DOI 10.4171/8ECM/27

From branching singularities in minimal surfaces to non-smoothness points in ice-water interfaces

Joaquim Serra

Abstract. We review some recent developments on the regularity theory of two classical free boundary problems: the obstacle problem and Stefan's problem.

We emphasize the similarities and differences between these recent results (for the obstacle problem and Stefan's problem) and the regularity theory of integer rectifiable area-minimizing currents (and related problems) developed during the XXth century.

1. Introduction

The aim of this note is to review some recent developments on the regularity theory of two classical free boundary problems: the obstacle problem and Stefan's problem.

We believe that these new developments are better understood and appreciated when one can recognize in them strong parallelisms, and yet crucial differences, with the regularity theories of Plateau's problem, Signorini's problem, and *Almgren's problem*. (Throughout the note, we will use the non-standard (but convenient) keyword *Almgren's problem* to refer to the analog of Plateau's problem in context of integer rectifiable area-minimizing currents of codimension 2 or higher, which was studied by Almgren in his famous work [4].) Consequently, we provide, in addition to a rather complete background on the former two problems, a (partial) historical overview of the latter three, focusing on their connections and analogies with the obstacle problem and Stefan's problem.

Finally, we describe the methods and results in recent works [25–27, 29] concerned with the fine structure of the singular sets in the obstacle problem and Stefan's problem.

2. Five "classical" problems

We begin by presenting—in chronological order of their first appearance—the five problems that will be discussed throughout the note: Plateau's problem (1760s),

2020 Mathematics Subject Classification. 35R35.
Keywords. Stefan's problem, obstacle problem, free boundary.

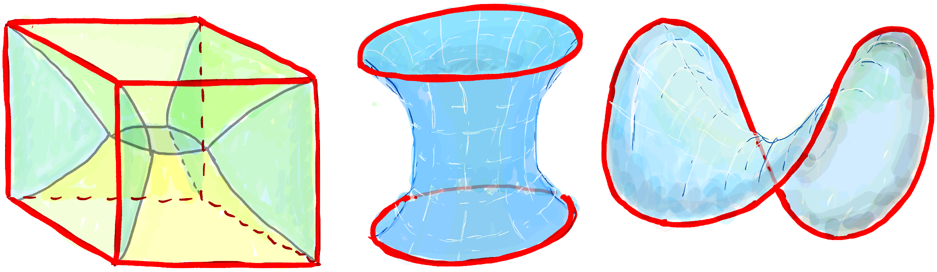

Figure 1. Soap films spanning the red "curves".

Stefan's problem (1890s), Signorini's problem (1950s), the obstacle problem (1960s), and Almgren's problem (1970s).

2.1. Plateau's problem: The elegant shapes of soap films

Given a curve in $\mathbb{R}^3$, can one find a surface with minimal area having it as boundary? Raised by Joseph-Louis Lagrange in the 1760s, this problem is one of the most classical and influential ones in the calculus of variations and geometry. It is named after the Belgian physicist, J. Plateau (1801–1883), who experimentally investigated the (physical) geometric laws of soap films and bubbles. By the effect of surface tension, soap films are natural examples of area-minimizing surfaces.

By a well-known classical computation going back to Lagrange, if a piece of an area-minimizing surface is smooth, then its mean curvature (sum of the principal curvatures) must be identically zero.

A difficulty of Plateau's problem is that area-minimizing "surfaces" may not be surfaces in the classical sense of differential geometry. For instance, physical soap films can take the shapes sketched in Figure 1, and while the center and right ones are smooth surfaces, the soap film on the left is not smooth (or not even locally homeomorphic to a planar disc!) near some of its points.

Between the 1930s and the 1970s, several well-known analysts and geometers, including Almgren, De Giorgi, Douglas, Federer, Fleming, Radó, Reifenberg, and Taylor, among others, yielded outstanding contributions to Plateau's problem, which shaped its modern theory; see for instance [1, 3, 4, 14, 15, 18, 20, 21, 28, 40–42, 53]. They addressed the following fundamental questions:

(i) Which mathematical objects, that are "surfaces" in some sense, allow for a rigorous solution of the area minimization problem?

(ii) Are such minimizers smooth, possibly outside of a certain singular set?

(iii) What can be said about the singular set? (e.g., is it lower dimensional?)

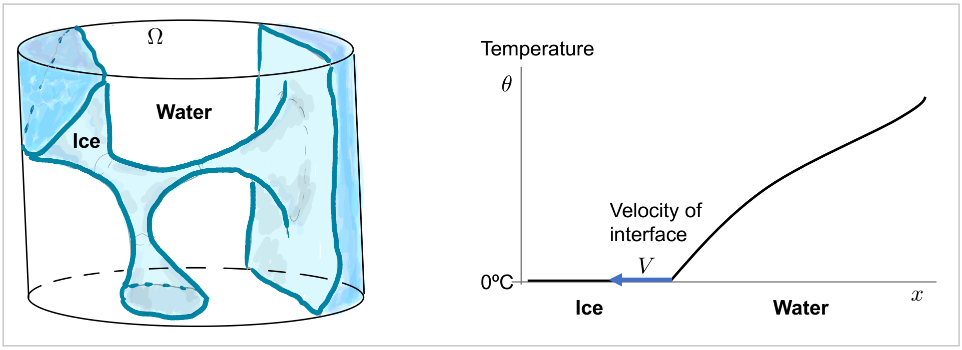

Figure 2. Stefan's problem: ice melting in water.

Thanks to intensive efforts during the XXth century, the answers to these questions are today well understood. In Section 3, we will review some key aspects of the regularity theory of Plateau's problem, i.e., the answers to questions (ii) and (iii). Some of them will have clear parallelisms in the other problems we will discuss.

2.2. Stefan's problem: Ice melting in water

Dating back to the XIXth century, Stefan's problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water. The problem is named after Josef Stefan, a Slovenian physicist who introduced it around 1890; see [52].

The most classical formulation of Stefan's problem (see e.g. [19, 24]) is as follows: let $\Omega \subset \mathbb{R}^3$ be some bounded domain. For concreteness, let us think that Ω is a "cylindric water tank" as drawn in Figure 2. We denote by $\theta = \theta(x,t)$ the temperature of the water at the point $x \in \Omega$ at time $t \in \mathbb{R}^+ := [0, +\infty)$. We assume that $\theta \geq 0$ in $\Omega \times \mathbb{R}^+$. The (nonnegative) temperature at the boundary of the tank is given, and we assume that $\theta = 0$ at $t = 0$.

The set $\{(x,t) \in \Omega \times \mathbb{R}^+ : \theta(x,t) > 0\}$, denoted for brevity by $\{\theta > 0\}$, represents the water while its complement, denoted by $\{\theta = 0\}$, represents the ice. The temperature θ satisfies the heat equation

$$\partial_t \theta - \Delta\theta = 0 \quad \text{in the region } \{\theta > 0\},$$

while in the complement θ is simply zero.

Determining the time-evolving domain $\{\theta > 0\}$ in which the heat equation holds is part of the problem. Equivalently, one must determine where the ice-water interface $\partial\{\theta > 0\}$, also called the *free boundary*, is. For it, an additional equation—so-called

Stefan's condition[1]—is needed:

$$\partial_t \theta = |\nabla_x \theta|^2 \quad \text{on } \partial\{\theta > 0\}. \tag{2.1}$$

It is not difficult to see that, in the previous setting, the ice $\{\theta = 0\}$ must shrink over time. More precisely, if at some point of the tank there is liquid water at some given time, then the same point remains occupied by liquid at all future times.

The relevant regularity questions for Stefan's problem are as follows:

(i) Is the problem well posed?[2]

(ii) Is the free boundary smooth, or may it have singularities?

(iii) If there are singularities, how often may they occur in space and time?

We will discuss the answers to these questions (which remained completely open until the 1970s!) later on in this note.

2.3. Signorini's problem (1950s)

Raised in 1959, Signorini's problem [50] consists in finding the (elastic) equilibrium configuration of an elastic body, resting on a rigid frictionless horizontal plane and subject only to its mass forces.

The difficulty of the problem lies on the fact that one needs to determine which points on the bottom surface of the body will be in contact with the plane (and what is the deformation at the points which are not in contact).[3]

In a very linearized situation, Signorini's problem is reduced to following a minimization problem in the half space $U := \mathbb{R}^3 \cap \{x_3 \geq 0\}$,

$$\min \left\{ \int_U |\nabla u|^2 \, dx \text{ among } u : U \to \mathbb{R} \text{ satisfying } u(x_1, x_2, 0) \geq g(x_1, x_2), \right. \\ \left. \lim_{x \to \infty} u(x) = 0 \right\}, \tag{2.2}$$

[1] This extra relation comes from two considerations. First, the normal velocity of the interphase, V, is proportional to the amount of heat absorbed by it (and used to melt the ice). In turn, this flow of heat "entering" the interphase is, by Fourier's law, proportional to the gradient of temperature. Hence, we have $|V| = C|\nabla\theta|$. Second, since $\theta = 0$ on the moving interphase, we obtain that, on it, V and $\nabla\theta$ are parallel and $(\partial_t + V \cdot \nabla)\theta = 0$. Combining the two previous equations and choosing the physical units to make $C = 1$, we obtain Stefan's condition.

[2] In the sense of Hadamard, i.e., given initial and boundary conditions, is there a unique solution which depends continuously on the given data?

[3] If one knew, for instance, that all points are in contact, then the initial and final position of all the boundary points of the body would be obviously determined, and resolving the body's deformation would be much simpler!

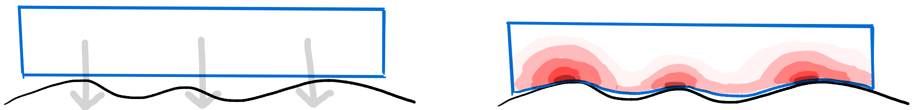

Figure 3. Signorini's problem: an elastic body lying on a surface.

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth prescribed function satisfying $\limsup_{x\to\infty} g < 0$. This problem is often called the *thin obstacle problem* and has other applications as well, such as in the modeling of semipermeable membranes. See [5, 13, 22, 38] and references therein for more information on this problem.

The model (2.2) can be used when the bottom surface of the (undeformed) elastic solid is a small perturbation of a horizontal plane. In order to derive it more easily, let us consider the following variant of the problem: Assume that the bottom surface of the (undeformed) elastic solid is (exactly) a horizontal plane and that, instead, the rigid surface on which the body will rest is a small perturbation of a horizontal plane. The horizontal surface is then described as $\{x_3 = \varepsilon g(x_1, x_2)\}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is some bounded function—which we assume to be smooth—and where $\varepsilon > 0$ is small. This situation is depicted in Figure 3. Let us suppose for simplicity that $g = -1 + \bar{g}$, where $\bar{g} : \mathbb{R} \to [0, 1]$ is smooth and compactly supported.

The undeformed body "suspended in air" corresponds to $U := \mathbb{R}^3 \cap \{x_3 \geq 0\}$. As we let it rest on the rigid surface, it experiences a deformation. For ε small, horizontal deformations may be neglected, and we can think that the displacements are only vertical. More precisely, there is a function $u : U \to \mathbb{R}$ and a constant $c \in (0, 1)$ such that the point of the solid before occupying the position $(x_1, x_2, x_3) \in U$ in the suspended configuration, now occupies the position $(x_1, x_2, x_3 + \varepsilon u)$ in the resting configuration. We are considering for simplicity the "boundary condition at infinity" $\lim_{x\to\infty} u = -c \in (-1, 0)$, but it would not be difficult to consider other (more realistic) boundary conditions by modifying (2.2) accordingly.

The elastic energy of the deformed body is proportional (at leading order in ε) to $\int_{\mathbb{R}^2} |\nabla u|^2$. Hence, up to replacing u and g by $u - c$ and $g - c$, we obtain (2.2).

Now, although the minimization problem (2.2) leads to a nonlinear Euler–Lagrange equation, the fact that the (convex) Dirichlet energy is minimized inside the convex set $\{u \in H^1_0(U) : u(\cdot, \cdot, 0) \geq g\}$ confers it a very nice mathematical structure. The study of Singorini's problem was the starting point for the study of other similar convex constrained minimization problems with free boundaries, initiating in the 1960s the field of *variational inequalities*.

