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Convex bodies all whose sections (projections) are equal

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Abstract. This work deals with the following question: if all hyperplane sections through the origin (orthogonal projections) of a convex body are "equal", is the convex body "equal" to the ball? where the notion of "equal" changes throughout the paper. Topology, Lie groups, Fourier analysis, and convex geometry interrelates in the solution and understanding of these problems.

1. Introduction

The purpose of this paper is to answer the following question:

If all hyperplane sections through the origin of a convex body are "equal", is the convex body "equal" to the ball?

The meaning of the notion "equal" will change in the course of this paper.

Similarly, we are interested in the following problem:

If all orthogonal projections of a convex body onto hyperplanes are "equal", is the convex body "equal" to the ball?

We believe that topology and convex geometry are deeply and beautifully interrelated in the solution and understanding of these problems.

A good reference for these problems and related problems is the book "Geometric Tomography" by Richard Gardner [11]. In particular, see Problems 3.3 and 7.4.

During this paper, unless otherwise stated, B is always an $(n+1)$-dimensional convex body with the origin as an interior point and $n \geq 2$.

2. Sections with the same area

The first meaning of "equal" is same "area".

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If all hyperplane sections through the origin of a convex body B have equal n-dimensional volume, does B have the $(n+1)$-dimensional volume of the corresponding ball?

The answer to this question is by far a resounding no. There exist counterexamples. However, if we add the symmetry hypothesis to the question, the answer becomes yes. More precisely, the following theorem holds.

Theorem 2.1. *If all hyperplane sections through the origin of a centrally symmetric convex body B have equal n-dimensional volume, then the convex body B is a ball centered at the origin.*

Proof. The proof of this theorem uses analysis. We give here a sketch of the proof using harmonic integration. See Falconer's paper [10] or Schneider's book [26].

First of all, let $f : \mathbb{S}^n \to \mathbb{R}$ be a continuous function such that

$$\int_{\langle x,y\rangle=0} f(x)\,dx = 0, \quad \text{for every } y \in \mathbb{S}^n,$$

where integration refers to the usual measure in the $(n-1)$-sphere. Then the classical theorem of Funk-Hecke on spherical harmonics (see [12]) implies that f is an odd function; that is, $-f(x) = f(-x)$, for almost every $x \in \mathbb{S}^n$.

Let now B_1 and B_2 be two $(n+1)$-dimensional convex bodies that are symmetric with center at the origin and assume that the corresponding parallel n-dimensional areas of their sections through the origin are equal. We shall show that $B_1 = B_2$. For this purpose, let $f_1, f_2 \colon \mathbb{S}^n \to \mathbb{R}$ be the radial functions of B_1 and B_2. Note that because B_1 and B_2 are centrally symmetric, f_1 and f_2 are even functions. Moreover, by hypothesis

$$\frac{1}{n}\int_{\langle x,y\rangle=0} f_1(x)^n\,dx = \frac{1}{n}\int_{\langle x,y\rangle=0} f_2(x)^n\,dx,$$

for every $y \in \mathbb{S}^{n-1}$.

By our first argument, $f_1^n - f_2^n$ is an odd function, but since f_1 and f_2 are even functions, $f_1^n = f_2^n$. Moreover, since $f_1 \geq 0$ and $f_2 \geq 0$, we obtain that $f_1 = f_2$ and hence that $B_1 = B_2$.

Suppose now that B is a centrally symmetric convex body with the property that all its hyperplane sections through the origin have equal n-dimensional volume, and let $G \in \mathrm{SO}_{n+1}$ be a linear isometry. Then, by the above $B = GB$, for every $G \in \mathrm{SO}_n$, and consequently B is a ball centered at the origin. ∎

3. Congruent and similar sections

The second meaning of "equal" is congruence.

Theorem 3.1 (Schneider's theorem). *If all hyperplane sections through the origin of a convex body B are congruent, then the convex body B is an $(n+1)$-ball centered at the origin.*

This time, the hypothesis of symmetry is not necessary. The theorem was proved by Süss [28] for $n = 2$. In 1970, using topological ideas, Mani [16] proved it for $n =$ even and, in 1979, Burton [7] proved it for $n = 3$. Finally, Rolf Schneider [25] in 1980, using analysis, proved it in general. In 1990, using the topological ideas of Hadwiger and Gromov, Montejano [20] proved the following result which, together with the false center theorem, allows an alternative proof of Schneider's theorem to be given.

Theorem 3.2. *If all hyperplane sections through the origin of a convex body B are affinely equivalent, then every hyperplane section of B through the origin is centrally symmetric.*

The proof of Theorem 3.2 uses topological ideas. Indeed, it uses the notion of field of convex bodies introduced by Hadwiger and developed by Mani in [16].

3.1. Fields of convex bodies

Let $\mathbf{K}^n$ be the space of all compact convex sets in $\mathbb{R}^n$ with the Hausdorff metric topology.

A *field of convex bodies tangent to* $\mathbb{S}^n$ is a continuous function

$$\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1},$$

such that $\kappa(u) \subset u + u^{\perp} \subset \mathbb{R}^{n+1}$, for every $u \in \mathbb{S}^n$, where $u^{\perp}$ denotes the subspace of $\mathbb{R}^{n+1}$ orthogonal to u.

If, in addition, $\kappa(u)$ is congruent (affinely equivalent) to the convex body $K \subset \mathbb{R}^n$, for every $u \in \mathbb{S}^n$, then we obtain a *field of convex bodies tangent to* $\mathbb{S}^n$ and congruent (affinely equivalent) to K. If, in addition, $\kappa(u) - u = \kappa(-u) + u$, then we have a *complete turning* of K in $\mathbb{R}^{n+1}$.

If all hyperplane sections through the origin of a convex body B are congruent (affinely equivalent), then there is a field of convex bodies tangent to $\mathbb{S}^n$ and congruent (affinely equivalent) to $\mathbb{R}^n \cap B$:

$$\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1},$$

defined as follows:

$$\kappa(u) = u + (u^{\perp} \cap B), \quad \text{for every } u \in \mathbb{S}^n.$$

Obviously, this field is a complete turning because $\kappa(u) - u = \kappa(-u) + u$, for every $u \in \mathbb{S}^n$.

Note that given a field of convex bodies $\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1}$ tangent to $\mathbb{S}^n$ and congruent to K, we may always assume without loss of generality that, for every $u \in \mathbb{S}^n$, the circumcenter of $\kappa(u)$ is the point $u \in (u + u^{\perp})$.

The link between Hadwiger's notion of field of convex bodies and the topology of Lie groups traces back to the work of Steenrod [27] and Gromov [13]. Every vector bundle $\xi : E \to \mathbb{S}^n$ with base the sphere $\mathbb{S}^n$, fiber $\mathbb{R}^m$, and structure group $\mathrm{GL}(m, \mathbb{R})$, can be obtained from $\mathbb{B}^n_+ \times \mathbb{R}^m$ disjoint union $\mathbb{B}^n_- \times \mathbb{R}^m$ by gluing the first copy $\mathbb{S}^{n-1} \times \mathbb{R}^m \subset \mathbb{B}^n_+ \times \mathbb{R}^m$ with the second copy $\mathbb{S}^{n-1} \times \mathbb{R}^m \subset \mathbb{B}^n_- \times \mathbb{R}^m$ via a fiber preserving homeomorphism

$$S^{n-1} \times \mathbb{R}^m \to S^{n-1} \times \mathbb{R}^m$$

that glue every fiber $\{x\} \times \mathbb{R}^m$ with the fiber $\{x\} \times \mathbb{R}^m$ using an element $g_x \in \mathrm{GL}(m, \mathbb{R})$, where $\mathbb{B}^n_+$ and $\mathbb{B}^n_-$ are respectively, the north and south closed hemisphere of $\mathbb{S}^n$. The map $g : \mathbb{S}^{n-1} \to \mathrm{GL}(m, \mathbb{R})$, given by $g(x) = g_x$, is called the *characteristic map* of the vector bundle ξ. It is not difficult to see that two vector bundles are equivalent (as fiber bundles) if and only if their corresponding characteristic maps are homotopic.

The existence of a field of convex bodies tangent to $\mathbb{S}^n$ and congruent to K implies that the tangent bundle $T\mathbb{S}^n$ can be obtained gluing the copies $\mathbb{B}^n_+ \times \mathbb{R}^m$ and $\mathbb{B}^n_- \times \mathbb{R}^m$ using only isometries that fix K. In other words, the following holds:

There exists a field of convex bodies tangent to $\mathbb{S}^n$ and congruent to K if and only if the characteristic map

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{\chi_n} & \mathrm{SO}_n \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & G_K & \end{array}$$

factorizes through

$$G_K = \{ g \in \mathrm{SO}_n \mid g(K) = K \}.$$

If this is so, then we say that the structure group of $T\mathbb{S}^n$ *reduces to* G_K.

The main idea in the proof of Theorem 3.2 is that a complete turning of K is only possible if K has a center of symmetry (indeed, if $n = 3, 7$, the fact that the tangent bundle of $\mathbb{S}^3$ and $\mathbb{S}^7$ is parallelizable implies that a complete turning of K is possible if and only if K has a center of symmetry).

Since vector bundles over contractible spaces are trivial, we are going to take advantage of the existence of the field of convex bodies $\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1}$, tangent to the sphere $\mathbb{S}^n$ and congruent to K, to construct a continuous map

$$\Phi : \mathbb{B}^n_+ \to \mathrm{SO}_n,$$

such that $\Phi(x)(K) = \kappa(x)$, for every $x \in \mathbb{B}^n_+$.

