationally equivalent, then*

$$H^1\big(G, \mathrm{Pic}(X)\big) \simeq H^1\big(G,\ \mathrm{Pic}(X')\big).$$

Surprisingly, in some cases, the invariant $H^1(G, \mathrm{Pic}(X))$ can be computed in terms of G-fixed locus.

Theorem 6.4 ([19]). *Let G be a cyclic group of prime order p and let X be a smooth projective rational G-surface. Assume that $\mathrm{F}^{\mathrm{nu}}(X, G)$ is a curve of genus g. Then*

$$H^1\big(G, \mathrm{Pic}(X)\big) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

This theorem was slightly generalized with a more conceptual proof in [97]. Another cohomological invariant which is called *Amitsur group* was introduced in [17].

As a consequence of Theorem 6.4, one can see that involutions from different families in Theorem 3.1 are not stably conjugate in $\mathrm{Cr}_2(\Bbbk)$. Note, however, that $H^1(G, \mathrm{Pic}(X))$ is a discrete invariant. For example, stable conjugacy of involutions whose curves $\mathrm{F}^{\mathrm{nu}}(X, G)$ are non-isomorphic but have the same genus is not known.

A natural question that arises here is to find examples of subgroups in $\mathrm{Cr}_n(\Bbbk)$ that are stably conjugate but not conjugate. This question is similar to the birational Zariski problem [11].

Example. Let $G = \mathfrak{S}_3 \times \boldsymbol{\mu}_2$. There are two embeddings of this group into the Cremona group $\mathrm{Cr}_2(\Bbbk)$ induced by the following actions:

(i) action on $\mathbb{P}^2 = \{x_1 + x_2 + x_3 = 0\} \subset \mathbb{P}^3$ by permutation and reversing signs;

(ii) action on the sextic del Pezzo surface $\{y_1 y_2 y_3 = y'_1 y'_2 y'_3\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by permutation and taking inverses.

It was shown in [65] that these two subgroups in $\mathrm{Cr}_2(\Bbbk)$ are stably conjugate; in fact, they are conjugate in $\mathrm{Cr}_4(\Bbbk)$. On the other hand, they are not conjugate [51].

Here is another example of this kind, which was pointed out to us by Yuri Tschinkel.

Example ([92]). Let V and W be faithful linear representations of G with $\dim(V) = \dim(W) = n$. Assume that the images of G in $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$ do not contain non-identity scalar matrices. Then by a variant of the no-name lemma [39], we have the following G-birational equivalences of G-varieties:

$$\mathbb{P}(V) \times \mathbb{k}^{n+1} \underset{\mathrm{bir}}{\sim} V \times W \underset{\mathrm{bir}}{\sim} \mathbb{P}(W) \times \mathbb{k}^{n+1},$$

where $\mathbb{k}^{n+1}$ is viewed as the trivial representation. Hence G-varieties V and W are stably G-birationally equivalent. On the other hand, it may happen that they are *not* G-birationally equivalent.

For example, Reichstein and Youssin [92] showed that the *determinant* of the action in the tangent space at a fixed point of a finite abelian group, up to sign, is a birational invariant of the action. This allowed them to produce nonbirational linear actions, e.g., of groups $\boldsymbol{\mu}_{p^n}$ on $\mathbb{P}^n$, with $p \geq 5$. Many new examples of nonbirational linear actions were given in [60, Section 10-11]; these are based on new invariants introduced in [61] (see also [46, 59]). These invariants take into account more refined information about the action on subvarieties with nontrivial abelian stabilizers.

A prime number p is said to be a *torsion prime* for the group $\mathrm{Bir}(X)$ if there is a finite abelian p-subgroup $G \subset \mathrm{Bir}(X)$ not contained in any algebraic torus of $\mathrm{Bir}(X)$ [76]. Note that if a group G is contained in an algebraic torus $T \subset \mathrm{Bir}(X)$, then for any smooth projective birational model Y of X on which T acts biregularly, we have $H^1(G, \mathrm{Pic}(Y)) = 0$. Then by Theorem 6.3, the inequality $H^1(G, \mathrm{Pic}(Y)) \neq 0$ for a finite p-subgroup $G \subset \mathrm{Aut}(Y)$ implies that a prime number p is a torsion prime for $\mathrm{Bir}(Y)$ and for $\mathrm{Bir}(Y \times \mathbb{P}^n)$ for any n. Using Theorem 6.4 and the classification [36], one can immediately see that the set of all torsion primes for $\mathrm{Cr}_2(\mathbb{k})$ is equal to $\{2, 3, 5\}$, and the numbers 2, 3, and 5 are torsion primes for $\mathrm{Cr}_n(\mathbb{k})$ for any $n \geq 2$. This fact was proved in [76] by using another argument. In the case $n \geq 3$, the collection of all torsion primes for $\mathrm{Cr}_n(\mathbb{k})$ is unknown.