2.4. The obstacle problem (1960s)

Conceived as a paradigmatic variational inequality, the obstacle problem originates in the papers [8,30,32,35]. The initial motivation of the problem (which gives its name) concerned Plateau's problem with an obstacle. Namely, given a concave function $\psi : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a convex smooth domain, and some boundary values $h : \partial\Omega \to \mathbb{R}$ satisfying $h \geq \psi|_{\partial\Omega}$, find a surface with minimal area among all graphs lying above the *obstacle* ψ and spanning the curve $\{(x, h(x)) : x \in \partial\Omega\}$. In other words,

$$\min\left\{\int_\Omega \sqrt{1+|\nabla v|^2}\,dx \text{ among } v \geq \psi \text{ satisfying } v|_{\partial\Omega} = h\right\}. \tag{2.3}$$

Determining where the surface will be in contact with the obstacle is part of the problem.

The obstacle problem is actually the "small perturbation version" of (2.3), namely, the same minimization problem where $\sqrt{1+|\nabla v|^2}$ is replaced by $\frac{1}{2}|\nabla v|^2$. Computing its first variation with respect to nonnegative perturbations, one finds that a minimizer u must satisfy the variational inequality

$$\int_\Omega \nabla v \cdot \nabla \xi\,dx \geq 0 \text{ in } \Omega, \quad \text{for all } \xi \in C_c^\infty(\Omega) \text{ such that } \xi \geq 0. \tag{2.4}$$

Using this one can show that v is lower-semicontinuous, and hence the set $\{x \in \Omega : v(x) > \psi\}$, denoted by $\{v > \psi\}$ for brevity, is open. Since inside the set $\{v > \psi\}$ the solution v can be slightly perturbed in both the upwards and downwards directions, its minimality yields

$$\int_\Omega \nabla v \cdot \nabla \eta\,dx = 0 \text{ in } \Omega, \quad \text{for all } \eta \in C_c^\infty\big(\{v > \psi\}\big). \tag{2.5}$$

Considering the new function $u := v - \psi$ and integrating by parts in (2.4)-(2.5), we obtain the PDE

$$\begin{cases} \Delta u = \max(0, -\Delta\psi), \\ u \geq 0, \end{cases} \tag{2.6}$$

which is also called the obstacle problem.

Although the original motivation of the obstacle problem does not seem very deep, much more interesting applications have been found in the last decades. A beautiful one concerns the configuration of a cloud of Coulomb charges (all with the same sign), which are kept together by a confining electric potential. In the asymptotic regime corresponding to a very large number of charges, the potential generated by them solves a problem of the form (2.6) (see for instance [43] or the introduction of [49]). Other well-known applications are the dam problem (fluid filtration) and optimal stopping problems (for Finance and Probability). As we will see, the obstacle problem in the particular case $-\Delta\psi \equiv 1$ is also closely connected with Stefan's problem.

2.5. Almgren's problem (1970s)

After their existence theory was established in the 1960s (see [20, 21]), the question of (partial) regularity for oriented area-minimizing m-dimensional surfaces in $\mathbb{R}^{m+k}$ (more precisely integer rectifiable area-minimizing m-currents), in codimension $k \geq 2$ was a very natural one. For the sake of brevity, throughout this note we will use the keyword *Almgren's problem* to refer to this problem. It is a convenient and probably fair name for the problem, since Almgren anticipated its mathematical significance and studied it in depth during the last two decades of his life. His complete resolution was published in a famous 950-page posthumous paper [4].

The details of Almgren's proof are so intricate that its correctness was rather a myth until De Lellis and Spadaro deciphered its key ideas and bridged them with shorter (and clearer) arguments in a series of recent papers (see [17] and references therein). In Section 5, we will describe (very roughly) some ideas from this monumental proof, since they have clear parallels in our recent results for the Stefan problem and the obstacle problem.

One aspect that makes codimension $k \geq 2$ area-minimizing surfaces particularly delicate is the phenomenon of branching. As it was already known before the 1970s (by a classical result of Wirtinger and Federer; see [16, Section 1.2] for details), holomorphic "curves" are area-minimizing 2-surfaces in $\mathbb{R}^4$. For example, we can consider $S := \{(x_3 + ix_4)^2 = (x_1 + ix_2)^3\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$. Note that S is not smooth at 0: it has a delicate type of singularity called *branching singularity*. While—as we will see in Section 3—singularities in soap-film-like area-minimizing surfaces in $\mathbb{R}^3$ (or in integer rectifiable codimension 1 currents in $\mathbb{R}^n$ for all n) are always of conical type, zooming in infinitely at a branching singularity, we always obtain a plane, just as in smoothness points. However, near branching singularities, the surface is really a multiple-valued graph over the tangent plane. As we will see, this feature makes the analysis of the problem in codimension $k \geq 2$ much harder than in the case $k = 1$.

3. Classical regularity theory for Plateau's problem (1960s)

In Section 2.1, we stated the three main questions (i)-(ii)-(iii) associated to Plateau's problem. Similar questions apply to all the problems considered here. Now, in the case of Plateau's problem, existence question (i) is very challenging, and the multiple (all valid) answers to it obtained during the XXth century were celebrated breakthroughs (see references in Section 2.1). However, the discussion of (i) does not reveal any parallelism between Plateau's problem and the obstacle problem or Stefan's problem. For this reason (and also because it would take too much space), we will not discuss (i) here and will focus on the regularity part: questions (ii) and (iii).

Although we do not discuss (i), it is perfectly possible to intuitively understand most of the main ideas in the regularity theory for Plateau's problem without giving a completely rigorous definition of "area-minimizing surface". For our purposes, it is enough to think of physical soap films.

In the rest of the section, Γ will denote some prescribed (reasonably regular) contour. We postulate the existence of an "area-minimizing surface" (a physical soap film) spanning Γ, which we denote by S. We review next some of the main ingredients of the regularity theory for such S.

3.1. Minimal surface equation (1760s)

As found by Lagrange in the 1760s, smooth pieces of an area-minimizing surface must have zero mean curvature. As a consequence, if a piece of the surface can be represented by a C^1 graph, then it solves a uniformly elliptic equation with continuous coefficients. Then, linear methods in elliptic PDE (Schauder estimates) can be used to show that the piece of surface must be analytic.

3.2. De Giorgi's "flatness implies smoothness" principle (1961)

One of the most fundamental results for area-minimizing surfaces is the following theorem of De Giorgi [14] (see also [31]). We give a slightly modified version of statement (not involving the excess) due to Savin [46].

Theorem 1 ([14]). *There exists $\varepsilon_\circ > 0$ dimensional such that the following holds. Assume that S has minimal perimeter inside B_1 (i.e., the curve Γ which the soap film S spans does not intersect B_1) such that $S \cap B_1 \subset \{|x_n| \le \varepsilon_\circ\}$. Then, ∂S is an analytic graph in $B_{1/2}$.*

It will become clear in the next subsections that this theorem is a fundamental pillar of the theory. Let us now recall the heuristic behind its proof: let $B_1' \subset \mathbb{R}^2$ be the unit ball and suppose that $S = \{x_3 = \varepsilon g(x_1, x_2)\}$ with $\varepsilon > 0$ tiny and $g : B_1' \to \mathbb{R}$ bounded. Then the area of S is given by

$$\int_{B_1'} \sqrt{1 + \varepsilon^2 |\nabla g|^2}\, dx_1\, dx_2 = \pi + \varepsilon^2 \int_{B_1'} \frac{|\nabla g|^2}{2}\, dx_1\, dx_2 + O(\varepsilon^4).$$

Hence, for $\varepsilon \downarrow 0$, the fact that $x_3 = \varepsilon g(x_1, x_2)$ has minimal area should imply that g (which is nothing but the x_3 coordinate on S, as a function of x_1, x_2, and divided by ε) must be, approximately, a minimizer of $\int_{B_1'} \frac{|\nabla g|^2}{2}\, dx_1\, dx_2$. In other words, g is approximately harmonic.[4] As a consequence (of this happening at every scale and near every point of S), the smoothness of the limiting harmonic functions as $\varepsilon \downarrow 0$ is inherited by S, which can be shown to be a $C^{1,\alpha}$-graph. The minimal surface equation then implies its analyticity.

3.3. Fleming's monotonicity formula and tangent cones (1962)

A very useful consequence of the minimality of S is the so-called "monotonicity formula". Fix $x \in S$. For $r > 0$, let $B_r(x) \subset \mathbb{R}^3$ denote the Euclidean ball of radius r centered at x. Given $r > 0$ such that $B_r(x) \cap \Gamma = \varnothing$ (recall that Γ is the contour spanned by S), let us consider the dimensionless quantity

$$a_x(r; S) := \frac{1}{r^2} \operatorname{Area}\big(S \cap B_r(x)\big). \tag{3.1}$$

Then, $a_x(r; S)$ is monotone nondecreasing in r (this was first shown in [21]).

To prove the monotonicity formula (at $x = 0$), one compares the area of S in $B_r(0)$ with the area of "competitors" S_t obtained glueing the rescaled surface tS for some $t \in (0, 1)$ inside $B_{tr}(0)$ with a "conical interpolation" $\{x \in \mathbb{R}^n : \frac{rx}{|x|} \in S\}$ in the annulus $B_r \setminus B_{tr}$ (note that in this way S_t coincides with S on ∂B_r).

One can show that $a_x(r; S)$ is constant between $r = 0$ and $r = R$ if and only if S is conical inside B_R, that is, if $t(S \cap B_R) = S \cap B_{tR}$ for all $t \in (0, 1)$.

The previous observation gives essential information on area-minimizing surfaces: they must have conical structure around each point "when looked at the microscope". More precisely, let us consider the "zoomed-in" (around x) surfaces $S^{x,r} := \frac{1}{r}(S - x)$ for $r > 0$. For any fixed $R > 0$, $a_0(R; S^{x,r}) = a_x(Rr; S)$ is monotone increasing (in r) and converges to the constant $a_x(0^+, S)$ as $r \downarrow 0$. Hence we have $0 \le a_0(R; S^{x,r}) - a_0(0^+; S^{x,r}) \downarrow 0$ as $r \downarrow 0$. As a consequence, one can prove that the surface $S^{x,r}$ must be closer and closer to some cone inside any fixed ball B_R, as $r \downarrow 0$. This crucial property was first noticed in [28].

3.4. The classification of minimal cones: Taylor, Almgren, and Simons

By the discussion in the previous subsection, for any given $x \in S$, the "zoomed-in" surface $S^{x,r} \cap B_1$ is arbitrarily close to some area-minimizing cone $\mathcal{C}$, provided that we take r small enough (possibly depending on x). This leads us to the question: what are the possible area-minimizing cones $\mathcal{C}$? The answer depends on the type of objects which we want to admit as "surfaces". As proven in [53], in the case of "soap-film-like minimal surfaces" (Reifenberg [41] or Almgren [3]), there are exactly three

[4]The actual proof of this kind of statement is, of course, more complicated than that: to start with S does not need to be a graph, so first, one must suitably approximate it by graphs, and then one needs to understand how to transfer the regularity of harmonic functions to S. But this gives a good enough idea on how the proof works.

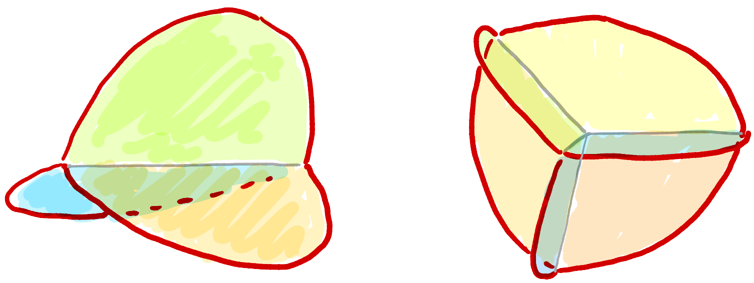

Figure 4. Possible singularities in soap-film-like minimal surfaces: Y type (left) and tetrahedron type (right).

possibilities: a plane, three half-planes meeting in Y shape with angles of 120°, or the cone generated by the edges of a regular tetrahedron centered at 0 (see Figure 4).