Suppose that K is not symmetric. We may assume without generality that there is a point x_0 in the boundary of K such that $-x_0 \notin K$ and hence for every $g \in \mathrm{SO}_n$, $g(x_0) \neq -x_0$.

Note that

$$\{\Phi(x)(x_0)\}$$

is a field of vectors tangent to $\mathbb{B}^n_+$. Furthermore, for every $u \in \mathbb{S}^{n-1}$, we have that $\Phi(u)(x_0) \neq -\Phi(-u)(x_0)$. We are going to add a small annulus to B^n_+ at the boundary to obtain a larger n-dimensional ball $\widetilde{B}^n$ and we are going to take advantage of this annulus to define on it a tangent vector field that coincides with the one we have in B^n and with an additional property. The idea is that for every point $u \in \mathbb{S}^{n-1}$, we will use the annulus to rotate from the vector $\Phi(u)(x_0)$ towards the vector $\Phi(-u)(x_0)$. Since $\Phi(u)(x_0) \neq -\Phi(-u)(x_0)$, we can do this unambiguously in such a way that at the end on the border of $\widetilde{B}^n$, the tangent vector at the point $u \in \partial\widetilde{B}^n$ coincides with the tangent vector at the point $-u \in \partial\widetilde{B}^n$. Using this procedure, we obtain a complete turning of a nonzero vector field in the sphere $\mathbb{S}^n$, which is a contradiction to the well-known result that there is not a section to the canonical vector bundle of n-subspaces in $\mathbb{R}^{n+1}$; see [27].

Suppose that K_1, K_2 are convex bodies who have as ellipsoid of minimal volume containing them the unit ball. It is easy to see that if K_1 and K_2 are affinely equivalent, then they are actually congruent. Suppose now that $K \subset \mathbb{R}^n$ is a convex body with the unit ball as the ellipsoid of minimal volume containing it, and let $\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1}$ be a field of convex bodies tangent to $\mathbb{S}^n$ and affinely equivalent to the convex body $K \subset \mathbb{R}^n$, then there is a field of convex bodies tangent to $\mathbb{S}^n$ congruent to K. For every $x \in \mathbb{S}^n$, let $E_x \subset x + x^\perp$ be the ellipsoid of minimal volume containing $\kappa(x)$ and let h_x be the affine map that translates and dilates the principal axes of E_x to obtain the unit ball. It is easy to observe that the affine map h_x varies continuously with x. Hence $\kappa' : \mathbb{S}^n \to \mathbf{K}^{n+1}$, given by $\kappa'(x) = h_x(\kappa(x))$, is a field of convex bodies tangent to $\mathbb{S}^n$ congruent to K. By all the above, *if* $\kappa : \mathbb{S}^n \to \mathbf{K}^{n+1}$ *is a field of convex bodies tangent to* $\mathbb{S}^n$ *and affinely equivalent, then, for every* $x \in \mathbb{S}^n$, $\kappa(x)$ *is symmetric.*

3.2. The proof of Schneider's theorem and similar sections

Summarizing, Theorem 3.2 is true because a complete turning of K is only possible if K has a center of symmetry. This result, in combination with Larman's beautiful false center theorem [14], gives rise to a topological proof of Schneider's theorem.

Theorem 3.3 (Larman's false center theorem). *If all hyperplane sections through the origin of a convex body* B *have a center of symmetry, then either* B *is an ellipsoid or* B *is symmetric with respect to the origin.*

The proof of Schneider's theorem (Theorem 3.1) goes as follows. If all hyperplane sections through the origin of B are congruent, by Theorem 3.2, then every hyperplane section through the origin is centrally symmetric. By Larman's false center theorem (Theorem 3.3), either B is symmetric with center the origin and Theorem 2.1 implies that B is a ball centered at the origin, or B is an ellipsoid in which case it is easy to see directly that B is again a ball centered at the origin.

The third meaning of equal is similarity. If B is an $(n+1)$-ball with the origin as an interior point but not necessarily centered at the origin, then all hyperplane sections of B through the origin are n-balls and hence all are similar. Our next theorem states that this is always the case.

Theorem 3.4 (Montejano). *If all hyperplane sections through the origin of a convex body B are similar, then the convex body B is an n-ball not necessarily centered at the origin.*

A sketch of the proof is the following. Since similarities are affine equivalences, by Theorem 3.2, all hyperplane sections of B through the origin have a center of symmetry. By Larman's false center theorem (Theorem 3.3), either the origin is the center of symmetry of B or B is an ellipsoid. Using a topological argument, it is possible to prove that, in the first case, all hyperplane sections of B through the origin are not only similar but actually congruent and hence, by Schneider's theorem (Theorem 3.1), B is a ball or, in the second case, if B is an ellipsoid, it is easy to directly verify that our hypothesis implies that B is actually a ball.

4. Affinely equivalent sections and the Banach conjecture

The fourth meaning of equal is affine equivalence.

Conjecture 4.1. *If all hyperplane sections through the origin of a convex body B are affinely equivalent, then the convex body B is an ellipsoid.*

It turns out that Conjecture 4.1 is equivalent to the Banach conjecture over the reals.

4.1. The Banach conjecture

In 1932, in his book [3], Stephan Banach asked the following question:

Let V be a Banach space, real or complex, finite or infinite dimensional, all of whose n-dimensional subspaces, for some fixed integer n, $2 \leq n < \dim(V)$, are isometric to each other. Is it true that V is a Hilbert space?

This conjecture was proved first for $n = 2$ and real V in 1935 by Auerbach, Mazur, and Ulam [2] and in 1959 for all $n \geq 2$ and infinite dimensional real V by A. Dvoretzky [9]. In 1967, M. Gromov [13] proved the conjecture for even n and all V, real or complex, for odd n and real V with $\dim(V) \geq n + 2$, and for odd n and complex V with $\dim(V) \geq 2n$. V. Milman [18] extended Dvoretzky's theorem to the complex case, in particular, reproving Banach's conjecture for infinite dimensional complex space V. Recently, in 2021, Bor, Hernández-Lamoneda, Jiménez-Desantiago, and Montejano [4] proved the Banach conjecture if V is real and $n \equiv 1 \bmod 4$, with the possible exception of $n = 133$, and a little later, Bracho and Montejano [6] proved the Banach conjecture if V is complex and $n \equiv 1 \bmod 4$. A thorough account of the history of this conjecture is found in the notes on Section 9 in [17]. We also recommend [24].

Our next goal is to prove that the Banach conjecture over the reals is equivalent to Conjecture 4.1. First note that Banach's conjecture is a codimension one problem: since every Banach space, all of whose subspaces of a fixed dimension $n \geq 2$ are Hilbert spaces, is itself a Hilbert space, which easily follows from the elementary characterization of a norm coming from an inner product via the "parallelogram law", an affirmative answer for n in codimension one implies immediately an affirmative answer for n in all codimensions.

Note next that two Banach spaces V_1 and V_2 are isometric if there is a linear isomorphism $f : V_1 \to V_2$ that preserves the norm. That is, two Banach spaces V_1 and V_2 are isometric if their unit balls are linearly equivalent. To conclude, note that a finite dimensional Banach space V is a Hilbert space if and only if V is isometric to the Euclidean space, that is, if and only if its unit ball is an ellipsoid.

Finally, in the solution of Conjecture 4.1, we may always assume that not only B but all hyperplane sections of B through the origin have as a center of symmetry the origin. This is so because by Theorem 3.2 every section of B has a center of symmetry and therefore by Larman's false center theorem (Theorem 3.3) either B is an ellipsoid or the origin is the center of B.

4.2. Topology of Lie groups

From now on, until the end of this section, suppose that B is a convex body with the property that all its hyperplane sections through the origin are affinely equivalent. Our first interest is to answer the following question:

What can we say about the sections of B?

For example, due to Theorem 3.2, we know that all these sections have a center of symmetry, but do these sections share some other property?

Choose a convex set $K \subset \mathbb{R}^n$ affinely equivalent to all hyperplane sections of B through the origin with the additional property that the ellipsoid of minimal volume containing K is the unit n-ball. Define G as the group of symmetries of K, that is, G is the subgroup of linear isomorphism in $\mathrm{GL}(n,\mathbb{R})$ keeping fixed K and with positive determinant. Note that every element of G fixes also the unit n-ball that therefore $G \subset \mathrm{SO}_n$. As we shall see, G is a compact Lie group relevant in the solution of our previous question.

As in the sketch of the proof of Theorem 3.2, in Section 3, there is a field of convex bodies tangent to $\mathbb{S}^n$ and affinely equivalent to K. This implies that the structure group of the tangent bundle of the sphere $\mathbb{S}^n$ can be reduced to G or, in other words, that the characteristic map of $T\mathbb{S}^n$

$$\chi_n : \mathbb{S}^{n-1} \to \mathrm{SO}_n$$

can be factorized through G. See Steenrod's book [27] or Mani's paper [16].