Maximal singularities method. The maximal singularities method is the most powerful tool to study birational maps between Mfs's. It goes back to the works of G. Fano and even earlier works of other Italian geometers. However, the first application of this technique with rigorous proofs appeared much later in the breakthrough paper of Manin and Iskovskikh [49]. For an introduction to the "standard," non-equivariant maximal singularities method, we refer to the book [90]. Below we outline very briefly an equivariant version of the method.

Definition ([40, Definition 7.10], [29, Definition 3.1.1]). A $G\mathbb{Q}$-Fano variety X is said to be *G-birationally rigid* if given birational G-map $\Phi : X \dashrightarrow X^\sharp$ to the total space of another G-Mfs $X^\sharp/Z^\sharp$, there exists a birational G-selfmap $\psi : X \dashrightarrow X$

such that the composition $\Phi \circ \psi : X \dashrightarrow X^\sharp$ is an isomorphism (in particular, $Z^\sharp$ is a point; i.e., $X^\sharp$ is also a $G\mathbb{Q}$-Fano variety).

A $G\mathbb{Q}$-Fano variety X is said to be *G-birationally superrigid* if any birational G-map $\Phi : X \dashrightarrow X^\sharp$ to the total space of another G-Mfs $X^\sharp/Z^\sharp$ is an isomorphism.

The maximal singularities method allows to check G-birational (super)rigidity using only internal geometry of the original variety, without considering all other G-Mfs's. We need the following technical definition which has become common nowadays.

Definition. Let X be a normal variety, let $\mathcal{M}$ be a linear system of Weil divisors on X without fixed components, and let λ be a rational number. We say that the pair $(X, \lambda\mathcal{M})$ is *canonical* if some multiple $m(K_X + \lambda M)$ is Cartier, where $M \in \mathcal{M}$, and for any birational morphism $f : Y \to X$, one can write

$$m(K_Y + \lambda\mathcal{M}_Y) = f^* m(K_X + \lambda\mathcal{M}) + \sum a_i E_i,$$

where $\mathcal{M}_Y$ is the birational transform of $\mathcal{M}$, E_i are prime exceptional divisors, and $a_i \geq 0$ for all i.

In the surface case, the canonical property is very easy to check: a pair $(X, \lambda\mathcal{M})$ is canonical if and only if

$$\operatorname{mult}_P(\mathcal{M}) \leq 1/\lambda$$

for any point $P \in X$.

Now, suppose that a $G\mathbb{Q}$-Fano variety X is not G-birationally superrigid. Then the Noether–Fano inequality [34, Theorem 4.2] implies the existence of a G-invariant linear system $\mathcal{M}$ on X without fixed components such that the pair $(X, \lambda\mathcal{M})$ is not canonical, where $\lambda \in \mathbb{Q}$ is taken so that $K_X + \lambda\mathcal{M}$ is numerically trivial. Moreover, any $\mathcal{M}$ as above defines a birational G-map $X \dashrightarrow X^\sharp$ to the total space of a G-Mfs $X^\sharp/Z^\sharp$. To show the existence or non-existence of such $\mathcal{M}$, one needs to analyze the geometry of the variety X carefully.

Example. Let X be a del Pezzo surface of degree 1. Assume that X is a G-del Pezzo with respect to some group $G \subset \operatorname{Aut}(X)$. This means that G acts on X so that $\operatorname{rk}\operatorname{Pic}(X)^G = 1$. For example, this holds for any subgroup $G \subset \operatorname{Aut}(X)$ containing the Bertini involution. Let $\mathcal{M}$ be a G-invariant linear subsystem without fixed components. Since $\operatorname{Pic}(X)^G = \mathbb{Z} \cdot K_X$, we have $\mathcal{M} \subset |-nK_X|$ for some $n > 0$. Suppose that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Then $\operatorname{mult}_P(\mathcal{M}) > n$. Since $\mathcal{M}$ has no fixed components,

$$n^2 = (-nK_X)^2 = \mathcal{M}^2 \geq \big(\operatorname{mult}_P(\mathcal{M})\big)^2 > n^2.$$

The contradiction shows that X is G-birationally superrigid.