An easy computation shows that

$$a_0(r;\mathscr{C}) \equiv \begin{cases} \ell_1 : \pi \approx 3.1 & \text{if } \mathscr{C} \text{ is a plane,} \\ \ell_2 := 3\pi/2 \approx 4.7 & \text{if } \mathscr{C} \text{ is of } Y \text{ type,} \\ \ell_3 := 3\arccos(-1/3) \approx 5.7 & \text{if } \mathscr{C} \text{ is of tetrahedron type.} \end{cases} \tag{3.2}$$

Hence, thanks to Fleming's monotonicity formula, for every point x in a soap film S (in $\mathbb{R}^3$), zoomed-in surface $S^{x,r}$ (r tiny) must be close to one of the previous three possible cones. Moreover, the type of cone is determined by the value of $a_x(0^+;S)$, which necessarily belongs to $\{\ell_1,\ell_2,\ell_3\}$, as (3.2).

Based on our experience when observing physical soap films, we would expect that the zoomed-in surfaces should look like a plane around "most points", but still one important idea is still needed to show this (see next subsection). Still, we can already start to devise the power of De Giorgi's theorem and Fleming's monotonicity formula combined. They imply that for any given $x \in S$, if there exists $r > 0$ such that $a_x(r;S) < \ell_2$, then S will be analytic in some neighborhood of x.[5] Such points x are called *regular points*. All other points are called *singular points*.

Let us close the subsection with an important remark: if instead of soap-film-like minimal surfaces we had considered *boundaries of sets of minimal perimeter* (resp. *integer rectifiable area-minimizing* 2*-currents*) in $\mathbb{R}^3$, then the only possible minimal cones would have been the planes. In particular, De Giorgi's theorem implies that such notions of area-minimizing surfaces in $\mathbb{R}^3$ are analytic unconditionally.

[5]Indeed, since a_x is monotone and $a_x(0^+,S)$ must take one of the three values in (3.2), the assumption $a_x(r,S) < \ell_2$ implies that the value at 0^+ can only be π. Hence for small enough scales, $S^{x,r}$ will be arbitrarily close to a plane, and then De Giorgi's theorem implies its analyticity.

On the other hand, the same strategy—described here for surfaces in $\mathbb{R}^3$—works for hypersurfaces in $\mathbb{R}^n$. In that case, Almgren [2] for $n = 4$ and Simons [51] for $5 \le n \le 7$ proved that if $\mathcal{C}$ is an area-minimizing (hyper-)cone in $\mathbb{R}^n$ and $\mathcal{C} \cap \mathbb{S}^{n-1}$ is smooth, then $\mathcal{C}$ must be a hyperplane. This classification result is important because one can deduce from it that *boundaries of sets of minimal perimeter (resp. integer rectifiable area-minimizing* $(n-1)$*-currents) are analytic in dimensions* $n \le 7$. This dimension 7 is sharp since Simons's cone $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$ is an example of area-minimizing surface (with respect to the two previous notions) in $\mathbb{R}^8$, as shown in [7].

3.5. Federer's dimension reduction principle and partial regularity theorems

In order to complete our heuristic overview of the classical regularity theory for area-minimizing surfaces in $\mathbb{R}^3$, a last key idea is missing: the dimension reduction principle. The first observation we need to make is that the map $\mathfrak{m} : S \to \{\ell_1, \ell_2, \ell_3\}$ defined as

$$\mathfrak{m}(x) := a_x(0^+, S) = \inf\{a_x(r; S) : r > 0\}$$

will be upper-semicontinuous, since it is an infimum of continuous functions. As a consequence, the set of tetrahedron type singular points $X_3 := \{\mathfrak{m} = \ell_3\}$ is closed.

In order to glimpse how Federer's dimension reduction argument works, let us show that X_3 is discrete. Indeed, assume by contradiction that $x_k \in X_3$ converges to $x_\infty \in \mathbb{R}^n \setminus (\bigcup_k \{x_k\} \cup \Gamma)$. Since X_3 is closed, x_∞ belongs to X_3.

Now given $\varepsilon > 0$ arbitrarily small, we can choose $r_\varepsilon > 0$ (depending on x_∞) such that $0 \le a_{x_\infty}(r_\varepsilon, S) - \ell_3 < \varepsilon/2$. On the other hand, since $a_x(r_\varepsilon, S)$ is continuous in x, there exist $\varrho_\varepsilon \in (0, r_\varepsilon)$ such that $0 \le a_x(r_\varepsilon; S) - \ell_3 < \varepsilon$ for all $x \in X_3 \cap B_{\varrho_\varepsilon}(x_\infty)$. Since $x_k \to x_\infty$, for k sufficiently large, we will have $r_k := |x_k - x_\infty| < \varrho_\varepsilon$.

Let us now zoom in: consider $S_k^* := S^{x_\infty, r_k}$ and define $x_k^* := (x_k - x_\infty)/r_k$. Note that, by definition, x_k^* belongs to $\mathbb{S}^2$. By scaling, we have $a_0(1; S_k^*) = a_{x_\infty}(r_k; S)$ and $a_{x_k^*}(1; S_k^*,) = a_{x_k}(r_k; S)$. Hence, by definition of ϱ_ε,

$$\ell_3 = a_0(0^+; S_k^*) \le a_0(S_k^*, 1) = a_{x_\infty}(r_k; S) < a_{x_\infty}(\varrho_\varepsilon; S) \le \ell_3 - \varepsilon$$

and

$$\ell_3 = a_{x_k^*}(0^+; S_k^*) \le a_{x_k^*}(1; S_k^*) = a_{x_k}(r_k; S) < a_{x_\infty}(\varrho_\varepsilon; S) \le \ell_3 - \varepsilon.$$

Hence choosing ε sufficiently small (and k sufficiently large), we find that

- S_k^* will be arbitrarily close to a cone of tetrahedron type (centered at 0),
- S_k^* will be arbitrarily close to a cone about the point $x_k^* \in \mathbb{S}^2$.

This gives an obvious contradiction since the cone of tetrahedron type is clearly not a cone about any of its points in $\mathbb{S}^2$.

A refined version of the same type of argument allows one to show that singular points of Y type have Hausdorff dimension[6] at most one.[7] This type of argument is often called Federer's dimension reduction and works in several contexts where "zoomed-in objects" have some conical structure. The basic principle can be summarized as follows: if $X \subset \mathbb{R}^n$ is at the same time a cone about 0 and about another point $x^* \neq 0$, then X must be translation invariant in the direction x^* (since $tX = X$ and $t(X - x^*) = X - x^*$ for all $t > 0$ imply that $X - (t-1)x^* = X$).

4. Stefan's problem and the obstacle problem during 1970s–2000s

4.1. Duvaut's transformation

From the XIX century formulation of the Stefan problem explained in Section 2.2, it was not even clear if the problem was well posed. A key development was obtained in 1973 by Duvaut [19], who revealed a hidden convex structure in the problem: recall that θ denotes the temperature and consider the transformation

$$u(x,t) := \int_0^t \theta(x,\tau)\, d\tau.$$

Duvaut showed that the new function

$$u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$$

satisfies

$$\begin{aligned} \partial_t u - \Delta u &= -\chi_{\{u>0\}}, \\ u &\geq 0, \\ \partial_t u &\geq 0, \end{aligned} \tag{4.1}$$

where χ_A denotes the characteristic function of the set A.

By the strong maximum principle, if u is Duvaut's transformation of a temperature solving the Stefan problem, then it also satisfies the strict monotonicity property

$$\partial_t u > 0 \quad \text{in } \{u > 0\}. \tag{4.2}$$

This seemingly qualitative property was never used in the regularity theory developed in the 1970s. Still, we state it here because it plays an important role in the recent results.

[6]We recall that a subset $X \subset \mathbb{R}^n$ is said to have Hausdorff dimension $\beta \in [0,n]$ if for all $\beta' > \beta$ and for all $\delta > 0$ there exist countably many balls $B_{r_i} z_i$ covering X such that $\sum_i (r_i)^\beta < \delta$. One can easily check from this definition that the Hausdorff dimension of an m-plane in $\mathbb{R}^n$ is m.

[7]Actually, Y points form analytic curves by the deep results in [34, 53].

Since we can easily recover θ from u by computing its time derivative, we see that (4.1) is an equivalent formulation of the Stefan problem. The new formulation is useful because (4.1) enjoys a convex structure: it is the L^2-*gradient flow*[8] of the convex functional

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \max(0, u) \right) dx. \tag{4.3}$$

As a consequence, questions such as the well-posedness of solutions become much simpler in the new formulation.

4.2. Stefan's problem as a parabolic obstacle problem

Let us notice that stationary (constant-in-time) solutions of (4.1) satisfy exactly the obstacle problem (2.6) in particular case $-\Delta\psi \equiv 1$, that is

$$\begin{aligned} \Delta u &= \chi_{\{u>0\}}, \\ u &\geq 0. \end{aligned} \tag{4.4}$$

In this respect, (4.1) is a parabolic version of the obstacle problem (4.4). Solutions to (4.4) are critical points (and hence minimizers, since the functional is convex) of $J(u)$. Note that such constant-in-time solutions of (4.1) are never solutions of the Stefan problem (never arise as Duvaut transforms of some temperature) since, for instance, they do not satisfy (4.2). Still, understanding the regularity of the free boundaries for the obstacle problem (4.4) is a logic first step before dealing with time-dependent solutions.

4.3. Obstructions to regularity of the free boundary: Schaeffer's examples (1977)

To study (4.4), a first thing one might try is to construct some explicit solutions. In most simple cases, the obtained free boundaries are smooth.

However, it is possible to find singular free boundaries even in two dimensions, as done by Schaeffer in [48]. He used complex variables to construct solutions of (4.4) in $\mathbb{R}^2$ in which the free boundary has a cusp represented by the curve (Figure 5, left)

$$x_2 = \pm x_1^{2k+\frac{1}{2}}, \quad 0 \leq x_1 \leq 1,$$

[8]The solution u satisfies, for infinitesimal $\tau > 0$,

$$u(\cdot, t+\tau) = \arg\min \left(J(v) + \frac{1}{2\tau} \|v - u(\cdot, t)\|^2_{L^2(\Omega)} \right) + o(\tau),$$

where the minimum is among functions $v : \Omega \to \mathbb{R}^n$ satisfying the prescribed boundary condition for u at time $t+\tau$, i.e., $v = u(\cdot, t+\tau)$ on $\partial\Omega$.

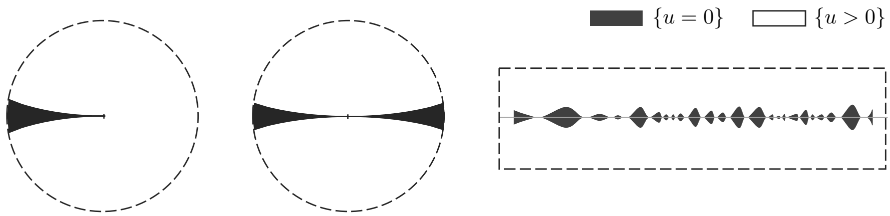

Figure 5. Schaeffer's examples of singular free boundaries.

where $k \in \mathbb{N}$. In this family of examples, the set $\{u > 0\}$ is actually the image of $\{|z| \le 1, \operatorname{Im} z > 0\}$ under the conformal mapping $f(z) = z^2 + i\, z^{4k+1}$, and u satisfies near the origin

$$u(z) \approx \frac{x_2^2}{2} + c_k \operatorname{Im}\left(z^{2k+\frac{3}{2}}\right) + \cdots ,$$

where $z = x_1 + ix_2$.

Another family type of singularities (two-sided cusps) was also constructed by Schaeffer (Figure 5, center). In this case, the free boundary is represented by the curves

$$x_2 = \pm |x_1|^{2k}, \quad -1 \le x_1 \le 1.$$

In the case of general smooth concave obstacles ψ, Schaeffer noticed that solutions to (2.6) may even have infinitely many cusps (Figure 5, right).

4.4. Caffarelli's breakthrough (1977)

It was not until 1977, with the groundbreaking paper of Caffarelli [9] (and with the paper [33]), that the "modern" regularity theory for (4.1) and (4.4) was initiated. Since, as explained before, (4.4) is a particular case of (4.1)—that of constant-in-time solutions—Caffarelli's results described next apply at the same time to both the obstacle problem and Stefan's problem.

The approach of Caffarelli to the regularity of free boundaries of (4.1)—or of (4.4)—has some rough similarities with the regularity theory of area-minimizing hypersurfaces described in Section 3. In Caffarelli's regularity theory (as in area-minimizing surfaces), *blow-ups* (limiting zoomed-in objects) are central actors. Informally speaking, Caffarelli looks at points on the free boundary through the microscope, and infers "macroscopic properties" of the free boundary from the "microscopic" ones.