If n is even and G is not transitive, the structure group of the tangent bundle of the sphere $\mathbb{S}^n$ cannot be reduced to G. This is so because if there is a map

$$f : \mathbb{S}^{n-1} \to \mathrm{SO}_n$$

homotopic to χ_n, such that $f(\mathbb{S}^{n-1}) \subset G$ and $e : \mathrm{SO}_n \to \mathbb{S}^n$ is the evaluation map (at any point), then ef is homotopic to $e\chi_n$. The non-transitivity of G implies that there are $x, y \in \mathbb{S}^n$ such that $g(x) \neq y$, for every $g \in \mathrm{SO}_n$. If $e : \mathrm{SO}_n \to \mathbb{S}^n$ is the evaluation at x, then the map e is not surjective and therefore ef is null homotopic. Thus, $e\chi_n$ is null homotopic, which is a contradiction in even dimensions, where we can easily calculate the even degree of $e\chi_n$. Consequently, if n is even, a field of convex bodies tangent to $\mathbb{S}^n$ affinely equivalent to K implies that G is transitive and consequently that K is an n-ball. In contrast, for $n = 3$, there is a field of convex bodies tangent to $\mathbb{S}^n$ and congruent to K, for every convex body $K \subset \mathbb{R}^n$, because $\mathbb{S}^3$ is parallelizable.

Summarizing, if n is even, the answer to our question: what can we say about the sections of B? is that all these sections are affinely equivalent to a ball and hence all of them are ellipsoids. This immediately implies that B is an ellipsoid, solving conjecture 1 when n is even and the Banach conjecture when n is even and V is a Banach space over the reals.

The case $n =$ odd is more complicated. First note that if $n = 3, 7$, this topological technique does not give us information about the sections of B, because $\mathbb{S}^3$ and $\mathbb{S}^7$ are parallelizable. We shall prove next that if $n \equiv 1 \bmod 4$, with the possible exception of $n = 133$, a field of convex bodies tangent to $\mathbb{S}^n$ affinely equivalent to K implies that K is an affine body of revolution.

Suppose that the characteristic map of the sphere χ_n factorizes through the maximal connected subgroup $G \subset \mathrm{SO}_n$, that is,

$$\mathbb{S}^{n-1} \to G \hookrightarrow \mathrm{SO}_n\,.$$

We have two cases:

(1) G is an irreducible representation, that is, the action of G does not fix any proper subspace, and

(2) the action of G fixes a proper subspace Γ^k; $1 \leq k \leq n-1$.

In the first case, mathematicians have extensively studied irreducible representations, in particular, those for which the structural group of the space tangent to the sphere can be reduced to them. In particular, Leonard [15] proved that if $G \subset \mathrm{SO}_n$ is a maximal connected irreducible representation and the characteristic map of the sphere χ_n factorizes through G, then G is a simple group.

If this is so, we have several options:

- G is a classical group; SO_k, SU_k, Sp_k,
- G is a spin group; Spin_k,
- G is one of the exceptional Lie groups, G_2, F_4, E_6, E_7 or E_8.

Furthermore, in 2006, Cadek and Crabb proved that under the same hypothesis for G, if $n \geq 8$, then G is not isomorphic to SO_k, SU_m, Sp_m, with $k \geq 4$, $m \geq 2$. If $n \equiv 1 \bmod 4$, this rules out the classical groups, with the exception of $n = 5$. We leave this exceptional case for the next section. Furthermore, it can be proved that every irreducible representation of Spin_k, which does not factor through SO_m, is even dimensional. In our case, it is clear that G does not factor through SO_m, so if n is odd, we can rule out the possibility of a spin group for G.

Suppose now that $n \equiv 1 \bmod 4$. If this is the case, $\dim(G)$ is not too small with respect to n and hence G is not an exceptional Lie group, with the possible exception of the Lie group $E_7 \subset O_{133}$. This is so because it can be proved that in this case, $\dim(G) \geq 2n - 3$ (see [8, Proposition 3.1]). Hence to rule out the exceptional groups, one can simply check (e.g., in Wikipedia) the following table in which we list the smallest irreducible representation for them, and the smallest irreducible representation congruent to 1 mod 4 is highlighted in red, verifying that in all the cases, with the exception of E_7, $\dim(G) \leq 2n - 4$.

Group	G_2	F_4	E_6	E_7	E_8
dim G	14	52	78	133	248
Irreps	7	26	27	56	248
	14	52	78	133	3875
	27	273	351	912	⋮
	64	⋮	2925	⋮	1763125
	77	⋮	⋮	⋮	⋮

All the above implies that if G is irreducible and $n \equiv 1 \bmod 4$, then G is E_7 or is conjugate to O_n. Consequently, in this last case, K must be a ball, all the sections must be ellipsoids, and B must be an ellipsoid, as we wished.

The second case is when the action of G fixes a proper subspace Γ^k; $1 \leq k \leq n-1$.

If $n = 4k + 1$, the tangent space of the sphere $T\mathbb{S}^n$ splits:

$$T\mathbb{S}^n = e^1 \oplus \eta^{4k},$$

where e^1 is a vector bundle of dimension 1 and η^{4k} is unsplittable.

From here, we deduce that Γ^k is either 1 or $(n-1)$-dimensional, and G is a subset of a conjugate copy of SO_{n-1}. Furthermore, using an argument very similar to the argument used in the proof that $G = SO_n$, when n is even (or see the case $n = 5$), it is possible to prove that G is actually a conjugate copy of SO_{n-1}. This gives rise to the case in which K is a body of revolution.

Summarizing, *suppose that B is an $(n+1)$-dimensional convex body with the property that all its hyperplane sections through the origin are affinely equivalent, $n \equiv 1 \bmod 4$, $n \neq 5, 133$. Then, every hyperplane section of B through the origin is an affine body of revolution.*

4.3. The case $n = 5$

This case is an exceptional case in our proof of the Banach conjecture but it is also interesting enough to illustrate the true complexity of the conjecture. This section will be dedicated to its complete proof.

Let B, K, and G be defined as in the previous section but this time B is a centrally symmetric convex body in $\mathbb{R}^6$, and $G = \{g \in SO_5 \mid g(K) = K\}$ is a compact Lie subgroup of SO_5. Furthermore, we know that the characteristic map of the tangent space of $\mathbb{S}^5$

$$\begin{array}{ccc} \mathbb{S}^4 & \xrightarrow{\chi_5} & SO_5 \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & G & \end{array}$$

factorizes through G.

Suppose first that G leaves invariant a proper subspace of $\mathbb{R}^5$. We shall prove that in this case K is a body of revolution.

By hypothesis, there is a k-dimensional subspace Λ invariant under G. This immediately implies that there is a continuous field of k-planes in $\mathbb{S}^5$. By [27, Theorem 27.18], we know that $\mathbb{S}^5$ admits a continuous field of k-planes if and only if $k = 1$ or $k = 4$. So, assume without loss of generality that $k = 1$, and therefore that Λ is a line invariant under G. Suppose without loss of generality that Λ is the line through the origin orthogonal to $\mathbb{R}^4$, in such a way that $G \subset SO_4$. We will prove that G acts transitively on $\mathbb{R}^4$, thus proving that K is a body of revolution.

Given any 5-dimensional plane through the origin in $\mathbb{R}^6$, it is easy to prove that there is a unique complex plane through the origin contained in it. It is for this reason that there is a field of complex planes tangent to $\mathbb{S}^5$. This implies that the structural group of $T\mathbb{S}^5$ can be reduced to SU_2. Thus, we may assume that $\chi_5 : \mathbb{S}^4 \to SU_2$ is the characteristic map of $T\mathbb{S}^5$. If $e : SO_4 \to \mathbb{S}^3$ is the evaluation, hence, $e\chi_5 : \mathbb{S}^4 \to \mathbb{S}^3$ is not null homotopic. To see this, note that SU_2 is homeomorphic to $\mathbb{S}^3$ and the evaluation $e : SU_2 \to \mathbb{S}^3$ is a homeomorphism. Therefore, if $e\chi_5 : \mathbb{S}^4 \to \mathbb{S}^3$ is homotopically trivial, then the same holds for $\chi_5 : \mathbb{S}^4 \to SU_2$, but this implies that the characteristic map of $T\mathbb{S}^5$ is homotopic to a constant, and therefore that $T\mathbb{S}^5$ is parallelizable which is a contradiction.

We know that the structural group of $T\mathbb{S}^5$ can be reduced to G. Therefore, the characteristic map $\chi_5 : \mathbb{S}^4 \to SU_2$ is homotopic on SO_4 to a map $f : \mathbb{S}^4 \to G$. This implies that $e\chi_5, ef : \mathbb{S}^4 \to \mathbb{S}^3$ are homotopic. If G does not act transitively on $\mathbb{R}^4$, hence ef is null homotopic, but this is a contradiction to the fact that $e\chi_5$ is not null homotopic. Consequently, G acts transitively on $\mathbb{R}^4$ and K is a body of revolution, as we wished.

Suppose now that $G \subset SO_5$ does not leave invariant a proper subspace of $\mathbb{R}^6$. That is, we must study the *irreducible representations on* $\mathbb{R}5$.

Consider S the collection of 3×3 real symmetric matrices with zero trace. Then, S is a real vector space of dimension 5 with the following natural interior product: given $A, B \in S$,

$$A \odot B = \mathrm{tr}(AB).$$

The group $G = SO_3$ defines the following representation: $g(A) = gAg^{-1} = gAg^t$, for every $g \in G$ and $A \in S$.