Similar arguments show that any G-del Pezzo surface X of degree ≤ 3 is G-birationally rigid. Moreover, it is G-birationally superrigid if and only if G has no orbits of length $\leq K_X^2 - 2$ on X. In particular, $\mathrm{PSL}_2(\mathbf{F}_7)$-del Pezzo surface from Theorem 3.2 is G-birationally superrigid.

Example. All the $G\mathbb{Q}$-Fano threefolds from Theorem 4.2 are G-birationally superrigid [17, 28, 30]. In particular, different embeddings of $\mathrm{PSp}_4(\mathbf{F}_3)$ and $\mathrm{PSL}_2(\mathbf{F}_{11})$ are not conjugate in $\mathrm{Cr}_3(\Bbbk)$.

There is another relevant and very important notion called G-solidity [25]. For Fano varieties without group action, this notion has been introduced earlier by Shokurov [98] (who called solid Fano varieties primitive) and by Ahmadinezhad and Okada [2].

Definition ([25]). A G-Fano variety X is G-*solid* if X is not G-birational to a G-Mfs with a positive dimensional base.

For example, a G-del Pezzo surface X of degree 4 is G-solid if and only if G has no fixed points on X [40, §8].

A part of the maximal singularities method is the so-called Sarkisov program [34, 45]. It allows us to decompose any birational map between Mfs's into a composition of elementary ones. Refer to [50] for an explicit description of this program in dimension two and to [31] for examples and applications.

7. Application: Essential dimension

The notion of the essential dimension of a finite group G, denoted by $\mathrm{ed}(G)$, was introduced by Buhler and Reichstein [21]. Informally, $\mathrm{ed}(G)$ is the minimal number of algebraic parameters needed to describe a faithful representation. More precisely, given a faithful linear representation V of G viewed as a G-variety, the *essential dimension* $\mathrm{ed}(G, V)$ is the minimal value of $\dim(X)$, where X is taken from the set of all G-varieties admitting dominant rational G-equivariant map $V \dashrightarrow X$. It can be shown that $\mathrm{ed}(G, V)$ does not depend on V, so we can omit V in the notation. It is easy to see that $\mathrm{ed}(G) = 1$ if and only if G is cyclic or dihedral of order $2n$ where n is odd. Finite groups of essential dimension ≤ 2 have been classified [43].

The essential dimension of symmetric groups $\mathfrak{S}_n$ is important because it is equal to the minimal number of parameters needed to describe the general polynomial of degree n modulo Tschirnhaus transformations [21]. The values of $\mathrm{ed}(\mathfrak{S}_n)$, as well as of $\mathrm{ed}(\mathfrak{A}_n)$, are known for $n \leq 7$, and bounds exist for any n as follows.

Theorem 7.1 ([21, 42]). *If $n \geq 6$, then*

$$n - 3 \geq \mathrm{ed}(\mathfrak{S}_n) \geq \lfloor n/2 \rfloor,$$

$$\mathrm{ed}(\mathfrak{S}_n) \geq \mathrm{ed}(\mathfrak{A}_n) \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 2\lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

In many cases, the computations of $\mathrm{ed}(G)$ use the machinery of G-varieties. As an example, following Serre [95], we show that $\mathrm{ed}(\mathfrak{A}_6) = 3$. Let V be the standard six-dimensional permutation representation of $\mathfrak{A}_6$. There exists an equivariant open embedding $V \subset (\mathbb{P}^1)^6$. On the other hand, the group $\mathrm{PSL}_2(\Bbbk)$ also acts on $(\mathbb{P}^1)^6$ so that the two actions commute. Hence we have a dominant rational $\mathfrak{A}_6$-map

$$V \hookrightarrow (\mathbb{P}^1)^6 \to (\mathbb{P}^1)^6 / \mathrm{PSL}_2(\Bbbk),$$

where $(\mathbb{P}^1)^6 / \mathrm{PSL}_2(\Bbbk)$ is a birational quotient. Since $\dim((\mathbb{P}^1)^6 / \mathrm{PSL}_2(\Bbbk)) = 3$, we have $\mathrm{ed}(\mathfrak{A}_6) \leq 3$. Thus it is sufficient to show that $\mathrm{ed}(\mathfrak{A}_6)$ is not equal to 2. If so, there exists a dominant rational G-map $V \dashrightarrow X$ to a surface which must be rational. According to Theorem 3.2, we may assume that $X = \mathbb{P}^2$. But in this case, a Sylow 3-subgroup $S \subset \mathfrak{A}_6$ is abelian and acts without fixed points on $\mathbb{P}^2$. On the other hand, S has a fixed point on V, and the same should be true for the image of any rational S-map to a projective variety [58]. Therefore, $\mathrm{ed}(\mathfrak{A}_6) = 3$ as claimed.