For (4.1) the natural scaling of the equation suggests considering, for given $(x_\circ, t_\circ) \in \partial\{u > 0\}$ and $r > 0$,

$$u^{x_\circ, t_\circ, r}(x,t) := \frac{1}{r^2} u(x_\circ + rx, t_\circ + r^2 t).$$

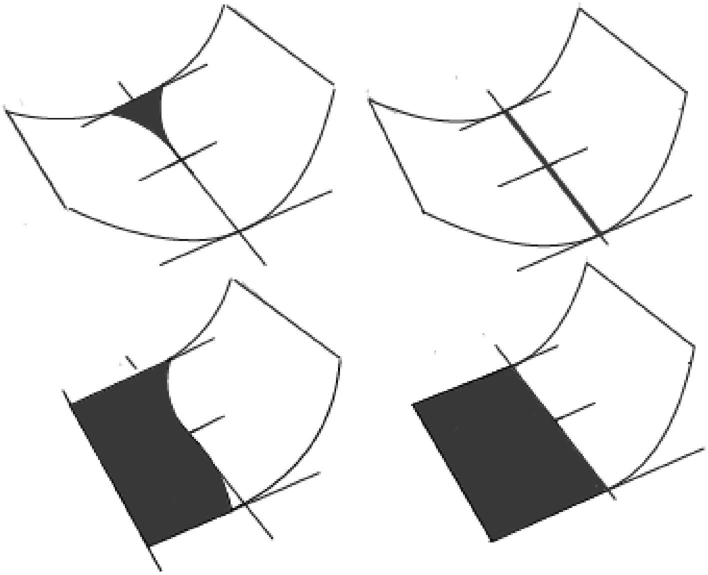

Figure 6. Illustration of the dichotomy in Caffarelli's theorem.

It is easy to see that $u^{x_\circ,t_\circ,r}$ is again a solution of (4.1). *Blow-ups* are defined as accumulation points as $r \downarrow 0$ of $u^{x_\circ,t_\circ,r}$.

The main result from Caffarelli [9], combined with the fundamental analyticity result for C^1 free boundaries of Kinderlehrer and Nirenberg [33], is stated next. (See Figure 6 for an illustration of the result.)

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \times (0,T) \to \mathbb{R}$ be a solution of* (4.1). *For every $(x_\circ, t_\circ)$ belonging to the free boundary $\partial\{u > 0\}$, one of the following two alternatives holds:*

(a) $u^{x_\circ,t_\circ,r} \to \frac{1}{2}(\max(0, e \cdot x))^2$ *as $r \downarrow 0$, for some $e \in \mathbb{S}^{n-1}$; and the free boundary is a (moving) smooth embedded $(n-1)$-surface near $(x_\circ, t_\circ)$;*

(b) $u^{x_\circ,t_\circ,r_k} \to \frac{1}{2} x \cdot Ax$ *for some $r_k \downarrow 0$ and some nonnegative definite matrix A with* $\operatorname{trace}(A) = 1$*; and the free boundary has a cusp-like singularity at $(x_\circ, t_\circ)$.*

Besides the idea of considering blow-ups, the methods used by Caffarelli to prove this result were rather different than those employed in area-minimizing surfaces. For instance, in Caffarelli's approach, a convexity property of blow-ups is crucially used in its classification, and his methods "based on the maximum principle" can be applied to more general non-variational problems, such as fully nonlinear obstacle problems. For a self-contained overview of Caffarelli's 1977 proof, we refer the reader to [23].

4.5. Weiss' epiperimetric inequality approach (1999)

In the paper [54], Weiss introduced a new monotonicity formula for the obstacle problem, which is in many respects analogous to the Federer–Fleming monotonicity formula for area-minimizing surfaces. Given a solution u of the obstacle problem (4.4), and $x_\circ \in \partial\{u > 0\}$, he introduced the *adjusted energy* (recall that the functional J was defined in (4.3))

$$W_{x_\circ}(r, u) = \frac{1}{r^{n+2}} J\big(u; B_r(x_\circ)\big) - \frac{1}{r^{n+3}} \int_{\partial B_r} u.$$

He proved that $W_{x_\circ}(r, u)$ is monotone nondecreasing in r, and constant if and only if u is 2-homogeneous (i.e., $u(tx) = t^2 u(x)$ for all $t > 0$).

Similarly to what happens with area-minimizing surfaces, this monotonicity formula follows from comparing the energy of the solution J of u in B_r with the energy of its natural competitor (defined for given $t \in (0, 1)$)

$$\tilde{u}(x) := \begin{cases} \frac{|x|^2}{r^2} u\big(\frac{rx}{|x|}\big) & x \in B_r \setminus B_{tr}, \\ t^2 u(tx) & x \in B_{tr}. \end{cases}$$

Using Weiss' monotonicity formula, one can show that blow-ups in the obstacle problem must be 2-homogeneous. Similarly, as we explained with $a_{x_\circ}(r, S)$ in the case of Plateau's problem, $W_{x_\circ}(0^+, u)$ can only take two different values: $\frac{|B_1|}{8(n+2)}$ and $\frac{|B_1|}{4(n+2)}$. The lowest possible value defines regular points, while the higher value is attained at singular points.

The paper [54] was the first to introduce methods for the obstacle problem which had a very strong parallelism with those for area-minimizing surfaces (e.g., an "epiperimetric inequality"), reinforcing the connection between the two theories.

4.6. First regularity results on the singular set and open questions

After the results of Caffarelli [9], a natural question was as follows: what else can be said about the singular set?

Besides some first results in two dimensions [11], there was no real progress on this question until 1991, when Sakai [44, 45] obtained a very precise description of singularities for the obstacle problem (4.4) in $\mathbb{R}^2$. He essentially proved that the cusps of Schaeffer (and small analytic perturbations of them) are the only ones which may appear for (4.4) in $\mathbb{R}^2$.

In dimensions $n \geq 3$, where complex analysis is of no use, the first results on the singular set were established again by Caffarelli in 1998 [10] (using the Alt–Caffarelli–Friedmann monotonicity formula) and by Monneau in 2003 [37] (using a new monotonicity formula based on the Weiss one). They established that a solution

u of (4.4) in $\mathbb{R}^n$ ($n \geq 3$) must satisfy at any singular point $x_\circ$,

$$u(x_\circ + x) = \frac{1}{2} x \cdot Ax + o\big(|x|^2\big). \tag{4.5}$$

As a consequence of (4.5) and Whitney's extension theorem, one obtains that the singular set enjoys spatial C^1-regularity, in the sense that they can be covered by $(n-1)$-manifolds of class C^1. Still, this result seems rather weak in the sense that it does not prevent the singular set from being as large as the regular part of the free boundary. In this direction, the following conjecture holds.

Conjecture 3 (Schaeffer [47]). *Generically, solutions of the obstacle problem have smooth free boundaries.*

In other words, the conjecture states that, generically, the free boundary has *no* singular points. Here the word "generically" must be interpreted as "for most boundary values". Until very recently (see Section 6), Conjecture 3 was only known to hold in the plane $\mathbb{R}^2$, a result of Monneau [37].

In the evolutionary case, the "parabolic analog" of Monneau's monotonicity formula [37] for solutions to (4.1) and its consequences were investigated in [6, 36].

5. Almgren's problem and the thin obstacle problem during 1970s–2000s

5.1. Branching singularities of holomorphic curves

As explained in Section 2.5, holomorphic curves such as $S_1 := \{w^2 = z^3\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$, where $w = x_3 + ix_4$ and $z = x_1 + ix_2$, are examples of area-minimizing 2-surfaces in $\mathbb{R}^4$. In the case of S_1, we can write x_3 and x_4 as "two valued functions" of x_1 and x_2: since $\zeta = \sqrt{z^3}$ involves a complex square root, there are two possible values of (x_3, x_4) for each pair (x_1, x_2).

Let us consider two further examples: $S_2 := \{(w - z^2)^2 = z^5\}$ and $S_3 := \{(w - z^2 + z^3)^3 = z^{11}\}$. Both have branching singularities at 0, but they look even more complicated than the case of S_1. In order to understand the singularity of S_2, we need to proceed as follows: we first consider the change of variables $\zeta(z, w) = w - z^2$ and notice that the coordinates (z, ζ) are diffeomorphic to (z, w) near the origin. In the new coordinates, we have $S_2 := \{\zeta^2 = z^5\}$, so we see that the singularity has again two branches (from the complex square root involved in $\zeta = \sqrt{z^5}$). Only after we rectify the coordinates, we can clearly see the structure of the branching singularity of S_2. Something similar happens for S_3. In that case, the new coordinates would be $\zeta(w, z) = w - z^2 + z^3$ and the model singularity $\{\zeta^3 = z^{11}\}$, with three branches from $\sqrt[3]{\cdot}$.

5.2. Almgren's regularity theorem

In [4], Almgren established the following theorem.

Theorem 4. *Let S be an oriented area-minimizing surface*[9] *of dimension $n \geq 2$ in $\mathbb{R}^{m+k}$, where $k \geq 2$.*

Then, S is an analytic submanifold in $\mathbb{R}^{m+k} \setminus \Gamma$, where Γ denotes the boundary[10] *of S, with the exception of a closed set* $\operatorname{Sing}(S)$ *of dimension at most $m-2$ (discrete if $m = 2$).*

The dimensional estimate for the singular set is optimal, as shown by holomorphic curves with branching points.

5.3. Q-valued harmonic functions, frequency formula

In Section 5.1, we saw examples of branching singularities in explicit holomorphic curves. Let us explain next in what sense general oriented area-minimizing surfaces resemble holomorphic curves.

Suppose that $S \subset \mathbb{R}^4$ is any area-minimizing oriented 2-surface[11] and that 0 is a non-smoothness point on it (e.g., an integer rectifiable area-minimizing current). Similarly, as in Section 3.3, $a_0(r; S)$ is monotone nondecreasing and the zoomed-in surfaces $S^{0,r}$ converge towards a cone $\mathcal{C}$, now a 2-surface in $\mathbb{R}^4$. It is not difficult to show that planes are the only possible area-minimizing oriented 2-cones in $\mathbb{R}^4$. The difficulty now is that $\mathcal{C}$ could be a plane with "multiplicity two or higher"; in other words, we could have $a_0(0^+; S) = Q\pi$, for some $Q \geq 2$ (as it happens in branching singularities of holomorphic curves). Note that this cannot happen for codimension 1 surfaces, thanks to De Giorgi's theorem.

What can one do at those "multiplicity points"? Assume first that, up to a rotation, $S^{0,r}$ is close to the plane $\{x_3 = x_4 = 0\}$. If $S^{0,r}$ happened to be (locally near 0) a very flat multiplicity one graph $x_i = \varepsilon f_i(x_3, x_4)$, $i = 1, 2$, then its surface area would be given by an integral of the type

$$\int \sqrt{\left(1 + \varepsilon^2 |\nabla f_1|^2\right)\left(1 + \varepsilon^2 |\nabla f_2|^2\right) - \varepsilon^4 (\nabla f_1 \cdot \nabla f_2)^2}\, dx_1\, dx_2$$

$$\approx \pi + \frac{\varepsilon^2}{2} \int \left(|\nabla f_1|^2 + |\nabla f_2|^2\right) dx_1\, dx_2.$$

Hence, both f_1 and f_2 would need to be approximate minimizers of the Dirichlet energy! Something similar happens when $\boldsymbol{f} = (f_1, f_2)$ is not a single-valued

[9] Rigorously, assume that S is an integer rectifiable area-minimizing current.
[10] More precisely, Γ is the support of the boundary of the current S.
[11] Integer rectifiable current.

map from $\mathbb{R}^2 \to \mathbb{R}^2$, but a multiple-valued one. More precisely, the pair of functions $\boldsymbol{f}(x_1, x_2)$ do not "return" a point in $\mathbb{R}^2$, but a Q-tuple of them: all the pairs $(x_3/\varepsilon, x_4/\varepsilon)$ for which (x_1, x_2, x_3, x_4) belongs to S. Still in this case, the area would be given by an analogous expression as above. And, as before, the fact that the S is area-minimizing should imply that, as $\varepsilon \downarrow 0$, the multiple-valued functions are approximate minimizers of the Dirichlet energy—appropriately generalized to the context of multiple-valued functions.