Clearly, G acts linearly on S and furthermore,

$$g(A) \odot g(B) = \mathrm{tr}(gAg^{-1}gBg^{-1}) = \mathrm{tr}(gABg^{-1}) = A \odot B.$$

It is well known that this is a faithful, irreducible, representation. That is, we may think G is a subgroup of SO_5 with the property that G does not leave invariant any proper subspace. Moreover, it is well known that any other irreducible representation on $\mathbb{R}^5$ factors through G.

The following lemma finally proves that *if B is a 6-dimensional convex body with the property that all its hyperplane sections through the origin are affinely equivalent, then every hyperplane section of B through the origin is an affine body of revolution.*

Lemma 4.2. *Let $\Omega \subset SO_5$ be a subgroup isomorphic to SO_3, Then, the structural group of $T\mathbb{S}^5$ cannot be reduced to Ω.*

Proof. Suppose that there is $f : \mathbb{S}^4 \to \Omega$ such that $i_\Omega f : \mathbb{S}^4 \to SO_5$ is homotopic to the characteristic map $\chi_5 : \mathbb{S}^4 \to SO_5$ of $T\mathbb{S}^5$, where $i_\Omega : \Omega \to SO_5$ is the inclusion. Let $\pi : \mathbb{S}^3 \to \Omega$ be the double covering map and let $g : \mathbb{S}^3 \to \Omega$ be such that $\pi g = f$.

Let $u : SU_2 \to SO_5$ be the inclusion. Hence $\pi_3(SO_5) = \mathbb{Z}$ (every compact, simple Lie group has $\pi_3 = \mathbb{Z}$) and $u_* : \pi_3(SU_2) \to \pi_3(SO_5)$ is an isomorphism. On the other hand, at the level of homology, $H_3(SO_5, \mathbb{Z}_2)$ is a directed sum of $\mathbb{Z}_2$'s and $u_* : H_3(SU_2, \mathbb{Z}_2) \to H_3(SO_5, \mathbb{Z}_2)$ is not zero. Let us consider $[i_\Omega \pi] \in \pi_3(SO_5) = \mathbb{Z}$. Suppose that $[i_\Omega \pi] = m \in \mathbb{Z}$ and let $\zeta : \mathbb{S}^3 \to SU_2$ such that the induced homomorphism in homotopy is $\zeta_*(1) = m \in \pi_3(SU_2) = \mathbb{Z}$. Consequently, $u\zeta : \mathbb{S}^3 \to SO_5$ is homotopic to $i_\Omega \pi : \mathbb{S}^3 \to SO_5$. In 3-dimensional homology, $(i_G \pi)_*(1) = 0$ which implies that $(u\zeta)_*(1) = 0$ and therefore, since $u_* : H_3(SU_2, \mathbb{Z}_2) \to H_3(SO_5, \mathbb{Z}_2)$ is not zero, that m is even.

Since m is even, the map $\zeta g : \mathbb{S}^4 \to SU_2$ is null homotopic, because $\zeta_* : \pi_4(\mathbb{S}^3) \to \pi_4(SU_2)$ is zero. This is a contradiction to the fact that $\mathbb{S}^5$ is not parallelizable. ∎

The intuitive claim that

$$u_* : H_3(SU_2, \mathbb{Z}_2) \to H_3(SO_5, \mathbb{Z}_2)$$

is not zero, used in the above proof, is not so easy to prove. Indeed, to justify it, it is necessary to use the Dynkin index.

4.4. Affine bodies of revolution

A convex body $K \subset \mathbb{R}^n$ is a *body of revolution* if it admits an *axis of revolution*; i.e., a 1-dimensional line L such that each section of K by an affine hyperplane Δ orthogonal to L is an $(n-1)$-dimensional Euclidean ball in Δ, centered at $\Delta \cap L$ (possibly empty or just a point). If L is an axis of revolution of K, then $L^\perp$ is the associated *hyperplane of revolution*. Clearly, a ball is a body of revolution and any line through its center serves as an axis of revolution.

An axis of revolution of a plane convex figure is an axis of symmetry (or reflexion). Of course, a convex figure may have two different axes of symmetry without being a disk. In dimension $n \geq 3$, the situation is different.

Theorem 4.3. *A convex body of revolution $K \subset \mathbb{R}^n$, $n \geq 3$, with two different axes of revolution must be a ball.*

Proof. Consider $G_K = \{g \in SO_n \mid g(K) = K\}$ the collection of orientation preserving isometries that fix K and suppose that $L \neq L'$ are two different axes of revolution of K. Without loss of generality, we may assume that L is the 1-dimensional subspace orthogonal to $\mathbb{R}^{n-1}$. Clearly, the collection of orientation preserving isometries of $\mathbb{R}^n$ that fix L also fix K and is equal to $SO_{n-1} \subset SO_n$. On the other hand, the group of orientation preserving isometries of $\mathbb{R}^n$ that fixes L' fixes also K and is equal to

$\mathrm{SO}'(n-1)$, a conjugate subgroup of SO_{n-1} in SO_n. Thus, our hypotheses imply that

$$\mathrm{SO}_{n-1} \subsetneq G_K \subset \mathrm{SO}_n,$$

but it is well known that SO_{n-1} is a *maximal connected* subgroup of SO_n (see [23, Lemma 4]). Therefore, $G_K = \mathrm{SO}_n$ and K must be a ball. ■

An *affine body of revolution* is a convex body affinely equivalent to a body of revolution. The images, under an affine equivalence, of an axis of revolution and its associated hyperplane of revolution of the body of revolution are an axis of revolution and associated hyperplane of revolution of the affine body of revolution (not necessarily perpendicular anymore). Clearly, an ellipsoid centered at the origin is an affine body of revolution and any hyperplane through the origin serves as a hyperplane of revolution.

As in the Euclidean case, a non-elliptical body of revolution admits a unique axis of revolution and a unique hyperplane of revolution.

Corollary 4.4. *An affine convex body of revolution $K \subset \mathbb{R}^n$, $n \geq 3$, with two different hyperplanes of revolution must be an ellipsoid.*

Proof. Let E be the ellipsoid of minimal volume containing K. By translation and dilatation of the principal axes of this ellipsoid, we obtain an affine isomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(E)$ is the unit ball of $\mathbb{R}^n$. Then since every affine isomorphism that fixes $f(K)$ also fixes $f(E)$, we have that $f(K)$ contains two different axes of revolution. By Lemma 4.3, $f(K)$ is a ball and consequently K is an ellipsoid. ■

4.4.1. Sections of affine bodies of revolution. It is not difficult to see that every section of a body of revolution is a body of revolution, that is why sections of affine bodies of revolution are affine bodies of revolution. Is the converse true? As far as I know, nobody knows the answer.

Conjecture 4.5. *Suppose that B is an $(n+1)$-dimensional convex body all whose hyperplane sections through the origin are affine bodies of revolution, $n \geq 3$. Then B is an affine body of revolution.*

We shall give a partial answer to this conjecture which will turn out to be sufficiently good for our purposes. Under the same hypothesis, we shall prove that at least one section of B through the origin is an ellipsoid. If this is so, and if, in addition, B satisfies the hypothesis that every two or its hyperplane section through the origin are affinely equivalent, then every section of B through the origin is an ellipsoid and consequently B is an ellipsoid. The proof of the existence of at least one elliptical section is a very interesting proof that combines ideas of convex geometry and algebraic topology. Before exposing it here, we require three intuitive lemmas, which we will state without proof. We ask the reader to include their own proofs.

Lemma 4.6. *Every hyperplane section $\Gamma \cap K$ of an affine body of revolution $K \subset \mathbb{R}^n$, $n \geq 3$, is an affine body of revolution. Furthermore, if H is the hyperplane of revolution of K, then either Γ is parallel to H or $\Gamma \cap H$ is a hyperplane of revolution of $\Gamma \cap K$.*

Lemma 4.7. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine body of revolution with axis of revolution the line L and let Γ be a hyperplane containing L. Suppose that $\Gamma \cap K$ is an ellipsoid. Then K is an ellipsoid.*

Lemma 4.8. *Let $B \subset \mathbb{R}^{n+1}$ be a centrally symmetric convex body, all of whose hyperplane sections through the origin are non-elliptical affine symmetric bodies of revolution. For each $x \in \mathbb{S}^n$, let L_x be the (unique) axis of revolution of $x^{\perp} \cap B$, where $x^{\perp}$ denotes the subspace orthogonal to x. Then $x \mapsto L_x$ is a continuous function $\mathbb{S}^n \to \mathbb{R}P^n$. Consequently,*

$$\{x + L_x\}_{x \in \mathbb{S}^n}$$

is a field of lines tangent to $\mathbb{S}^n$.

Since every field of tangent lines gives rise to a trivial 1-dimensional fiber bundle over $\mathbb{S}^n$, then there is

$$\psi : \mathbb{S}^n \to \mathbb{S}^n,$$

such that for every $x \in \mathbb{S}^n$

$$L_x \cap \mathbb{S}^n = \{-\psi(x), \psi(x)\}.$$

Note that, for every $x \in \mathbb{S}^n$, $\psi(x)$ is orthogonal to x and hence $\psi(x) \neq -x$. This implies that $\psi : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the identity map and therefore that ψ is surjective.