Using similar arguments and the classification of embeddings of $\mathfrak{A}_7$ to groups of birational transformations of rationally connected threefolds (Theorem 4.2), A. Duncan proved that $\mathrm{ed}(\mathfrak{A}_7) = \mathrm{ed}(\mathfrak{S}_7) = 4$ [42].

Denote by $\mathrm{rdim}(G)$ (resp. $\mathrm{cdim}(G)$) the minimal dimension of faithful representations of G (resp. the smallest n such that G is embeddable to $\mathrm{Cr}_n(\Bbbk)$). It immediately follows from the definition that

$$\mathrm{ed}(G) \leq \mathrm{rdim}(G).$$

If G is a p-group, then the equality holds $\mathrm{ed}(G) = \mathrm{rdim}(G)$ [54]. In general, this equality fails, but there is a bound in terms of Jordan constants.

Theorem 7.2 ([91]). $\mathrm{rdim}(G) \leq \mathrm{ed}(G) \cdot \mathrm{j}(\mathrm{ed}(G))$*, where* $\mathrm{j}(n)$ *is the Jordan constant.*

I. Dolgachev conjectured that $\mathrm{ed}(G) \geq \mathrm{cdim}(G)$ (see [44]). It would be interesting to test this conjecture for the group $G = \mathrm{PSL}_2(\mathbf{F}_{11})$. In fact, we have

$$3 \leq \mathrm{ed}\left(\mathrm{PSL}_2(\mathbf{F}_{11})\right) \leq 4$$

by Theorem 3.2 and because the group $\mathrm{PSL}_2(\mathbf{F}_{11})$ is simple and has a faithful five-dimensional representation. Assuming Dolgachev's conjecture, by Theorem 4.2 we would have $\mathrm{ed}(\mathrm{PSL}_2(\mathbf{F}_{11})) = 4$. But this is unknown. See [44] for interesting discussions. The computation of the essential dimension of $\mathrm{PSL}_2(\mathbf{F}_{11})$ should complete Beauville's classification of finite simple groups of essential dimension ≤ 3 [10].

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References

[1] D. Abramovich and J. Wang, Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.* **4** (1997), no. 2-3, 427–433 Zbl 0906.14005 MR 1453072

[2] H. Ahmadinezhad and T. Okada, Birationally rigid Pfaffian Fano 3-folds. *Algebr. Geom.* **5** (2018), no. 2, 160–199 Zbl 1407.14038 MR 3769891

[3] M. Alberich-Carramiñana, *Geometry of the Plane Cremona Maps*. Lecture Notes in Math. 1769, Springer, Berlin, 2002 Zbl 0991.14008 MR 1874328

[4] A. Avilov, Automorphisms of threefolds that can be represented as an intersection of two quadrics. *Mat. Sb.* **207** (2016), no. 3, 3–18 Zbl 1370.14036 MR 3507481

[5] A. Avilov, Automorphisms of singular three-dimensional cubic hypersurfaces. *Eur. J. Math.* **4** (2018), no. 3, 761–777 Zbl 1423.14096 MR 3851116

[6] A. A. Avilov, Existence of standard models of conic bundles over algebraically non-closed fields. *Mat. Sb.* **205** (2014), no. 12, 3–16 Zbl 1317.14091 MR 3309386

[7] T. Bandman and Y. G. Zarhin, Jordan groups, conic bundles and abelian varieties. *Algebr. Geom.* **4** (2017), no. 2, 229–246 Zbl 1388.14047 MR 3620637

[8] L. Bayle and A. Beauville, Birational involutions of $\mathbf{P}^2$. *Asian J. Math.* **4** (2000), no. 1, 11–17; Kodaira's issue Zbl 1055.14012 MR 1802909

[9] A. Beauville, p-elementary subgroups of the Cremona group. *J. Algebra* **314** (2007), no. 2, 553–564 Zbl 1126.14017 MR 2344578

[10] A. Beauville, Finite simple groups of small essential dimension. In *Trends in Contemporary Mathematics*, pp. 221–228, Springer INdAM Ser. 8, Springer, Cham, 2014 Zbl 1386.14173 MR 3586401

[11] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, Variétés stablement rationnelles non rationnelles. *Ann. of Math. (2)* **121** (1985), no. 2, 283–318 Zbl 0589.14042 MR 786350

[12] E. Bertini, Ricerche sulle trasformazioni univoche involutorie nel piano. *Annali di Mat. Pura Appl.* **8** (1877), 254–287 Zbl 09.0578.02