The Q-valued Dirichlet minimizers $\boldsymbol{f} : \mathbb{R}^m \to (\mathbb{R}^k)^Q/\sim$, where $\sim$ identifies Q-tuples of points in $\mathbb{R}^k$ which are equal up to reordering, are a main object in Almgren's theory. Interesting minimizers such as $x_3 + ix_4 = \sqrt{(x_1 + ix_2)^3}$ have branched structure, where the multiple graphs are "knotted" to one another. In Almgren's theory, the singularities of minimal surfaces are shown to correspond to the singularities of multiple-valued minimizers of the Dirichlet energy, also called *multiple-valued harmonic functions*.

A crucial ingredient in Almgren's theory is the *frequency formula*: if $\boldsymbol{f} : \mathbb{R}^m \to (\mathbb{R}^k)^Q/\sim$ $\boldsymbol{f}(x_\circ) = \mathbf{0}$ is Dirichlet-minimizer, then the dimensionless quantity

$$\phi_{x_\circ}(r; \boldsymbol{f}) := \frac{r \int_{B_r(x_\circ)} |\nabla \boldsymbol{f}|^2}{\int_{\partial B_r(x_\circ)} \boldsymbol{f}^2} \tag{5.1}$$

is monotone nondecreasing in r. Moreover, $\phi_{x_\circ}(r; \boldsymbol{f}) \equiv \lambda$ for some $\lambda \geq 0$ ($\phi_{x_\circ}$ is constant in r) if and only if $\boldsymbol{f}(x_\circ + \cdot)$ is λ-homogeneous. As a consequence of the frequency formula, whenever $\boldsymbol{f}$ is a multiple-valued harmonic function and $\boldsymbol{f}(x_\circ) = \mathbf{0}$, "blow-up" sequences

$$\boldsymbol{f}^{x_\circ, r_k} := \frac{\boldsymbol{f}(x_\circ + r_k \cdot)}{\left(r^{1-m} \int_{\partial B_r} \boldsymbol{f}^2\right)^{1/2}}, \quad r_k \downarrow 0,$$

converge (up to subsequences) towards some homogeneous multiple-valued harmonic function $\boldsymbol{f}^*$.

5.4. Dimension reduction and center manifold

With the frequency formula at hand, we can explain (roughly and naively) some other key ideas in the proof of Theorem 4, which will later have parallels in the obstacle problem and Stefan's problem. Assume that S is a minimal surface (current) of dimension m inside $\mathbb{R}^{m+k}$ that has a singular (or non-smoothness) point at 0. As discussed before, near 0, S will be well approximated by a multiple-valued Dirichlet minimizer $\boldsymbol{f} : \mathbb{R}^m \to (\mathbb{R}^k)^Q/\sim$, where $Q \in \mathbb{N}$ is given by $Q = a_0(0^+; S)/|B_1^n|$ (here $|B_1^m|$ denotes the m-dimensional volume of the unit ball of $\mathbb{R}^m$).

Let us only discuss for simplicity the case $m = k = 2$. In that case, we want to show that singular points are isolated. So, assume by contradiction that there was a sequence of singular points $x_k \to 0$ and let $r_k := |x_k|$ be their norms. Consider the blow-up sequence $\boldsymbol{f}^{0,r_k}$, which will converge (up to a subsequence) towards some λ-homogenous (possibly multiple-valued) function $\boldsymbol{f}^*$, where $\lambda = \phi_0(0^+; \boldsymbol{f})$. Now $\boldsymbol{f}^* : \mathbb{R}^2 \to (\mathbb{R}^k)^Q/\sim$ must be of the form $\boldsymbol{f}(r\cos\theta, r\sin\theta) = r^\lambda \boldsymbol{g}(\theta)$, where $\boldsymbol{g}$ is some Q-valued curve. The fact that $\boldsymbol{f}$ is harmonic (Dirichlet minimizer) imposes very strong restrictions on $\boldsymbol{g}$ (e.g., locally each branch $g : \mathbb{R} \to \mathbb{R}^k$ must satisfy the ODE $g'' = \lambda^2 g$). This strong rigidity helps in classifying all possibilities for $\boldsymbol{f}^*$, and one can show that it must be exactly given by a holomorphic curve, e.g.,

$$x_3 + ix_4 = (x_1 + ix_2)^3 \quad \text{or} \quad (x_3 + ix_4)^Q = (x_1 + ix_2)^{Q+1}.$$

Now, if $\boldsymbol{f}^*$ has a (multiplicity Q) branching singularity at 0, then—since we now know that $\boldsymbol{f}^*$ is a homogeneous holomorphic curve—it must be isolated. Hence $\boldsymbol{f}^*$ must be smooth away from 0. This fact—thanks to Allard's version of Theorem 1, which only applies to multiplicity one points $x \in S$ (i.e., $a_x(0^+; S)/\pi = 1$)—implies that S will not have any other singularity in a (sufficiently small) neighborhood of 0.

Still, there is the possibility that—as it happens in the examples S_2 and S_3 given from Section 5.1—, $\boldsymbol{f}^*$ may be a harmonic polynomial. In such cases, the branching singularity will only show itself after we "rectify" $\boldsymbol{f}$, subtracting from it the (single-valued) harmonic polynomial P, which "best fits" $\boldsymbol{f}$ near 0. This idea leads to the notion of *center manifold*: in order to see the branching structure, we must consider the deviation of S, not from the tangent plane, but from the "best fitting" smooth single-valued minimal graph near 0. The frequency function on $\boldsymbol{f} - P$—more precisely $\phi_0(r; \boldsymbol{f} - P)$—is also monotone, and $(\boldsymbol{f} - P)(r_k\,\cdot)$ divided by its L^2 norm on ∂B_1 converges to some new homogeneous blow-up $\boldsymbol{f}^*$. Now, by construction f^* cannot be single-valued, so it must have a branching singularity.

In order to prove the result in higher dimensions, we need an appropriate variant of Federer's dimension reduction principle (previously discussed in the context of area-minimizing surfaces in $\mathbb{R}^3$). The dimension reduction is based on the following simple property: if a function $f : \mathbb{R}^m \to \mathbb{R}^k$ is at the same time λ-homogeneous with respect to 0 and μ-homogeneous with respect to $x_\circ \neq 0$, then necessarily $\lambda = \mu$ and f is translation invariant in the direction $x_\circ$. A similar property holds for multiple-valued functions $\boldsymbol{f}$.

5.5. Almgren's methods applied to Signorini's problem

In [5], the authors devised how to apply the methods introduced by Almgren in the context of area-minimizing currents to Signorini's problem. This leads to a very important progress, as described next.

In order to get rid of superfluous technical details, instead of (2.2), the authors consider the "cleaner" zero obstacle problem: Let $n \geq 2$, and consider $u : B_1 \to \mathbb{R}$

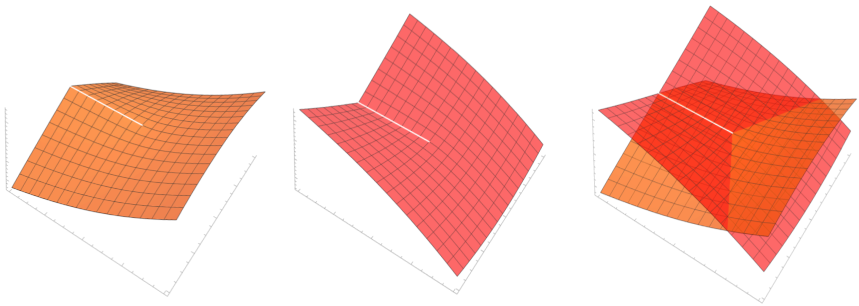

Figure 7. From Signorini's problem to "2-valued harmonic functions".

($B_1 \subset \mathbb{R}^n$ is the unit ball) satisfying

$$\int_{B_1} |\nabla v|^2 \geq \int_{B_1} |\nabla u|^2 \text{ for all } v \in H^1(B_1) \text{ with } v \geq 0 \text{ on } \{x_n = 0\} \text{ and } v = u \text{ on } \partial B_1. \quad (5.2)$$

A key contribution of [5] was to show that functions satisfying (5.2) behave essentially identically to Dirichlet-minimizing 2-valued functions. As a matter of fact, for $n = 2$, explicit examples of minimizers to (5.2) can be obtained, and they are conspicuously related to the examples of branching singularities of holomorphic curves discussed before. For example, a very important explicit solution of (5.2) for $n = 2$ is $u(x_1, x_2) = \operatorname{Re} \sqrt{(x_1 + ix_2)^3}$, where now $\sqrt{\cdot}$ selects only the principal branch. This is clearly related to the branching singularity $(x_3 + ix_4)^2 = (x_1 + ix_2)^3$.

Heuristically, if u is a minimizer of (5.2), then the 2-valued function $(u(x), -u(x))$ can be thought of as "2-valued harmonic function" (see Figure 7).

Among the multiple analogies, the frequency formula $\phi_{x_\circ}(r, u)$—defined exactly as in (5.1) replacing f with u—is also monotone nondecreasing for every point such $x_\circ \in \{x_n = 0\} \cap \{u = 0\}$. The main contribution [5] was to show that if $\lambda := \phi_{x_\circ}(r, u) < 2$ at some *free boundary point* $x_\circ \in \partial\{u > 0\} \cap \{x_n = 0\}$, then either $\lambda \leq 3/2$ or $\lambda \geq 2$. Moreover, they proved that the set of points where the first alternative holds is open and is an $(n-2)$-manifold of class C^1 inside $\{x_n = 0\}$.

6. The singular set in the obstacle problem (2017–2021)

In 2015, more than 30 years after Caffarelli's breakthrough [9] for the obstacle problem, the following important questions remained essentially open in dimensions $n \geq 3$:

- Can we obtain some precise description of singularities in the obstacle problem?
- Is the singular set "small" in some sense? How small?

As we discussed before, satisfactory answers to these questions had been only obtained (through complex variable methods) in dimension $n = 2$ by Sakai [44, 45]. Sakai's methods did not work in higher dimensions, and improving Caffarelli's result required new ideas.

6.1. A finer analysis of the singular set

The first new result in this direction for $n \geq 3$ was established by Colombo, Spolaor, and Velichkov in [12]. By refining the methods of Weiss [54], they proved that at every singular point $x_\circ$, the expansion

$$u(x_\circ + x) = \frac{1}{2} x \cdot Ax + \omega(x). \tag{6.1}$$

holds with a quantitative logarithmic estimate for the error $|\omega(x)| \leq C|x|^2(\log|x|)^{-\gamma}$, where $\gamma > 0$. Caffarelli in [10] had obtained a qualitative control $|\omega(x)| \leq o(|x|^2)$ using the Alt–Caffarelli–Friedmann monotonicity formula—a different proof of the same qualitative estimate was given later in [37]. Sakai had found in [44,45] the (optimal) rate $|\omega(x)| \leq C|x|^3$ in dimensions $n = 2$. In the proofs of [12], one can glimpse some delicate obstructions to obtaining such a strong result in dimensions $n \geq 3$, although it was not clear if they were only of technical nature (no counterexample was known).

Independently and with different methods, Figalli and the author proved in [27] the following:

Theorem 5 ([27]). *Let u be a solution of the obstacle problem (4.4) in a ball of $\mathbb{R}^n$. For all singular points outside some "anomalous" (relatively open) set of Hausdorff dimension $\leq n-3$, (6.1) holds with $|\omega(x)| \leq C|x|^3$.*

Moreover, there exist examples in $\mathbb{R}^3$ of isolated singular points for which

$$\big|\omega(x)\big| \gg |x|^{2+\varepsilon} \quad \text{as } |x| \to 0 \text{ for all } \varepsilon > 0.$$

The previous theorem suggests that one might be able to give a much more precise description of the solutions than Caffarelli's near "most" singular points. However, not for all of them: the existence, already in $\mathbb{R}^3$, *anomalous* singular points for which $|\omega(x)| \gg |x|^{2+\varepsilon}$ for all $\varepsilon > 0$ is to be kept in mind as a warning of the arduousness of the problem.