From now on, let $B \subset \mathbb{R}^{n+1}$ be a centrally symmetric convex body, all of whose hyperplane sections through the origin are non-elliptical affine symmetric bodies of revolution, and remember that for every $u \in \mathbb{S}^n$, we denote by $u^{\perp}$ the n-dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to u. Furthermore, by Lemma 4.4, denote by L_u the unique affine axis of revolution of $u^{\perp} \cap B$, and by H_u the corresponding $(n-1)$-dimensional hyperplane of revolution of $u^{\perp} \cap B$. Note that the line L_u contains the origin. The fact that $u^{\perp} \cap B$ is symmetric implies that the origin is the center of the ellipsoid $H_u \cap B$ and therefore that the origin lies in L_u.

Lemma 4.9. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body with center at origin, $n \geq 4$, and suppose that every hyperplane section of B through the origin is a non-elliptical affine convex body of revolution. Suppose that $L_v \subset u^{\perp}$ for some $u, v \in \mathbb{S}^n$. Then*

$$H_v \cap \Gamma_u = H_u \cap \Gamma_v = H_v \cap H_u$$

are highlighted in red.

Proof. Consider $u^{\perp} \cap v^{\perp}$, the $(n-1)$-dimensional subspace of $v^{\perp}$. By hypothesis, $v^{\perp} \cap B$ is a non-elliptical affine body of revolution with affine axis of revolution L_v. Therefore, since $L_v \subset u^{\perp} \cap v^{\perp}$, we have that $u^{\perp} \cap v^{\perp} \cap B$ is an affine body of revolution with affine axis of revolution L_v. Furthermore, by Lemma 4.7, $u^{\perp} \cap v^{\perp} \cap B$ is not an ellipsoid. Moreover, the principal affine subspace of revolution of $u^{\perp} \cap v^{\perp} \cap B$ is $H_v \cap u^{\perp}$.

On the other hand, $u^{\perp} \cap v^{\perp}$ is an $(n-1)$-dimensional subspace of $u^{\perp}$. Note that $u^{\perp} \cap v^{\perp} \neq H_u$, otherwise $u^{\perp} \cap v^{\perp} \cap B = H_u \cap B$ would be an ellipsoid, contradicting our previous assumption. Since $u^{\perp} \cap B$ is a non-elliptical affine body of revolution and $u^{\perp} \cap v^{\perp} \neq H_u$, then, by Lemma 4.6, $u^{\perp} \cap v^{\perp} \cap B$ is an affine body of revolution with principal affine subspace of revolution $H_u \cap v^{\perp}$. Consequently, by Lemma 4.4, we have that $H_v \cap u^{\perp} = H_u \cap v^{\perp}$. ■

Our main result regarding affine bodies of revolution is the following theorem.

Theorem 4.10. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body with center at origin, $n \geq 4$, and suppose that every hyperplane section of B through the origin is an affine body of revolution. Then there is a hyperplane section through the origin of B which is an ellipsoid.*

Proof. Suppose not, suppose that B is a symmetric convex body with center at the origin and with the property that every hyperplane section of B through the origin is a non-elliptical affine convex body of revolution.

Let us fix a point $x_0 \in H_{u_0} \cap \mathbb{S}^n$. Since $\psi : \mathbb{S}^n \to \mathbb{S}^n$ is suprayective, let $v_0 \in \mathbb{S}^n$ such that $\psi(v_0) = x_0$. This implies that

$$L_{v_0} \subset H_{u_0}.$$

This is a contradiction to Lemma 4.9 because clearly $L_{v_0} \subset {u_0}^{\perp}$, hence,

$$L_{v_0} \subset H_{u_0} \cap {v_0}^{\perp} = H_{v_0} \cap {u_0}^{\perp} \subset H_{v_0},$$

which is impossible. ■

Theorem 4.10 is also true when $n = 2$. Indeed, in [21] Montejano proved that if B is a 3-dimensional convex body which contains the origin as interior point and every section through the origin is a figure that has a line of reflection (symmetry), then there is a section through the origin that is a disk. The proof also uses topology but it is intrinsically different to the proof of Theorem 4.10. The case $n = 3$ remains open.

With this, we have finished exposing the solution to the Banach conjecture over the reals given by Gromov [13], when $n =$ even and by Bor–Hernández Lamoneda–Jiménez-Desantiago–Montejano [4] when $n \equiv 1 \bmod 4$, $n \neq 133$. We summarize the results below in the following theorem.

Theorem 4.11 (Main theorem). *If all hyperplane sections through the origin of an $(n+1)$-dimensional convex body B are affinely equivalent, $n \equiv 0, 1, 2 \bmod 4$, $n \neq 133$, then the convex body B is an ellipsoid.*

4.5. The Banach conjecture, when n is odd and $\dim V \geq n+2$

As we have mentioned before, the cases of the Banach conjecture that have yet to be solved are those in which $n \equiv 3 \bmod 4$. That is, the first unsolved case from the Banach conjecture is the following.

Conjecture 4.12. *If all hyperplane sections through the origin of a 4-dimensional convex body B are affinely equivalent, then the convex body B is an ellipsoid.*

Indeed, Gromov in his original paper [13], using topology but a complete different sort of ideas, proved the Banach conjecture over the reals, when $n \equiv 3 \bmod 4$ and $\dim V > n+1$ and the Banach conjecture over the complex numbers, when $n \equiv 3 \bmod 4$ and $\dim V > 2n-1$.

The purpose of this section is to introduce these deep ideas. Let us prove the Banach conjecture over the reals, when $n > 1$ is odd and $\dim V \geq n+2$.

Theorem 4.13 (Gromov). *Let B be an $(n+2)$-dimensional convex body with the origin as interior point and suppose that all n-sections through the origin are linearly equivalent, for $n > 1$ odd. Then the convex body B is an ellipsoid.*

Denote by $V_{n,k}$ the space of all orthonormal k-frames $(e_1, \ldots, e_k)$, where $e_i \in \mathbb{R}^n$, $n \geq k$. For our purpose, consider the space of 4-frames (e_1, e_2, e_3, e_4) in $\mathbb{R}^{n+2}$ and also the two fiber bundles

$$p_1 : V_{n+2,4} \to V_{n+2,2}, \quad p_2 : V_{n+2,4} \to V_{n+2,2},$$

where

$$p_1(e_1, e_2, e_3, e_4) = (e_1, e_2), \quad p_2(e_1, e_2, e_3, e_4) = (e_3, e_4).$$

The fiber in both cases is *the Stiefel Manifold* $V_{n,2}$. For more about Stiefel fiber bundles, see the book [19].

Consider now a nonempty closed subset $V \subset V_{n+2,2}$ and denote

$$\widetilde{V} = p_1^{-1}(V) = \big\{(e_1, e_2, e_3, e_4) \in V_{n+2,4} \mid (e_1, e_2) \in V\big\}.$$

The following lemma is Proposition 3 of Gromov's paper [13].

Lemma 4.14. *If n is odd and the restriction $p_2| : \widetilde{V} \to V_{n+2,2}$ is a fiber bundle, then $V = V_{n+2,2}$.*

We give only a brief sketch of the main ideas of the proof. We must consider an arbitrary fiber $V_{n,2}$ of p_2 and prove that the intersection $V' = \widetilde{V} \cap V_{n,2}$ coincides with $V_{n,2}$. Note that the dimension of $V_{n,2}$ is equal to $2n-3$. In fact, if $V' \neq V_{n,2}$, then $H^{2n-3}(V;\mathbb{Q})=0$ and for $p+q=2n-3$, the second term $E_2^{p,q}$ in the spectral sequence of the fiber bundle $p_2|:\widetilde{V}\to V_{n+2,2}$ is trivial, which implies that $H^{2n-3}(\widetilde{V};\mathbb{Q})$ is trivial, contradicting an old result of Borel in [5, p. 192] that claims that for $n=$ odd, the homomorphism induced by the inclusion

$$H^{2n-3}(V_{n+4,2};\mathbb{Q})\to H^{2n-3}(V_{n,2};\mathbb{Q})$$

is non-zero.

Proof of Theorem 4.13. By hypothesis, there is a convex body $K\subset\mathbb{R}^n$ with the property that the ellipsoid of minimal volume containing K is the unit ball of $\mathbb{R}^n$ and such that every n-dimensional section of B through the origin is linearly equivalent to K. Let us denote, as usual, by G_K the Lie group of all linear isomorphisms of $\mathbb{R}^n$ that keep K fixed. Of course, $G_K\subset O_n$.

Let us fix a 2-dimensional plane Δ in $\mathbb{R}^{n+2}$ through the origin and define $V\subset V_{n+2,2}$ as the set of 2-frames (e_1,e_2) in $\mathbb{R}^{n+2}$ such that if $\langle e_1,e_2\rangle$ is the subspace spanned by e_1 and e_2, then the section $\langle e_1,e_2\rangle\cap B$ is linearly equivalent to the section $\Delta\cap B$. Furthermore, let $V'\subset V_{n,2}$ be the set of 2-frames (e_1,e_2) in $\mathbb{R}^n$ such that $\langle e_1,e_2\rangle\cap K$ is linearly equivalent to $\Delta\cap B$. Finally, let

$$\widetilde{V}=p_1^{-1}(V)=\big\{(e_1,e_2,e_3,e_4)\in V_{n+2,4}\mid(e_1,e_2)\in V\big\}.$$

We shall first prove that the restriction $p_2|:\widetilde{V}\to V_{n+2,2}$ is a locally trivial bundle with fiber V'. For that purpose, consider U an open contractible subset of $V_{n+2,2}$. Then, using the contractibility of U and the existence of a field of convex bodies lineally equivalent to K, contained in the fibers of the canonical vector bundle of n-subspaces in $\mathbb{R}^{n+2}$, it is possible to construct a continuous map $\Lambda:U\to\mathrm{GL}(n,n+2)$ satisfying the following properties:

(1) for every $(e_3,e_4)\in U$, $\Lambda_{e_3,e_4}:\mathbb{R}^n\to\mathbb{R}^{n+2}$ is a linear embedding,

(2) for every $(e_3,e_4)\in U$, $\Lambda_{e_3,e_4}(\mathbb{R}^n)$ is orthogonal to both e_3 and e_4,

(3) for every $(e_3,e_4)\in U$, $\Lambda_{e_3,e_4}(K)=\Lambda_{e_3,e_4}(\mathbb{R}^n)\cap B$.