[13] C. Birkar, Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)* **193** (2021), no. 2, 347–405 Zbl 1469.14085 MR 4224714

[14] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468 Zbl 1210.14019 MR 2601039

[15] J. Blanc, Linearisation of finite abelian subgroups of the Cremona group of the plane. *Groups Geom. Dyn.* **3** (2009), no. 2, 215–266 Zbl 1170.14009 MR 2486798

[16] J. Blanc, Elements and cyclic subgroups of finite order of the Cremona group. *Comment. Math. Helv.* **86** (2011), no. 2, 469–497 Zbl 1213.14029 MR 2775137

[17] J. Blanc, I. Cheltsov, A. Duncan, and Y. Prokhorov, Finite quasisimple groups acting on rationally connected threefolds. 2018, arXiv:1809.09226

[18] J. Blanc, S. Lamy, and S. Zimmermann, Quotients of higher-dimensional Cremona groups. *Acta Math.* **226** (2021), no. 2, 211–318 Zbl 07378146 MR 4281381

[19] F. Bogomolov and Y. Prokhorov, On stable conjugacy of finite subgroups of the plane Cremona group, I. *Cent. Eur. J. Math.* **11** (2013), no. 12, 2099–2105 Zbl 1286.14016 MR 3111709

[20] L. Braun, S. Filipazzi, J. Moraga, and R. Svaldi, The Jordan property for local fundamental groups. *Geom. Topol.* **26** (2022), no. 1, 283–319 Zbl 07525902 MR 4404879

[21] J. Buhler and Z. Reichstein, On the essential dimension of a finite group. *Compositio Math.* **106** (1997), no. 2, 159–179 Zbl 0905.12003 MR 1457337

[22] S. Cantat, The Cremona group in two variables. In *European Congress of Mathematics*, pp. 211–225, Eur. Math. Soc., Zürich, 2013 Zbl 1364.14009 MR 3469123

[23] S. Cantat, The Cremona group. In *Algebraic Geometry: Salt Lake City 2015*, pp. 101–142, Proc. Sympos. Pure Math. 97, Amer. Math. Soc., Providence, RI, 2018 Zbl 1451.14037 MR 3821147

[24] S. Cantat and S. Lamy, Normal subgroups in the Cremona group. *Acta Math.* **210** (2013), no. 1, 31–94 Zbl 1278.14017 MR 3037611

[25] I. Cheltsov, A. Dubouloz, and T. Kishimoto, Toric G-solid Fano threefolds. 2020, arXiv:2007.14197

[26] I. Cheltsov, V. Przyjalkowski, and C. Shramov, Burkhardt quartic, Barth sextic, and the icosahedron. *Int. Math. Res. Not. IMRN* **2019** (2019), no. 12, 3683–3703 Zbl 1454.14036 MR 3973105

[27] I. Cheltsov and C. Shramov, Three embeddings of the Klein simple group into the Cremona group of rank three. *Transform. Groups* **17** (2012), no. 2, 303–350 Zbl 1272.14013 MR 2921069

[28] I. Cheltsov and C. Shramov, Five embeddings of one simple group. *Trans. Amer. Math. Soc.* **366** (2014), no. 3, 1289–1331 Zbl 1291.14060 MR 3145732

[29] I. Cheltsov and C. Shramov, *Cremona Groups and the Icosahedron*. Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2016 Zbl 1328.14003 MR 3444095

[30] I. Cheltsov and C. Shramov, Finite collineation groups and birational rigidity. *Selecta Math. (N.S.)* **25** (2019), no. 5, Paper No. 71 Zbl 1440.14061 MR 4036497

[31] I. A. Cheltsov, Two local inequalities. *Izv. Ross. Akad. Nauk Ser. Mat.* **78** (2014), no. 2, 167–224 Zbl 1329.14021 MR 3234821

[32] B. Conrad, Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *Enseign. Math. (2)* **52** (2006), no. 1-2, 37–108 Zbl 1133.14028 MR 2255529

[33] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, $\mathbb{ATLAS}$ *of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups. With computational assistance from J. G. Thackray*. Oxford University Press, Eynsham, 1985 Zbl 0568.20001 MR 827219

[34] A. Corti, Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.* **4** (1995), no. 2, 223–254 Zbl 0866.14007 MR 1311348

[35] A. Corti, Del Pezzo surfaces over Dedekind schemes. *Ann. of Math. (2)* **144** (1996), no. 3, 641–683 Zbl 0902.14026 MR 1426888

[36] T. de Fernex, On planar Cremona maps of prime order. *Nagoya Math. J.* **174** (2004), 1–28 Zbl 1062.14019 MR 2066103