The methods introduced in [27] are strongly connected with Almgren's ones for minimal currents. The link between the two (a priori unrelated) problems, found in [27], is as follows. Let u be a solution of the obstacle problem (4.4) in $B_1 \subset \mathbb{R}^n$ with a singular point at 0. In other words, assume that (6.1) holds at $x_\circ = 0$ with $\omega(x) = o(|x|^2)$. We then consider $w(x) = u(x) - \frac{1}{2}x \cdot Ax$. In [27], it was found that (surprisingly!) $\phi_0(r; w)$ is monotone increasing in r, where ϕ_0 is, as before, Almgren's frequency formula. This property allows one to study the so-called *second blow-ups*, namely accumulation of points of the type

$$q(x) = \lim_{r_k \to 0} \frac{w(rx)}{\|w(r_k \cdot)\|_{L^2(\partial B_1)}}.$$

Thanks to the monotonicity of ϕ on w, such second blow-ups q are λ-homogeneous—that is $q(tx) = t^\lambda q(x)$ for all $t \geq 0$—where $\lambda = \phi_0(0^+; w)$. Moreover, in [27], it is found that, outside of an $n-3$ dimensional set of singular points, the second blow-ups have homogeneity $\lambda \geq 3$ and are either harmonic or solutions of the thin obstacle problem (5.2). This allows for a full classification of possible second blow-ups in two dimensions, and in higher dimensions, allows us to perform dimension reduction arguments à la Federer based on the frequency, similarly to Almgren's work for area-minimizing currents in codimensions ≥ 2.

Another insightful result from [27] is that, for all singular points outside some $(n-2)$-dimensional set we have, after rotation, the improved expansion

$$u(x_\circ + x) = \frac{1}{2}x_n^2 + x_n Q(x) + o\big(|x|^3\big), \tag{6.2}$$

where Q is some quadratic polynomial satisfying $\Delta(x_n Q) \equiv 0$. By analogy with Almgren's center manifold, this invites to subtract the polynomial $x_n Q$ in order to investigate higher order expansions (this turned out to be a quite delicate task, and the missing tools in order to perform it were only developed later in [25, 29]).

6.2. Generic regularity: Schaeffer's conjecture in low dimensions

Building on the methods of [27], we could recently obtain a positive answer to (Schaeffer's) Conjecture 3 in low dimensions:

Theorem 6 ([25]). *Conjecture 3 holds true in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

More precisely, we can consider 1-parameter monotone (and continuous) families of boundary data $g : \partial\Omega \times (0, 1) \to \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, satisfying $g(x; \tau') - g(x; \tau) \geq c(\tau' - \tau)$ for all $0 < \tau < \tau' < 1$. We let u^τ be the solution of (4.4) with boundary data $u^\tau = g(\cdot\,; \tau)$ on ∂B_1. The"generic regularity" question we want to understand can be phrased as follows: if we choose $\tau \in (0, 1)$ randomly (with a uniform distribution), will the free boundary of u^τ be analytic with probability one? We can answer positively this question in dimensions 3 and 4 (the positive answer in two dimensions had already been obtained by Monneau in [37] for $g(x; \tau) = g(x) + \tau$).

Our strategy towards this theorem is reminiscent of *Sard's theorem* in analysis. We aim to prove that the set of "singular values" $\tau \in (0, 1)$ has measure zero by improving, at most singular points, the order of approximation of certain polynomial expansions for u^τ. This is a delicate and long proof because the singular set needs to be split into several different subsets and, in each of them, the corresponding set of singular values has measure zero for a different reason.

In order to prove the conjecture in four dimensions, we need to consider the set of all points $x_\circ \in \Omega$, which are singular for some of the solutions u^τ in the family. We then show that, after to removing $(n-2)$-dimensional set, for all the other $x_\circ$, we have an expansion of the type

$$u^\tau(x_\circ + x) = \mathcal{P}(x) + O\big(|x|^5\big),$$

where $\tau = \tau(x_\circ)$ is the value of the parameter for which $x_\circ$ is singular. Here $\mathcal{P}$ is a polynomial of the form (in some orthonormal coordinates depending of $x_\circ$)

$$\mathcal{P}(x) := \frac{1}{2}\left(x_n + \sum_{\alpha=1}^{n-1} \frac{a_\alpha}{2} x_\alpha^2 + \frac{(\sum a_\alpha)}{6} x_n^2 + \sum_{\alpha=1}^{n-1}\left(a_\alpha^2 - \frac{a_\alpha(\sum a_\alpha)}{3}\right)\left(\frac{x_n^3}{12} - \frac{x_\alpha^2 x_n}{2}\right)\right)^2,$$

for some $a_\alpha \geq 0$ ($\alpha = 1, \ldots, n-1$). We call $\mathcal{P}$, the "Ansatz", and whose structure is found imposing $\Delta\mathcal{P} = 1 + O(|x|^3)$. In many respects, $\mathcal{P}$ plays an analogous role to Almgren's center manifold: also here the idea is that, only after subtracting a very smooth "tangent object", one is able to see branching-type patterns which can only occur on lower dimensional sets.

We then manage to obtain an approximate monotonicity of (a truncated version of) the frequency function ϕ_0 for the remainder

$$w := u^\tau(x_\circ + \cdot) - \mathcal{P},$$

and perform dimension reduction type of arguments à la Federer–Almgren. However, an interesting feature of the dimension reduction arguments in [25] (which is completely new with respect to Almgren's) is that we need to work not with one single solution but with an increasing family of them (which do not have any other link between them than the monotonicity). And the dimension bounds that we obtain for the union in τ of all "bad points" for the family $\{u^\tau\}_\tau$ are as precise as the estimate one single u^τ.

The existence of solutions with an $(n-3)$-dimensional set of "anomalous points" where the expansion is quadratic, and not better, prevents us from using the same kind of methods for Schaeffer's conjecture in dimensions 5 or higher.

6.3. C^∞ partial regularity

Building on the methods of [25] (and [26]), F. Franceschini and W. Zatoń obtained in [29] the following extremely detailed (and essentially optimal) result:

Theorem 7 ([29]). *Let u be a solution of the obstacle problem* (4.4) *in the unit ball of $\mathbb{R}^n$ and let Σ denote its singular set. There exists a closed set $\Sigma^\infty \subset \Sigma$ such that*

(i) $\dim_{\mathcal{H}}(\Sigma \setminus \Sigma^\infty) \le n-2$ *(countable, if $n = 2$);*

(ii) *locally, Σ^∞ is contained in one $(n-1)$-dimensional C^∞ manifold, and at every point $x \in \Sigma^\infty$ the solution u has a polynomial expansion of arbitrarily large order. Moreover, these are consistent from one point to another in the sense of Whitney's extension theorem.*

A key contribution from [29] was to show almost-optimal Lipchitz estimates (in terms of their L^2 norms in a small ball) for the differences $u(x_\circ + \cdot) - \mathcal{P}$, where $x_\circ$ is a singular point, and $\mathcal{P}$ is an Ansatz of arbitrarily large order (nonnegative polynomial satisfying $\Delta\mathcal{P} \approx 1$). Such Lipchitz estimates are needed to prove that Almgren's frequency formula on $w = u - \mathcal{P}$ is monotone. With the previous approach from [25], such estimates had necessarily errors of size $O(|x|^5)$, which was blocking the expansion at order 5. The smarter (and more natural, a posteriori) approach from [29] allows the authors to obtain similar Lipchitz estimates with an error of arbitrarily high order. As a consequence, they obtain a beautiful C^∞ partial regularity result for the singular set: something that seemed inconceivable only a few years ago.

7. The singular set in the Stefan problem (2019–2021)

After Caffarelli's 1977 breakthrough, a main question on the structure of the free boundaries in Stefan's problem remained open: how large may the singular set be? Very simple examples—such as a one-dimensional solution $u(x_3, t)$ for which the ice region is $\{|x_3| \le f(t)\}$ for some f decreasing—show that the singular set in Stefan's problem (in $\mathbb{R}^3$) may be as large as 2-dimensional; at least for some times. The regular part of the free boundary is a moving 2-surface in $\mathbb{R}^3$, so at such "bad" times, the singular set is as large as the regular part! However, in the examples, this may happen only for a very exceptional set of times. This suggests that the singular set should be "smaller" than the set of smooth points as a subset of *spacetime* $\mathbb{R}^3 \times \mathbb{R}$.

In order to measure the dimension of subsets of spacetime in Stefan's problem, it is natural to introduce a Hausdorff dimension associated to the "parabolic scaling" (which leaves the equation invariant). Namely, for a set $E \subset \mathbb{R}^n \times \mathbb{R}$, we write $\dim_{\mathrm{par}}(E) \le \beta$, when for all $\beta' > \beta$, E can be covered by countably many *parabolic cylinders* $B_{r_i}(x_i) \times (t_i - r_i^2, t_i)$, making $\sum_i r_i^{\beta'}$ arbitrarily small. Notice that, if we denote by $\dim_{\mathcal{H}}(E)$ the standard Hausdorff dimension of a set

$$E \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R},$$

then $\dim_{\mathcal{H}}(E) \le \dim_{\mathrm{par}}(E)$. On the other hand, the time axis has parabolic Hausdorff dimension 2, while it has standard Hausdorff dimension 1.

The only known dimensional bound on the singular set $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ for solutions to (4.1)-(4.2) in dimensions $n \geq 2$ was the following rather rough estimate: as a consequence of the results in [6, 9], at every singular point $(x_\circ, t_\circ)$, the qualitative expansion

$$u(x_\circ + x, t_\circ + t) = \frac{1}{2} x \cdot Ax + o\big(|x|^2 + |t|\big) \tag{7.1}$$

holds, where $A = A_{x_\circ, t_\circ}$ is a nonnegative definite matrix, satisfying $\mathrm{tr}(A) = 1$, which depends on $(x_\circ, t_\circ)$. As a consequence of (7.1), the set of singular points can be decomposed as $\Sigma = \bigcup_{m=0}^{n-1} \Sigma_m$, where

$$\Sigma_m := \big\{(x_\circ, t_\circ) \in \Sigma : \dim\big(\ker(A_{x_\circ, t_\circ})\big) = m\big\}, \quad m = 0, \dots, n-1.$$

Moreover, for each m, the set $\Sigma_m \cap \{t = t_\circ\}$ can be covered by a C^1 manifold of dimension m. Unfortunately, the previous expansion implies only $C^{1/2}$ regularity in time for the covering manifolds. As shown in [36], (7.1) also implies a (very rough) bound on the parabolic Hausdorff dimension of Σ:

$$\dim_{\mathrm{par}}(\Sigma) \leq n + \frac{1}{2}. \tag{7.2}$$

Since the parabolic dimension of the regular part of the free boundary has dimension $(n-1) + 2 = n + 1$, the previous bound shows that, in some weak sense, the singular set is smaller than the regular one. However, the bound (7.2) does not even rule out the existence of pathological solutions with singular points at every time (not even in two dimensions)!

7.1. Almgren meets Stefan

After the works [25, 27], it was very natural to apply the same kind of arguments to the Stefan's problem (4.1)-(4.2). Given a singular point $(x_\circ, t_\circ)$, let us consider

$$w(x) := u(x_\circ + x, t_\circ + x) - \frac{1}{2} x \cdot Ax.$$

In order to extend our methods from [27] to the parabolic setting, Poon's [39] parabolic version of Almgren's frequency formula plays an important role. Namely, denoting $G(x,t) = (4\pi t)^{-n/2} e^{\frac{|x|^2}{4t}}$ the time-reversed heat kernel, the functional

$$\phi(r, w) := \frac{r^2 \int_{\{t=-r^2\}} |\nabla w|^2 G \, dx}{\int_{\{t=-r^2\}} w^2 G \, dx},$$

can be shown to be monotone in r.[12]

[12] Actually, we need to employ a (new) suitable truncated version of ϕ (which we call ϕ^γ), and its monotonicity can be proved only up to exponentially small errors. But these are technical details.

Thanks to this fact, we can prove that

$$\frac{w(rx, r^2 t)}{\|w\|_r} \to q(x,t) \quad \text{as } r \to 0, \tag{7.3}$$

along subsequences and in compact subset of $\{t < 0\}$, where q is a parabolically homogeneous function: namely $q(rx, r^2 t) = r^\lambda q(x,t)$ for all $r > 0$, where $\lambda = \phi(0^+, w)$. In (7.3), we denote by $\|w\|_r$ the quantity $(\fint_{\mathcal{C}_r} w^2)^{1/2}$, which measures the "size" of w in the parabolic cylinder $\mathcal{C}_r := B_r \times (-r^2, 0)$.