Given a pair of linearly independent vectors (w_1,w_2), denote by $(\mathrm{GS}^1(w_1,w_2),\mathrm{GS}^2(w_1,w_2))$ the 2-frame obtained from (w_1,w_2) by the Gram–Schmidt procedure in such a way that $\langle w_1,w_2\rangle=\langle\mathrm{GS}^1(w_1,w_2),\mathrm{GS}^2(w_1,w_2)\rangle$.

Define the fiber preserving map

$$\Phi:U\times V'\to V_{n+2,4},$$

given by

$$\begin{aligned}&\Phi\big((e_3,e_4),(e_1,e_2)\big)\\&\quad=\big(\mathrm{GS}^1\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),\mathrm{GS}^2\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),e_3,e_4\big).\end{aligned}$$

First of all, by (2),

$$\left(\mathrm{GS}^1\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),\mathrm{GS}^2\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),e_3,e_4\right)\in V_{n+2,4}.$$

Moreover, by (1),

$$\left(\mathrm{GS}^1\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),\mathrm{GS}^2\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right)\right)\in V$$

and therefore

$$\left(\mathrm{GS}^1\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),\mathrm{GS}^2\left(\Lambda_{e_3,e_4}(e_1),\Lambda_{e_3,e_4}(e_2)\right),e_3,e_4\right)\in p_2|^{-1}(U).$$

Hence, we obtain a fiber preserving homeomorphism:

$$\begin{array}{ccc} U\times V' & \xhookrightarrow{\Phi} & p_2|^{-1}(U) \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle p_2|} \\ U & \xrightarrow[\mathrm{id}]{} & U \end{array}$$

thus proving that $p_2|:\widetilde{V}\to V_{n+2,2}$ is a locally trivial bundle with fiber V'. Furthermore, p_2 is a fiber bundle with structure group G_K. If this is so, by Lemma 4.14, $V=V_{n+2,2}$. This implies that for every two planes through the origin, the corresponding sections of B are linearly equivalent and hence that B is an ellipsoid. ■

4.6. The complex Banach conjecture

The fifth meaning of equal is complex affinely equivalence.

Let V be a finite dimensional Banach space over the complex numbers all of whose hyperplane subspaces are isometric to each other. Is it true that V is a Hilbert space?

Our next purpose is to prove that the above problem is equivalent to the following geometric problem. We need first some definitions.

Let $\mathbb{S}^1$ be the space of all unit complex numbers $\mathbb{C}$. Let A be a subset of complex space $\mathbb{C}^n$. We say that A is *complex symmetric* if and only if there is a translated copy A' of A such that $\xi A'=A'$, for every $\xi\in\mathbb{S}^1$. In this case, if $A'=A-x_0$, we say that x_0 is the center of complex symmetry of A. If $-A$ is a translated copy of A, then we just say that A is *symmetric*. It will be useful to consider the empty set as a complex symmetric set. Note that a *compact convex set $A\subset\mathbb{C}^n$ is complex symmetric with center at x_0 if and only if for every complex line L through x_0, the section $L\cap A$ is a disk centered at x_0*. Of course, any complex k-plane or a ball in a finite dimensional Banach space over the complex numbers is complex symmetric. A complex ellipsoid is the image of a ball under a complex affine transformation. Thus, balls of finite dimensional Hilbert spaces are complex ellipsoids. Of course, complex ellipsoids are complex symmetric sets. With this definition in mind, we may state the following problem equivalent to the complex Banach conjecture:

If all complex hyperplane sections through the origin of a convex body $B \subset \mathbb{C}^{n+1}$ with the origin as center of complex symmetry are complex linearly equivalent, is the convex body B a complex ellipsoid?

As was already mentioned, this problem has a positive answer when $n =$ even (Gromov [13]) and when $n \equiv 1 \bmod 4$ (Bracho and Montejano [6]). The purpose of this section is to give a brief summary of the ideas and techniques used in the proof.

This time, unlike the real case in which we use the principal bundle

$$\mathrm{SO}_n \hookrightarrow \mathrm{SO}_{n+1} \to \mathbb{S}^n,$$

we will use the corresponding principal bundle $\mathrm{SU}_n \hookrightarrow \mathrm{SU}_{n+1} \to \mathbb{S}^{2n+1}$. Here SU_n is the group of complex isometries of determinant 1 in $\mathbb{C}^n$ and we say that the structure group of the principal bundle $\mathrm{SU}_n \hookrightarrow \mathrm{SU}_{n+1} \to \mathbb{S}^{2n+1}$ can be reduced to $G \subset \mathrm{SU}_n$ if the characteristic map $\chi_n : \mathbb{S}^{2n} \to \mathrm{SU}_n$ of the complex bundle factorizes through G, that is, there is a map $f : \mathbb{S}^{2n} \to G$ such that the following diagram commutes up to homotopy, where $i : G \to \mathrm{SU}_n$ is the inclusion

$$\begin{array}{ccc} \mathbb{S}^{2n} & \xrightarrow{\chi_n} & \mathrm{SU}_n \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & G. & \end{array}$$

Denote by $\mathrm{GL}'_n(\mathbb{C})$ the group of complex linear isomorphisms of $\mathbb{C}^n$ with determinant a positive real number. Note that if K_1 and K_2 are complex symmetric convex bodies in $\mathbb{C}^n$ which are complex linearly equivalent, then there is $g \in \mathrm{GL}'_n(\mathbb{C})$ such that $g(K_1) = K_2$.

Given a complex symmetric convex body $K \subset \mathbb{C}^n$, let

$$G_K := \left\{g \in \mathrm{GL}'_n(\mathbb{C}) \mid g(K) = K\right\}$$

be the *group of complex linear isomorphisms of K with positive real determinant*. By Lemma 1 of Gromov [13], there exists a complex ellipsoid of minimal volume containing K centered at the origin. Suppose now that this minimal ellipsoid is the $(2n-1)$-dimensional unit ball, then every $g \in G_K$ is actually an element of SU_n, because it fixes the unit ball, so in this case, $G_K := \{g \in \mathrm{SU}_n \mid g(K) = K\}$.

The link between our geometric problem and the topology is via the following lemma.

Lemma 4.15. *Let $B \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a complex symmetric convex body with center at the origin all of whose complex hyperplane sections through the origin are complex linearly equivalent. Then there exists a complex symmetric convex body $K \subset \mathbb{C}^n$ with center at the origin and with the property that every complex hyperplane section of B is complex linearly equivalent to K and such that the structure group of the principal fiber bundle $\mathrm{SU}_n \hookrightarrow \mathrm{SU}_{n+1} \to \mathbb{S}^{2n+1}$ can be reduced to $G_K \subset \mathrm{SU}_n$.*

Our main interest naturally lies in studying the structure groups of the principal bundle ξ_n: $\mathrm{SU}_n \hookrightarrow \mathrm{SU}_{n+1} \to \mathbb{S}^{2n+1}$. In particular, if $n \equiv 0 \bmod 2$, ξ_n cannot be reduced to a proper subgroup of SU_{n-1} (see Leonard [15, Theorem 1B]). Therefore, under the hypothesis of Lemma 4.15, G_K must be SU_n, and hence K must be a ball. This implies that every section of B is a complex ellipsoid. Of course, every section of a complex symmetric body $B \subset \mathbb{C}^{n+1}$ is a complex ellipsoid only if B is a complex ellipsoid; see [6, Lemma 3.3]. This proves the complex Banach conjecture, when n is even.

For the case $n \equiv 1 \bmod 4$, the proof requires first studying the case in which $G_K \subset \mathrm{SU}_n$ is irreducible. If so, the topology of compact Lie groups over the complex numbers is simpler than over the real numbers and then it is possible to prove, in a similar way to the real case, that $G_K = \mathrm{SU}_n$. If this is the case, then every section of B is an ellipsoid and consequently B is also an ellipsoid. If $G_K \subset \mathrm{SU}_n$ is not irreducible but G_K is a proper subgroup of SU_n, then we can prove that $G_K = \mathrm{SU}_{n-1}$. To understand the convex geometry of the consequences of this result, we need the following definition:

A *complex body of revolution* is a complex symmetric convex body $K \subset \mathbb{C}^n$ for which there exists a 1-dimensional complex subspace L of $\mathbb{C}^n$, called its *axis of revolution*, such that for every affine complex hyperplane H orthogonal to L, we have that $H \cap K$ is either empty, a single point, or a $(2n-2)$-dimensional ball centered at $H \cap L$. Of course, K is a convex body of revolution if and only if $G_K = \mathrm{SU}_{n-1}$.

With this in mind, it is very clear that what we have obtained is the following theorem.