[37] O. Debarre and A. Kuznetsov, Gushel–Mukai varieties: classification and birationalities. *Algebr. Geom.* **5** (2018), no. 1, 15–76 Zbl 1408.14053 MR 3734109

[38] J. Déserti, *Some Properties of the Cremona Group*. Ensaios Mat. 21, Sociedade Brasileira de Matemática, Rio de Janeiro, 2012 Zbl 1276.14020 MR 2934616

[39] I. V. Dolgachev, Rationality of fields of invariants. In *Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, pp. 3–16, Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 1987 Zbl 0659.14009 MR 927970

[40] I. V. Dolgachev and V. A. Iskovskikh, Finite subgroups of the plane Cremona group. In *Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin. Vol. I*, pp. 443–548, Progr. Math. 269, Birkhäuser, Boston, MA, 2009 Zbl 1219.14015 MR 2641179

[41] I. V. Dolgachev and V. A. Iskovskikh, On elements of prime order in the plane Cremona group over a perfect field. *Int. Math. Res. Not. IMRN* **2019** (2009), no. 18, 3467–3485 Zbl 1188.14007 MR 2535007

[42] A. Duncan, Essential dimensions of A_7 and S_7. *Math. Res. Lett.* **17** (2010), no. 2, 263–266 Zbl 1262.14057 MR 2644373

[43] A. Duncan, Finite groups of essential dimension 2. *Comment. Math. Helv.* **88** (2013), no. 3, 555–585 Zbl 1300.14044 MR 3093503

[44] A. Duncan and Z. Reichstein, Versality of algebraic group actions and rational points on twisted varieties. *J. Algebraic Geom.* **24** (2015), no. 3, 499–530 Zbl 1327.14210 MR 3344763

[45] C. D. Hacon and J. McKernan, The Sarkisov program. *J. Algebraic Geom.* **22** (2013), no. 2, 389–405 Zbl 1267.14024 MR 3019454

[46] B. Hassett, A. Kresch, and Y. Tschinkel, Symbols and equivariant birational geometry in small dimensions. In *Rationality of Varieties*, pp. 201–236, Progr. Math. 342, Birkhäuser, Cham, 2021 MR 4383699

[47] O. Haution, Fixed point theorems involving numerical invariants. *Compos. Math.* **155** (2019), no. 2, 260–288 Zbl 1441.14156 MR 3905117

[48] H. P. Hudson, *Cremona Transformations in Plane and Space*. Cambridge University Press, 1927

[49] V. A. Iskovskih and J. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. Sb. (N.S.)* **86(128)** (1971), 140–166 MR 0291172

[50] V. A. Iskovskikh, Factorization of birational mappings of rational surfaces from the point of view of Mori theory. *Uspekhi Mat. Nauk* **51** (1996), no. 4(310), 3–72 Zbl 0914.14005 MR 1422227

[51] V. A. Iskovskikh, Two nonconjugate embeddings of the group $S_3 \times Z_2$ into the Cremona group. *Tr. Mat. Inst. Steklova* **241** (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 105–109 Zbl 1078.14015 MR 2024046

[52] V. A. Iskovskikh and Y. G. Prokhorov, Fano varieties. In *Algebraic Geometry, V*, pp. 1–247, Encyclopaedia Math. Sci. 47, Springer, Berlin, 1999 Zbl 0912.14013 MR 1668579

[53] M. C. Jordan, Mémoire sur les équations différentielles linéaires à intégrale algébrique. *J. Reine Angew. Math.* **84** (1878), 89–215 Zbl 09.0096.01 MR 1581645

[54] N. A. Karpenko and A. S. Merkurjev, Essential dimension of finite p-groups. *Invent. Math.* **172** (2008), no. 3, 491–508 Zbl 1200.12002 MR 2393078

[55] Y. Kawamata, Boundedness of **Q**-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, pp. 439–445, Contemp. Math. 131, Amer. Math. Soc., Providence, RI, 1992 Zbl 0785.14024 MR 1175897

[56] J. Kollár, Y. Miyaoka, and S. Mori, Rationally connected varieties. *J. Algebraic Geom.* **1** (1992), no. 3, 429–448 Zbl 0780.14026 MR 1158625

[57] J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi, Boundedness of canonical **Q**-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.* **76** (2000), no. 5, 73–77 Zbl 0981.14016 MR 1771144

[58] J. Kollár and E. Szabó, Fixed points of group actions and rational maps. Appendix to "Essential dimensions of algebraic groups and a resolution theorem for G-varieties" by Z. Reichstein and B. Youssin. *Canad. J. Math.* **52** (2000), no. 5, 1054–1056 Zbl 1044.14023 MR 1782331