We then show the following:

(i) If $(x_\circ, t_\circ) \in \Sigma_m$ with $m \leq n-2$, then the function q is *always* a *quadratic* caloric polynomial. This means that the expansion (7.1) cannot be improved at any of these points! To obtain an improved dimensional bound on Σ_m, we employ a barrier argument in the spirit of [25] to show that $u > 0$ in $B_r(x_\circ) \times (t_\circ + r^{2-\varepsilon}, \infty)$. In other words, a ball of radius r around one of these singular points will be completely occupied by water after increment of time of size $r^{2-\varepsilon}$. This gives

$$\dim_{\mathrm{par}}(\Sigma_m) \leq m, \quad 0 \leq m \leq n-2.$$

(ii) If $(x_\circ, t_\circ) \in \Sigma_{n-1}$, then q is a homogeneous solution of the parabolic thin obstacle problem. We denote by $\Sigma_{n-1}^{<3}$ the subset at which the homogeneity is less than 3.

(a) If $(0,0) \in \Sigma_{n-1}^{<3}$, we show that $\partial_t q \not\equiv 0$ and that q is *convex* in all directions that are tangential to $\{p_2 = 0\}$. This allows us to perform a dimension reduction that, combined with a barrier argument similar to that in (i), yields

$$\dim_{\mathrm{par}}(\Sigma_{n-1}^{<3}) \leq n-2.$$

(b) If $(0,0) \in \Sigma_{n-1} \setminus \Sigma_{n-1}^{<3}$, we show that q is always 3-homogeneous, hence

$$u(x_\circ + \cdot, t_\circ + \cdot) = \frac{1}{2} x \cdot Ax + O\big(|x|^3 + |t|^{3/2}\big). \tag{7.4}$$

This (and a barrier argument similar to the one before) implies that

$$\dim_{\mathrm{par}}(\Sigma_{n-1} \setminus \Sigma_{n-1}^{<3}) \leq n-1.$$

Combining these estimates in [26], we obtain the following theorem.

Theorem 8. *The singular set of solutions to* (4.1)-(4.2) *has a parabolic dimension* $n-1$.

Therefore, it is natural to ask ourselves if a similar result holds in the physical space $\mathbb{R}^3$ and, more in general, how often singular points may appear.

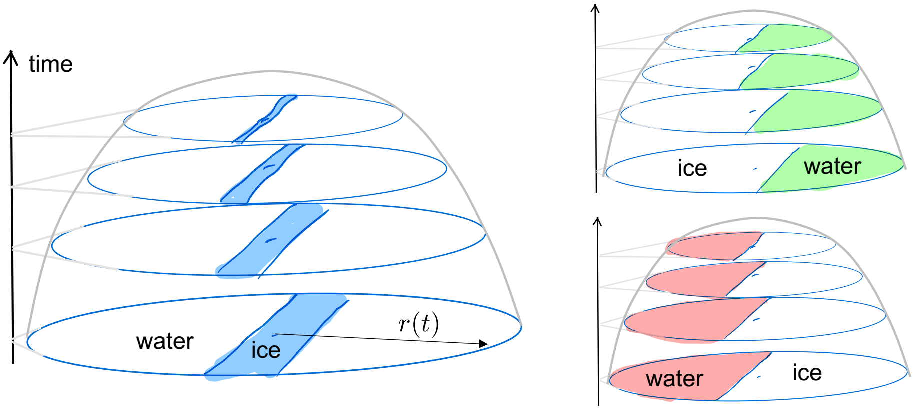

Figure 8. Inside of the shrinking ball $B_{r(t)}(x_\circ)$, the free boundary consists of two fronts, which evolve independently until they meet at time $t_\circ$.

7.2. Cubic expansions and their heuristic interpretation

With a bit of extra work, we can obtain a complete parabolic analog of the main result in [27]: for all singular points $(x_\circ, t_\circ)$ outside of a set of parabolic dimension $n-2$, the following expansion holds[13]

$$u(x_\circ + x, t_\circ + t) = \frac{1}{2}x_n^2 + a|x_n|\left(t + \frac{1}{6}x_n^2\right) + \left[\begin{matrix}\text{3-homogeneous caloric}\\ \text{polynomials}\end{matrix}\right] + o\big((|x| + |t|^{1/2})^3\big), \tag{7.5}$$

for some $a > 0$. The fact that this coefficient is positive, which turns out to be consequence of (4.2), is crucial.

Indeed, (7.5) implies that, if we look at the free boundary at time $t < t_\circ$ inside a ball of radius $\sqrt{t_\circ - t}$ centered at $x_\circ$, we will see two almost-parallel "independent" fronts which move one towards the other. More precisely, for $t < t_\circ$, we have

$$x_n = \pm 2a\,(t_\circ - t) + o(t_\circ - t) \quad \text{on } \partial\{u > 0\} \cap B_{\sqrt{t_\circ - t}}(x_\circ) \times \{t\}.$$

In this direction, let us (informally) define $r(t)$ as "the largest" radius for which the ice inside $B_\varrho(x_\circ)$ has two connected components for times before t—see Figure 8 (left). The expansion (7.5) actually implies $r(t) \gg \sqrt{t_\circ - t}$, as $t \uparrow t_\circ$.

Now, it is interesting to observe the following: suppose that $r(t)$ happened to stay bounded away from zero as $t \uparrow t_\circ$. Then, inside of some (small) parabolic cylinder $B_\varrho(x_\circ) \times (t_\circ - \varrho^2, t_\circ)$, the "positivity set" $\{u > 0\}$ would consist of exactly two

[13] After choosing an appropriate orthonormal frame depending on $(x_\circ, t_\circ)$.

connected components 1 and 2. We could then define $u^{(i)}$, $i = 1, 2$, as u multiplied by the characteristic function of the component i. Doing so, the two new functions $u^{(i)}$ would both solve (4.1)! Moreover, both functions would have a thick contact set $\{u^{(i)} = 0\}$, so the point $(x_\circ, t_\circ)$ would be regular for the two of them—see Figure 8 (right). Hence the free boundaries of $u^{(i)}$, $i = 1, 2$ (which correspond to the two fronts of u) would be smooth inside $B_\varrho(x_\circ)$ up to the final time $t = t_\circ$. At this final time, $t = t_\circ$, the two fronts $\{x_n = g^{(i)}(x')\}$ would be ordered, smooth, and tangent at least at $x_\circ$. Then, their tangency points in $B_\varrho(x_\circ)$ would necessarily be of one of the following two types.

- Infinite order tangency points of the two functions $g^{(i)}$: near such points the ice would be extremely thin, and hence they should become immediately surrounded by water after $t_\circ$.
- Lower dimensional tangency points: the subset of $g^{(1)} = g^{(2)}$ where the two functions disagree at some finite order k would be automatically contained in a smooth $(n-2)$-dimensional manifold (being contained in the transversal intersection of the graphs of certain $(k-1)$-derivatives of $g^i(x')$).

Of course, the difficulty is that we cannot expect $r(t)$ to be bounded away from zero at typical free boundary points. But it turns out that (with much extra effort) we can improve the bound $r(t) \gg \sqrt{t_\circ - t}$ to $r(t) \geq (t_\circ - t)^{\frac{1}{2+\beta}}$, for some tiny $\beta > 0$ (as $t \uparrow t_\circ$), at "most" singular points. This amounts to proving an expansion like (7.5) but with an error of size $O((|x| + |t|^{1/2})^{3+\beta})$. As we will see, such apparently small improvement is "breaking the parabolic scaling", and will allow us to obtain the same type of conclusions as if $r(t)$ stayed bounded away from zero! But such strong conclusions are not cheap to obtain: in order to improve (7.5) by a tiny positive β, we need to introduce completely new techniques. We need to go beyond Almgren.

7.3. Improving cubic expansions: Life beyond Almgren

Arguably, the most delicate point in [26] is to show that, for all singular points $(x_\circ, t_\circ)$ outside of a set of parabolic dimension $n-2$, the following expansion holds:

$$u(x_\circ + x, t_\circ + t) = \frac{1}{2}x_n^2 + a|x_n|\left(t + \frac{1}{6}x_n^2\right) + [\text{3-hom. cal. pol.}] + O\Big(\big(|x| + |t|^{1/2}\big)^{3+\beta}\Big), \tag{7.6}$$

for some $\beta > 0$ (which may depend on the point).

Given a singular point $(x_\circ, t_\circ)$ where (7.5) holds, it is natural to consider

$$w(x,t) := u(x_\circ + x, t_\circ + t) - \frac{1}{2}x_n^2 - a|x_n|\left(t + \frac{1}{6}x_n^2\right) - [\text{3-hom. cal. pol.}].$$

Now, one could naively try to show that Almgren frequency is again monotone on such w (this is the first we tried and, as a matter of fact, we thought for a long time that this was the way to go). Unfortunately, since $a > 0$, the cubic term is never a caloric polynomial and the frequency function $\phi(r, w)$ is never (almost) monotone.

In order to improve (7.5), we need a completely new strategy based on barriers, compactness, and certain ad-hoc monotonicity properties, which are much weaker than Almgren's (but which still give some nonempty information).

Our new approach consists in showing, essentially,[14] that

$$\|w\|_{L^\infty(B_r\times(-r^2,0))} \le \omega(r),$$

where ω satisfies the following alternative with $\varepsilon > 0$ arbitrarily small. For all $r > 0$ sufficiently small, we have either

$$\begin{aligned}&\Sigma \text{ is } (\varepsilon r)\text{-close to an } (n-2)\text{-plane inside } B_r(x_\circ)\\ &\quad \text{for } t \in (t_\circ - r^2, t_\circ) \text{ and } \omega\left(\frac{r}{2}\right) \le \frac{\omega(r)}{2^{3-\varepsilon}};\end{aligned} \tag{7.7}$$

or else, we have

$$\omega\left(\frac{r}{2}\right) \le \frac{\omega(r)}{2^{3+\frac{1}{2}}}. \tag{7.8}$$

In view of the previous alternative, it seems to look at dyadic scales $r = 2^{-i}$ and consider the "upper density" of scales at which (7.7) holds:

$$\vartheta := \limsup_{\ell\to\infty} \frac{\#\{i \le \ell : (7.7) \text{ holds at the scale } r = 2^{-i}\}}{\ell} \in [0,1].$$

Now, if $\vartheta = 1$, then as we zoom in around $(x_\circ, t_\circ)$, we see "enough scales" at which the singular set is close to an $(n-2)$-plane to conclude that "Σ is $(n-2)$ dimensional at $(x_\circ, t_\circ)$" (this requires new delicate GMT-type covering arguments). On the other hand, if $\vartheta < 1$, then for a positive $(1-\vartheta)$-proportion of scales, we have (7.8), while for the other scales, we have $\omega(\frac{r}{2}) \le 2^{-3+\varepsilon}$. Taking ε small, we can choose $\beta > 0$ such that $(1-\vartheta)\frac{1}{2} + \vartheta(-\varepsilon) = 3 + \beta$. We then see that

$$\omega(2^{-\ell}) \lesssim 2^{(-3-\frac{1}{2})(1-\theta)\ell} 2^{(-3+\varepsilon)\theta\ell}\omega(1) = 2^{-(3+\beta)\ell}\omega(1).$$

This gives (7.6) at such points.

[14] The (over)simplified statement given here is not strictly correct, but it gives a very good approximated idea on how the argument goes. The actual statement is much more involved (see [26, Proposition 11.3]). Although some of the subtleties in the actual statement are important and not mere technicalities, we cannot discuss them here.

7.4. C^∞ partial regularity and optimal dimensional bounds on the singular set

Once we have proven (7.6), we are ready to push the expansion to higher order. For this, we show with a barrier argument that the set $\{u > 0\}$ splits into two separate connected components inside the set $\Omega^\beta := \{|x|^{2+\beta} < -t\}$—here $(x_\circ, t_\circ) = (0, 0)$.

Note that, under the parabolic scaling $(x, t) \to (rx, r^2 t)$, the set Ω^β converges to $\mathbb{R}^n \times (-\infty, 0)$ as $r \to 0$. In other words, we have "broken the parabolic scaling". We then show a C^∞ regularity result (at $(0, 0)$) for the free boundary of solutions of (4.4) in Ω^β which have a "regular point" at $(0, 0)$. Here the difference with respect to the Caffarelli and Kinderlehrer–Nirenberg result is that in our case the domain Ω^β is not a parabolic cylinder: for every time slice space, the equation holds in ball, but its radius goes to zero as $t \uparrow 0$. Nevertheless, we manage to prove a C^∞ regularity which is robust enough to work in our setting. More precisely, to show that if $\bar{u}$ is a solution of the Stefan problem such that $\{\bar{u} = 0\}$ is sufficiently close to $\{x_n \le 0\}$ inside Ω^β, then we have a C^∞ expansion for $\bar{u}$ at $(0, 0)$. We then apply this result to our solution u multiplied by the characteristic functions of each of the two connected components of $\{u > 0\}$ inside Ω^β. In this way, we obtain a C^∞-type regularity for u.