Theorem 4.16. *Let $B \subset \mathbb{C}^{n+1}$, $n \equiv 1 \bmod 4$, $n \geq 5$, be a complex symmetric convex body with center at the origin all of whose complex hyperplane sections through the origin are complex linearly equivalent. Then, there exists a complex body of revolution $K \subset \mathbb{C}^n$ with center at the origin and with the property that every complex hyperplane section of B through the origin is $\mathbb{C}$-linearly equivalent to K.*

To conclude, we need to know what are the geometric consequences of all the complex hyperplane sections of a convex body being complex affine bodies of revolution.

Theorem 4.17. *A complex symmetric convex body $B \subset \mathbb{C}^{n+1}$ with center at the origin, $n \geq 4$, all of whose complex hyperplane sections through the origin are complex affine bodies of revolution, has at least one complex hyperplane section through the origin which is a complex ellipsoid.*

The proof of Theorem 4.17 is similar to the proof of Theorem 4.10 except this time the proofs are just technically more complicated. This concludes an sketch of the proof of the complex Banach conjecture when $n \equiv 0, 1, 2 \mod 4$, because by Theorems 4.16 and 4.17, every hyperplane section of B through the origin is a complex ellipsoid and therefore, by [6, Theorem 3.3] we obtain that B is a complex ellipsoid as we wished.

The following theorem follows immediately from Theorems 4.16 and 4.17. It proves the Banach conjecture over the complex numbers for $n \equiv 0, 1, 2 \mod 4$, and $\dim V > n$.

Theorem 4.18 (Bracho–Montejano [6]). *If all complex hyperplane sections through the origin of a complex symmetric convex body $B \subset \mathbb{C}^{n+1}$ are linearly equivalent, $n \equiv 0, 1, 2 \mod 4$, then the convex body B is a complex ellipsoid.*

5. Convex bodies all whose orthogonal projections are equal

The purpose of this section is to answer the following question:

If all orthogonal projections of a convex body onto hyperplanes are "equal", is the convex body "equal" to the ball?

5.1. Equal area, congruence, and affine equivalence

The first meaning of "equal" is same "area". In 1937, A. D. Aleksandrov [1] proved that if all orthogonal projections of a symmetric convex body have the same area, then not only does the body have the same volume of the corresponding ball but it is actually a ball.

Theorem 5.1 (Aleksandrov's projection theorem [1]). *If all orthogonal projections onto hyperplanes of a symmetric convex body $B \subset \mathbb{R}^{n+1}$ have equal n-dimensional volume, then the convex body B is a ball.*

Without the hypothesis of symmetry, Theorem 5.1 is false. However, a symmetric convex body all whose orthogonal projections have the same area not only has the volume of the corresponding ball but also it is actually a ball. For every $v \in \mathbb{S}^n$, denote by $B|v$ the orthogonal projection of B onto $v^{\perp}$ and let $\nu(B|v)$ be the n-dimensional volume of $B|v$. The proof of Theorem 5.1 follows immediately from the following Aleksandrov result (see [17, Theorem 2.11.1]). Given two convex bodies $B^1, B^2 \subset \mathbb{R}^{n+1}$ symmetric with respect to the origin and such that $\nu(B^1|v) = \nu(B^2|v)$, for every $v \in \mathbb{S}^n$, then B^1 is a translated copy of B^2. The proof of this result is analytic and a little more complicated than the proof of Theorem 2.1.

Using harmonic integration, it can be proved that a centrally symmetric convex body all whose $(n-1)$-dimensional perimeter areas are equal must be a ball. The proof is similar to the proof of Theorem 2.1 but using the support functions instead of the radial functions (see [10, Theorem 4]). Of course, without the symmetry hypothesis, the result is false as it can be observed with 3-dimensional convex bodies of constant width 1, in which the perimeter of all their orthogonal projections is π.

The next meaning of "equal" is congruence. That is, assume that all orthogonal projections onto hyperplanes of the convex body $B \subset \mathbb{R}^{n+1}$ are congruent.

The collection of orthogonal projections of $B \subset \mathbb{R}^{n+1}$,

$$\{B|v\}_{v\in\mathbb{S}^n}$$

give rise, not only to a field of convex bodies congruent to $B|e_1$ and tangent to $\mathbb{S}^n$, but also mainly to a complete turning of $B|e_1$, where $e_1 = \{1, 0, \dots,\} \in \mathbb{R}^{n+1}$. We know that a complete turning is only possible for symmetric convex bodies (see Section 3). So, $B|v$ is symmetric for every $v \in \mathbb{S}^n$ and, consequently, it is not very difficult to prove that B is symmetric, but in this last case Aleksandrov's theorem (Theorem 5.1) implies that B is also a ball. That is, we have the following theorem.

Theorem 5.2. *If all orthogonal projections onto hyperplanes of a convex body $B \subset \mathbb{R}^{n+1}$ are congruent, then the convex body B is a ball.*

Suppose now all orthogonal projections onto hyperplanes of the convex body $B \subset \mathbb{R}^{n+1}$ are affinely equivalent to a convex body K and suppose without loss of generality that the ellipsoid of minimal volume containing K is the unit ball. Denote $G_K := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g(K) = K \text{ and } \det(g) \text{ is positive}\} \subset \mathrm{SO}_n$. As in the case of the hyperplane sections, we have that the existence of the collection of projections $\{B|v\}_{v\in\mathbb{S}^n}$ gives rise directly to the following lemma which is the link between the topology and the geometric problem. Note that from the arguments given in the preceding paragraph and Theorem 3.2, we may assume without loss of generality that B and K are symmetric with center at the origin.

Lemma 5.3. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a symmetric convex body all of whose orthogonal projections onto hyperplanes are linearly equivalent. Then there exists a symmetric convex body $K \subset \mathbb{R}^n$, with the property that every orthogonal projection of B onto a hyperplane is linearly equivalent to K and such that the structure group of the principal fiber bundle $\mathrm{SO}_n \hookrightarrow \mathrm{SO}_{n+1} \to \mathbb{S}^n$ can be reduced to $G_K \subset \mathrm{SO}_n$.*

Once we have this technical lemma, we are in a position to know, using the topological arguments from Section 4.2, how the projections of B are. That is, we have the following theorem.

Theorem 5.4. *Let $B \subset \mathbb{R}^{n+1}$, $n \equiv 0,1,2 \bmod 4$, $n \geq 2$, $n \neq 133$, be a convex body all of whose orthogonal projections onto hyperplanes are affinely equivalent. Then, there exists a body of revolution $K \subset \mathbb{R}^n$, with the property that every orthogonal projection of B is affinely equivalent to K.*

To conclude, we need to know the geometric consequences of all orthogonal projections of a convex body being affine bodies of revolution. Every orthogonal projection of a body of revolution is a body of revolution, this is why projections of affine bodies of revolution are affine bodies of revolution. Is the converse true? As far as I know, nobody knows the answer. The following geometric question is of great interest. *Suppose that B is an $(n+1)$-dimensional convex body all whose orthogonal projections are affine bodies of revolution, $n \geq 3$. Is B an affine body of revolution?*

We shall give a partial answer to this question which will turn out to be sufficiently good for our purposes. Under the same hypothesis of the above question, we shall prove that at least one orthogonal projection of B is an ellipsoid. If this is so, and if, in addition, B satisfies the hypothesis that every two of its orthogonal projections are affinely equivalent, then every orthogonal projection of B is an ellipsoid and consequently B is an ellipsoid. The proof of the next theorem is very similar to the proof of Theorem 4.10, with the different adjustments that are always necessary when trying to adapt a proof for sections to one for projections.

Theorem 5.5. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, $n \geq 4$, and suppose that every orthogonal projection onto hyperplanes of K is an affine body of revolution. Then there is an orthogonal projection of B which is an ellipsoid.*

This result, together with Theorem 5.4, immediately implies the following characterization of the ellipsoid first proved by Montejano in [22].

Theorem 5.6. *Let $B \subset \mathbb{R}^{n+1}$, $n \equiv 0,1,2 \bmod 4$, $n \geq 2$, $n \neq 133$, be a convex body all of whose orthogonal projections onto hyperplanes are affinely equivalent. Then B is an ellipsoid.*

5.2. The codimension 2 case for orthogonal projections

In this section, we will adapt Gromov's ideas from Section 4.5 to the context of orthogonal projections.

We need first a technical lemma.

Lemma 5.7. *Given a linear embedding $h : \mathbb{R}^n \to \mathbb{R}^m$, $2 < n < m$, there is a continuous map $h^* : V_{n,2} \to V_{m,2}$ such that, for every $u \in V_{n,2}$, (i) $\langle h^*(u) \rangle \subset h(\mathbb{R}^n)$ and (ii) $h(\langle u \rangle^{\perp})$ is orthogonal to $\langle h^*(u) \rangle$, where $\langle u \rangle$ denotes the plane generated by u.*

Furthermore, h^* varies continuously with h, while h varies in the space of linear embeddings from R^n to R^m.