[59] M. Kontsevich, V. Pestun, and Y. Tschinkel, Equivariant birational geometry and modular symbols. 2019, arXiv:1902.09894

[60] A. Kresch and Y. Tschinkel, Equivariant Burnside groups and representation theory. 2021, arXiv:2108.00518

[61] A. Kresch and Y. Tschinkel, Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **23** (2022), no. 2, 1013–1052

[62] I. Krylov, Families of embeddings of the alternating group of rank 5 into the Cremona group. 2020, arXiv:2005.07354

[63] A. G. Kuznetsov, Y. G. Prokhorov, and C. A. Shramov, Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. *Jpn. J. Math.* **13** (2018), no. 1, 109–185 Zbl 1406.14031 MR 3776469

[64] A. A. Kuznetsova, Finite 3-Subgroups in the Cremona Group of Rank 3. *Mat. Zametki* **108** (2020), no. 5, 725–749 Zbl 1469.14028 MR 4169699

[65] N. Lemire, V. L. Popov, and Z. Reichstein, Cayley groups. *J. Amer. Math. Soc.* **19** (2006), no. 4, 921–967 Zbl 1103.14026 MR 2219306

[66] K. Loginov, Standard models of degree 1 del Pezzo fibrations. *Mosc. Math. J.* **18** (2018), no. 4, 721–737 Zbl 1420.14037 MR 3914112

[67] K. Loginov, A note on 3-subgroups in the space Cremona group. *Comm. Algebra* **50** (2022), no. 9, 3704–3714 MR 4442466

[68] J. I. Manin, Rational surfaces over perfect fields. II. *Mat. Sb. (N.S.)* **72 (114)** (1967), 161–192 Zbl 0182.23701 MR 0225781

[69] H. Minkowski, Zur Theorie der positiven quadratischen Formen. *J. Reine Angew. Math.* **101** (1887), 196–202 Zbl 19.0189.01 MR 1580123

[70] Y. Miyaoka and S. Mori, A numerical criterion for uniruledness. *Ann. of Math. (2)* **124** (1986), no. 1, 65–69 Zbl 0606.14030 MR 847952

[71] J. Moraga, On a toroidalization for klt singularities. 2021, arXiv:2106.15019

[72] S. Mori, On 3-dimensional terminal singularities. *Nagoya Math. J.* **98** (1985), 43–66 Zbl 0589.14005 MR 792770

[73] S. Mori, Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.* **1** (1988), no. 1, 117–253 Zbl 0649.14023 MR 924704

[74] Y. Namikawa, Smoothing Fano 3-folds. *J. Algebraic Geom.* **6** (1997), no. 2, 307–324 Zbl 0906.14019 MR 1489117

[75] V. L. Popov, On the Makar–Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In *Affine Algebraic Geometry*, pp. 289–311, CRM Proc. Lecture Notes 54, Amer. Math. Soc., Providence, RI, 2011 Zbl 1242.14044 MR 2768646

[76] V. L. Popov, Tori in the Cremona groups. *Izv. Ross. Akad. Nauk Ser. Mat.* **77** (2013), no. 4, 103–134 Zbl 1278.14065 MR 3135700

[77] Y. Prokhorov, p-elementary subgroups of the Cremona group of rank 3. In *Classification of Algebraic Varieties*, pp. 327–338, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011 MR 2779480

[78] Y. Prokhorov, Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.* **21** (2012), no. 3, 563–600 Zbl 1257.14011 MR 2914804

[79] Y. Prokhorov, G-Fano threefolds, I. *Adv. Geom.* **13** (2013), no. 3, 389–418 Zbl 1291.14024 MR 3100917

[80] Y. Prokhorov, G-Fano threefolds, II. *Adv. Geom.* **13** (2013), no. 3, 419–434 Zbl 1291.14025 MR 3100918

[81] Y. Prokhorov, 2-elementary subgroups of the space Cremona group. In *Automorphisms in Birational and Affine Geometry*, pp. 215–229, Springer Proc. Math. Stat. 79, Springer, Cham, 2014 Zbl 1327.14070 MR 3229353

[82] Y. Prokhorov and C. Shramov, Jordan property for groups of birational selfmaps. *Compos. Math.* **150** (2014), no. 12, 2054–2072 Zbl 1314.14022 MR 3292293

[83] Y. Prokhorov and C. Shramov, Jordan property for Cremona groups. *Amer. J. Math.* **138** (2016), no. 2, 403–418 Zbl 1343.14010 MR 3483470

[84] Y. Prokhorov and C. Shramov, Jordan constant for Cremona group of rank 3. *Mosc. Math. J.* **17** (2017), no. 3, 457–509 Zbl 1411.14018 MR 3711004

[85] Y. Prokhorov and C. Shramov, Finite groups of birational selfmaps of threefolds. *Math. Res. Lett.* **25** (2018), no. 3, 957–972 Zbl 1423.14094 MR 3847342

[86] Y. Prokhorov and C. Shramov, p-subgroups in the space Cremona group. *Math. Nachr.* **291** (2018), no. 8-9, 1374–1389 Zbl 1423.14099 MR 3817323

[87] Y. G. Prokhorov, On birational involutions of $\mathbb{P}^3$. *Izv. Ross. Akad. Nauk Ser. Mat.* **77** (2013), no. 3, 199–222 Zbl 1282.14025 MR 3098794

[88] Y. G. Prokhorov, Singular Fano manifolds of genus 12. *Mat. Sb.* **207** (2016), no. 7, 101–130 Zbl 1372.14032 MR 3535377

[89] Y. G. Prokhorov, Equivariant minimal model program. *Uspekhi Mat. Nauk* **76** (2021), no. 3(459), 93–182 Zbl 07402603 MR 4265398

[90] A. Pukhlikov, *Birationally Rigid Varieties*. Math. Surveys Monogr. 190, American Mathematical Society, Providence, RI, 2013 Zbl 1297.14001 MR 3060242

[91] Z. Reichstein, The Jordan property of Cremona groups and essential dimension. *Arch. Math. (Basel)* **111** (2018), no. 5, 449–455 Zbl 06951230 MR 3859426

[92] Z. Reichstein and B. Youssin, A birational invariant for algebraic group actions. *Pacific J. Math.* **204** (2002), no. 1, 223–246 Zbl 1054.14062 MR 1905199

[93] M. Reid, Young person's guide to canonical singularities. In *Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, pp. 345–414, Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 1987 Zbl 0634.14003 MR 927963

[94] J.-P. Serre, A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.* **9** (2009), no. 1, 193–208 Zbl 1203.14017 MR 2567402

[95] J.-P. Serre, Le groupe de Cremona et ses sous-groupes finis. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. *Astérisque* **332** (2010), 75–100 Zbl 1257.14012 MR 2648675

[96] J.-P. Serre, Problems for the Edinburgh workshop on Cremona groups. 2010

[97] E. Shinder, The Bogomolov–Prokhorov invariant of surfaces as equivariant cohomology. *Bull. Korean Math. Soc.* **54** (2017), no. 5, 1725–1741 Zbl 1398.14042 MR 3708807

[98] V. V. Shokurov, Problems about Fano varieties. In *Birational Geometry of Algebraic Varieties, Open Problems*, pp. 30–32, Katata, 1988

[99] V. V. Shokurov, Prelimiting flips. *Tr. Mat. Inst. Steklova* **240** (2003), 82–219 Zbl 1082.14019 MR 1993750

[100] A. Shramov, Birational automorphisms of Severy–Brauer surfaces. *Mat. Sb.* **211** (2020), no. 3, 169–184 Zbl 1445.14025 MR 4070054

[101] C. Shramov, Automorphisms of cubic surfaces without points. *Internat. J. Math.* **31** (2020), no. 11, Article No. 2050083 Zbl 1461.14057 MR 4163640

[102] C. Shramov, Finite groups acting on Severi–Brauer surfaces. *Eur. J. Math.* **7** (2021), no. 2, 591–612 Zbl 1473.14025 MR 4256964

[103] V. I. Tsygankov, Equations of G-minimal conic bundles. *Mat. Sb.* **202** (2011), no. 11, 103–160 Zbl 1261.14006 MR 2907201

[104] J. Xu, A remark on the rank of finite p-groups of birational automorphisms. *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 7, 827–829 Zbl 1454.14037 MR 4174816

[105] E. Yasinsky, The Jordan constant for Cremona group of rank 2. *Bull. Korean Math. Soc.* **54** (2017), no. 5, 1859–1871 Zbl 1428.14024 MR 3708815

[106] Y. G. Zarhin, Theta groups and products of abelian and rational varieties. *Proc. Edinb. Math. Soc. (2)* **57** (2014), no. 1, 299–304 Zbl 1311.14018 MR 3165026

Yuri Prokhorov
Steklov Mathematical Institute, 8 Gubkina street, 119991 Moscow; and AG Laboratory, HSE, 6 Usacheva str., 119048 Moscow, Russia; prokhoro@mi-ras.ru