Proof. Let $H \subset h(\mathbb{R}^n)$ be the plane such that H is orthogonal to $h(\langle u\rangle^{\perp})$ and let $\pi : h(\mathbb{R}^n) \to H$ be the orthogonal projection. Then, given $u = (u_1, u_2) \in V_{n,2}$, let

$$h^*(u_1, u_2) = \left(\mathrm{GS}^1\left(\pi(u_1), \pi(u_2)\right), \mathrm{GS}^2\left(\pi(u_1), \pi(u_2)\right)\right) \in V_{m,2},$$

where given a pair of linearly independent vector (w_1, w_2), denote by $(\mathrm{GS}^1(w_1, w_2), \mathrm{GS}^2(w_1, w_2))$ the 2-frame obtained from (w_1, w_2) by the Gram–Schmidt procedure in such a way that $\langle w_1, w_2\rangle = \langle \mathrm{GS}^1(w_1, w_2), \mathrm{GS}^2(w_1, w_2)\rangle$. ∎

Here is the analogue of Theorem 4.13 for orthogonal projections:

Theorem 5.8 (Montejano). *Let B be an $(n+2)$-dimensional convex body and suppose that all orthogonal projections onto n-planes are linearly equivalent, for $n > 1$ odd. Then the convex body B is an ellipsoid.*

Proof. There is a convex body $K \subset \mathbb{R}^n$ with the property that the minimal ellipsoid containing K is the unit ball of $\mathbb{R}^n$ and such that all orthogonal projections of B onto an n-dimensional subspace are linearly equivalent to K. Let us fix a 2-dimensional plane $\Delta \subset \mathbb{R}^{n+2}$ through the origin and define $V \subset V_{n+2,2}$ to be the set of 2-frames (e_1, e_2) in $\mathbb{R}^{n+2}$ such that the orthogonal projection of B onto $\langle e_1, e_2\rangle$ is linearly equivalent to the orthogonal projection of B onto Δ. Furthermore, let $V' \subset V_{n,2}$ be the set of 2-frames (e_1, e_2) in $\mathbb{R}^n$ such that the orthogonal projection of K onto $\langle e_1, e_2\rangle$ is linearly equivalent to the orthogonal projection of B onto Δ. Finally, let $\widetilde{V} = p_1^{-1}(V) = \{(e_1, e_2, e_3, e_4) \in V_{n+2,4} \mid (e_1, e_2) \in V\}$.

We shall first prove that the restriction $p_2| : \widetilde{V} \to V_{n+2,2}$ is a locally trivial bundle with fiber V'. For that purpose, consider U an open contractible subset of $V_{n+2,2}$. Then, using the contractibility of U and the existence of a field of convex bodies, lineally equivalent to K, contained in the fibers of the canonical vector bundle of n-subspaces in $\mathbb{R}^{n+2}$, it is possible to construct a continuous map $\Lambda : U \to \mathrm{GL}(n, n+2)$ satisfying the following properties:

(1) for every $(e_3, e_4) \in U$, $\Lambda_{e_3,e_4} : \mathbb{R}^n \to \mathbb{R}^{n+2}$ is a linear embedding,

(2) for every $(e_3, e_4) \in U$, $\Lambda_{e_3,e_4}(\mathbb{R}^n)$ is orthogonal to both e_3 and e_4,

(3) for every $(e_3, e_4) \in U$, $\Lambda_{e_3,e_4}(K)$ is the orthogonal projection of B onto $\Lambda_{e_3,e_4}(\mathbb{R}^n)$.

Define the fiber preserving map

$$\Phi : U \times V' \to V_{n+2,4}$$

given by $\Phi((e_3, e_4), (e_1, e_2)) = (h^*(e_1, e_2), e_3, e_4)$.

First of all, by (2), $(h^*(e_1, e_2), e_3, e_4) \in V_{n+2,4}$. Moreover, by (1) and Lemma 5.7, $(h^*(e_1, e_2)) \in V$ and therefore $(h^*(e_1, e_2), e_3, e_4) \in p_2|^{-1}(U)$. Hence, we obtain a fiber preserving homeomorphism

$$\begin{array}{ccc} U \times V' & \overset{\Phi}{\hookrightarrow} & p_2|^{-1}(U) \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle p_2} \\ U & \underset{\mathrm{id}}{\longrightarrow} & U. \end{array}$$

Thus proving that $p_2| : \widetilde{V} \to V_{n+2,2}$ is a locally trivial bundle with fiber V'. If this is so, by Lemma 4.14, $V = V_{n+2,2}$. This implies that every two orthogonal projections onto 2-dimensional planes are linearly equivalent and hence, by Theorem 5.6, for $n = 2$, that K is an ellipsoid. ∎

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References

[1] A. D. Aleksandrov, Zur Theorie der gemischten Volumina von konvexen Körpern, II: Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen. *Mat. Sbornik N. S.* **2** (1937), 1205–1238

[2] H. Auerbach, S. Mazur, and S. Ulam, Sur une propriété caractéristique de l'ellipsoïde. *Monatsh. Math. Phys.* **42** (1935), no. 1, 45–48 Zbl 0011.22208 MR 1550413

[3] S. Banach, *Théorie des opérations linéaires*. Monografie Matematyczne 1, PWN – Panstwowe Wydawnictwo Naukowe, Warszawa, 1932; English translation in *Theory of Linear Operations*, Vol. 38, Elsevier, 1987 Zbl 0005.20901

[4] G. Bor, L. Hernández Lamoneda, V. Jiménez-Desantiago, and L. Montejano, On the isometric conjecture of Banach. *Geom. Topol.* **25** (2021), no. 5, 2621–2642 Zbl 07396004 MR 4310896

[5] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)* **57** (1953), 115–207 Zbl 0052.40001 MR 51508

[6] J. Bracho and L. Montejano, On the complex Banach conjecture. *J. Convex Anal.* **28** (2021), no. 4, 1211–1222 Zbl 07470575 MR 4374354

[7] G. R. Burton, Congruent sections of a convex body. *Pacific J. Math.* **81** (1979), no. 2, 303–316 Zbl 0373.52004 MR 547601

[8] M. Čadek and M. Crabb, G-structures on spheres. *Proc. London Math. Soc. (3)* **93** (2006), no. 3, 791–816 Zbl 1110.55008 MR 2266967

[9] A. Dvoretzky, A theorem on convex bodies and applications to Banach spaces. *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 223–226; a detailed proof appeared in Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), 123–160 Zbl 0088.31802 MR 105652

[10] K. J. Falconer, Applications of a result on spherical integration to the theory of convex sets. *Amer. Math. Monthly* **90** (1983), no. 10, 690–693 Zbl 0529.52001 MR 723941

[11] R. J. Gardner, *Geometric Tomography*. 2nd edn., Encyclopedia Math. Appl. 58, Cambridge University Press, New York, 2006 Zbl 1102.52002 MR 2251886

[12] H. Groemer, Fourier series and spherical harmonics in convexity. In *Handbook of Convex Geometry, Vol. A, B*, pp. 1259–1295, North-Holland, Amsterdam, 1993 Zbl 0799.52001 MR 1243009

[13] M. L. Gromov, On a geometric hypothesis of Banach. *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967), 1105–1114 Zbl 0162.44402 MR 0217566

[14] D. G. Larman, A note on the false centre problem. *Mathematika* **21** (1974), 216–227 Zbl 0298.52005 MR 362048

[15] P. Leonard, G-structures on spheres. *Trans. Amer. Math. Soc.* **157** (1971), 311–327 Zbl 0217.49201 MR 275468

[16] P. Mani, Fields of planar bodies tangent to spheres. *Monatsh. Math.* **74** (1970), 145–149 Zbl 0189.52901 MR 259753

[17] H. Martini, L. Montejano, and D. Oliveros, *Bodies of Constant Width. An Introduction to Convex Geometry with Applications*. Birkhäuser/Springer, Cham, 2019 Zbl 1468.52001 MR 3930585

[18] V. D. Milman, A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.* **5** (1971), no. 4, 28–37 MR 0293374

[19] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*. Ann. of Math. Stud. 76, Princeton University Press, Princeton, NJ, 1974 Zbl 0298.57008 MR 0440554

[20] L. Montejano, Convex bodies with homothetic sections. *Bull. London Math. Soc.* **23** (1991), no. 4, 381–386 Zbl 0746.52009 MR 1125866

[21] L. Montejano, Two applications of topology to convex geometry. *Tr. Mat. Inst. Steklova* **247** (2004), no. Geom. Topol. i Teor. Mnozh., 182–185 Zbl 1104.52001 MR 2168169

[22] L. Montejano, Convex bodies with affinely equivalent projections and affine bodies of revolution. *J. Convex Anal.* **28** (2021), no. 3, 871–877 Zbl 07470554 MR 4374323

[23] D. Montgomery and H. Samelson, Transformation groups of spheres. *Ann. of Math. (2)* **44** (1943), 454–470 Zbl 0063.04077 MR 8817

[24] A. Pełczyński, On some problems of Banach. *Russ. Math. Surv.* **28** (1973), no. 6, 67–75 Zbl 0288.46016

[25] R. Schneider, Convex bodies with congruent sections. *Bull. London Math. Soc.* **12** (1980), no. 1, 52–54 Zbl 0401.52001 MR 565484

[26] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*. 2nd expanded edn., Encyclopedia Math. Appl. 151, Cambridge University Press, Cambridge, 2014 Zbl 1287.52001 MR 3155183

[27] N. Steenrod, *The Topology of Fibre Bundles*. Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999 Zbl 0942.55002 MR 1688579

[28] W. Süss, Kennzeichnende Eigenschaften der Kugel als Folgerung eines Brouwerschen Fixpunktsatzes. *Comment. Math. Helv.* **20** (1947), 61–64 Zbl 0029.32001 MR 21339

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