As a corollary of this C^∞ expansion, we are able to prove that, outside an $(n-2)$-dimensional set, if $(x_\circ, t_\circ)$ and (x_1, t_1) are singular points, then

$$|t_\circ - t_1| = o\big(|x_\circ - x_1|^k\big) \quad \text{for every } k \gg 1.$$

This allows to finally establish the following theorem.

Theorem 9 ([26]). *Let $\Omega \subset \mathbb{R}^n$, and let $u \in L^\infty(\Omega \times (0, T))$ solve the Stefan problem (4.1)-(4.2). Then there exists $\Sigma^\infty \subset \Sigma$ (recall that $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ denotes the singular set) such that*

$$\dim_{\rm par}(\Sigma \setminus \Sigma^\infty) \le n - 2, \quad \dim_{\mathcal{H}}\big(\{t \in (0, T) : \exists\, (x, t) \in \Sigma^\infty\}\big) = 0,$$

and $\Sigma^\infty \subset \Omega \times (0, T)$ can be covered by countably many $(n-1)$-dimensional submanifolds in $\mathbb{R}^{n+1}$ of class C^∞.[15]

In a sense, this result says that the singular set can be split into two separate pieces: one is very smooth and extremely rare in time (the set Σ^∞), and one lower dimensional (of parabolic dimension at most $n - 2$).

This is a very precise result. Indeed, it is easy to construct radial examples of solutions to (4.1)-(4.2) for which the singular set contains some $(n-1)$-sphere for countably many times. Such spheres would be covered by the set Σ^∞ in Theorem 9.

[15]Here, the $(n-1)$-submanifolds that cover Σ^∞ are of class C^∞ as subset of $\mathbb{R}^{n+1}$ with the usual Euclidean distance, not with respect to the parabolic distance. So, our statement is much stronger than the previously known results (for instance, [36] proved C^1 regularity of Σ with respect to the parabolic distance, which implies only $C^{1/2}$ regularity in time).

Now, for general solutions, we cannot prove that $\pi_t(\Sigma^\infty)$ is countable as in such examples, but we do prove that it must be a 0-dimensional set (and Hausdorff dimension cannot distinguish between countable and 0-dimensional sets, so the result is sharp in this sense). On the other hand, the complement of Σ^∞ inside Σ instead, is a set of "bad" singular points. These "bad" points do not enjoy a priori any extra spatial regularity, but in exchange, their parabolic dimension is bounded by $n-2$. The fact that points where Caffarelli's quadratic expansion cannot be improved exist (and may be $n-2$ dimensional) can be easily shown by considering any radial solutions in $\mathbb{R}^2$ with a singular point at $(0,0)$.

An important consequence of Theorem 9 is the following very precise bound for the physical case (three spatial dimensions):

Corollary 10 ([26]). *The set of singular times for Stefan's problem in* $\mathbb{R}^3$ *has Hausdorff dimension at most* $1/2$. *In particular, it has measure zero.*

Also, Theorem 9 implies that in $\mathbb{R}^2$, the set of singular times for Stefan's problem has zero Hausdorff dimension (prior to our results, it was not even known that in $\mathbb{R}^2$, the set of singular times had measure zero).

In summary, these new results provide a very good picture about how the singular set of the Stefan problem behaves.

Funding. Research supported by Swiss NSF Ambizione Grant PZ00P2 180042 and by the European Research Council (ERC) under Grant Agreement No. 948029.

References

[1] W. K. Allard, On the first variation of a varifold. *Ann. of Math. (2)* **95** (1972), 417–491 Zbl 0252.49028 MR 307015

[2] F. J. Almgren Jr., Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math. (2)* **84** (1966), 277–292 Zbl 0146.11905 MR 200816

[3] F. J. Almgren Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math. (2)* **87** (1968), 321–391 Zbl 0162.24703 MR 225243

[4] F. J. Almgren Jr., *Almgren's Big Regularity Paper. q-Valued Functions Minimizing Dirichlet's Integral and the Regularity of Area-Minimizing Rectifiable Currents up to Codimension* 2. World Sci. Monogr. Ser. Math. 1, World Scientific Publishing, River Edge, NJ, 2000 Zbl 0985.49001 MR 1777737

[5] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, The structure of the free boundary for lower dimensional obstacle problems. *Amer. J. Math.* **130** (2008), no. 2, 485–498 Zbl 1185.35339 MR 2405165

[6] A. Blanchet, On the singular set of the parabolic obstacle problem. *J. Differential Equations* **231** (2006), no. 2, 656–672 Zbl 1121.35145 MR 2287901

[7] E. Bombieri, E. De Giorgi, and E. Giusti, Minimal cones and the Bernstein problem. *Invent. Math.* **7** (1969), 243–268 Zbl 0183.25901 MR 250205

[8] H. R. Brezis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France* **96** (1968), 153–180 Zbl 0165.45601 MR 239302

[9] L. A. Caffarelli, The regularity of free boundaries in higher dimensions. *Acta Math.* **139** (1977), no. 3-4, 155–184 Zbl 0386.35046 MR 454350

[10] L. A. Caffarelli, The obstacle problem revisited. *J. Fourier Anal. Appl.* **4** (1998), no. 4-5, 383–402 Zbl 0928.49030 MR 1658612

[11] L. A. Caffarelli and N. M. Rivière, Asymptotic behaviour of free boundaries at their singular points. *Ann. of Math. (2)* **106** (1977), no. 2, 309–317 Zbl 0364.35041 MR 463690

[12] M. Colombo, L. Spolaor, and B. Velichkov, A logarithmic epiperimetric inequality for the obstacle problem. *Geom. Funct. Anal.* **28** (2018), no. 4, 1029–1061 Zbl 1428.49042 MR 3820438

[13] D. Danielli and S. Salsa, Obstacle problems involving the fractional Laplacian. In *Recent Developments in Nonlocal Theory*, pp. 81–164, De Gruyter, Berlin, 2018 Zbl 1408.35226 MR 3824211

[14] E. De Giorgi, *Frontiere orientate di misura minima. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960–61*. Editrice Tecnico Scientifica, Pisa, 1961 MR 0179651

[15] E. De Giorgi, Una estensione del teorema di Bernstein. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **19** (1965), 79–85 Zbl 0168.09802 MR 178385

[16] C. De Lellis, Almgren's Q-valued functions revisited. In *Proceedings of the International Congress of Mathematicians. Volume III*, pp. 1910–1933, Hindustan Book Agency, New Delhi, 2010 Zbl 1226.49041 MR 2827872

[17] C. De Lellis and E. Spadaro, Regularity of area minimizing currents III: blow-up. *Ann. of Math. (2)* **183** (2016), no. 2, 577–617 Zbl 1345.49053 MR 3450483

[18] J. Douglas, Solution of the problem of Plateau. *Trans. Amer. Math. Soc.* **33** (1931), no. 1, 263–321 Zbl 0001.14102 MR 1501590

[19] G. Duvaut, Résolution d'un problème de Stefan (fusion d'un bloc de glace à zéro degré). *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), A1461–A1463 Zbl 0258.35037 MR 328346

[20] H. Federer, The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Amer. Math. Soc.* **76** (1970), 767–771 Zbl 0194.35803 MR 260981

[21] H. Federer and W. H. Fleming, Normal and integral currents. *Ann. of Math. (2)* **72** (1960), 458–520 Zbl 0187.31301 MR 123260

[22] X. Fernández-Real, The thin obstacle problem: a survey. *Publ. Mat.* **66** (2022), no. 1, 3–55 Zbl 07473077 MR 4366205

[23] A. Figalli, Free boundary regularity in obstacle problems. In *Journées équations aux dérivées partielles*, p. Exposé no. 2, 2018

[24] A. Figalli, Regularity of interfaces in phase transitions via obstacle problems—Fields Medal lecture. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pp. 225–247, World Sci. Publ., Hackensack, NJ, 2018 MR 3966728

[25] A. Figalli, X. Ros-Oton, and J. Serra, Generic regularity of free boundaries for the obstacle problem. *Publ. Math. Inst. Hautes Études Sci.* **132** (2020), 181–292 Zbl 1456.35234 MR 4179834

[26] A. Figalli, X. Ros-Oton, and J. Serra, The singular set in the Stefan problem. 2021, arXiv:2103.13379

[27] A. Figalli and J. Serra, On the fine structure of the free boundary for the classical obstacle problem. *Invent. Math.* **215** (2019), no. 1, 311–366 Zbl 1408.35228 MR 3904453

[28] W. H. Fleming, On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)* **11** (1962), 69–90 Zbl 0107.31304 MR 157263

[29] F. Franceschini and W. Zatoń, C^∞ partial regularity of the singular set in the obstacle problem. 2021, arXiv:2102.00923

[30] M. Giaquinta and L. Pepe, Esistenza e regolarità per il problema dell'area minima con ostacoli in n variabili. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **25** (1971), 481–507 Zbl 0283.49032 MR 305205

[31] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*. Monogr. Math. 80, Birkhäuser, Basel, 1984 Zbl 0545.49018 MR 775682

[32] D. Kinderlehrer, The coincidence set of solutions of certain variational inequalities. *Arch. Rational Mech. Anal.* **40** (1970/71), 231–250 Zbl 0219.49014 MR 271799

[33] D. Kinderlehrer and L. Nirenberg, Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), no. 2, 373–391 Zbl 0352.35023 MR 440187

[34] D. Kinderlehrer, L. Nirenberg, and J. Spruck, Regularity in elliptic free boundary problems. *J. Analyse Math.* **34** (1978), 86–119 Zbl 0402.35045 MR 531272

[35] H. Lewy and G. Stampacchia, On the regularity of the solution of a variational inequality. *Comm. Pure Appl. Math.* **22** (1969), 153–188 Zbl 0167.11501 MR 247551

[36] E. Lindgren and R. Monneau, Pointwise regularity of the free boundary for the parabolic obstacle problem. *Calc. Var. Partial Differential Equations* **54** (2015), no. 1, 299–347 Zbl 1327.35455 MR 3385162

[37] R. Monneau, On the number of singularities for the obstacle problem in two dimensions. *J. Geom. Anal.* **13** (2003), no. 2, 359–389 Zbl 1041.35093 MR 1967031

[38] A. Petrosyan, H. Shahgholian, and N. Uraltseva, *Regularity of Free Boundaries in Obstacle-Type Problems*. Grad. Stud. Math. 136, Amer. Math. Soc., Providence, RI, 2012 Zbl 1254.35001 MR 2962060

[39] C.-C. Poon, Unique continuation for parabolic equations. *Comm. Partial Differential Equations* **21** (1996), no. 3-4, 521–539 Zbl 0852.35055 MR 1387458

[40] T. Radó, On Plateau's problem. *Ann. of Math. (2)* **31** (1930), no. 3, 457–469 MR 1502955

[41] E. R. Reifenberg, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type. *Acta Math.* **104** (1960), 1–92 Zbl 0099.08503 MR 114145

[42] E. R. Reifenberg, On the analyticity of minimal surfaces. *Ann. of Math. (2)* **80** (1964), 15–21 Zbl 0151.16702 MR 171198

[43] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*. Grundlehren Math. Wiss. 316, Springer, Berlin, 1997 Zbl 0881.31001 MR 1485778

[44] M. Sakai, Regularity of a boundary having a Schwarz function. *Acta Math.* **166** (1991), no. 3-4, 263–297 Zbl 0728.30007 MR 1097025

[45] M. Sakai, Regularity of free boundaries in two dimensions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **20** (1993), no. 3, 323–339 Zbl 0851.35022 MR 1256071

[46] O. Savin, Minimal surfaces and minimizers of the Ginzburg–Landau energy. In *Symmetry for Elliptic PDEs*, pp. 43–57, Contemp. Math. 528, Amer. Math. Soc., Providence, RI, 2010 Zbl 1225.35227 MR 2759034

[47] D. G. Schaeffer, An example of generic regularity for a non-linear elliptic equation. *Arch. Rational Mech. Anal.* **57** (1975), 134–141 Zbl 0319.35036 MR 387810

[48] D. G. Schaeffer, Some examples of singularities in a free boundary. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), no. 1, 133–144 Zbl 0354.35033 MR 516201

[49] S. Serfaty and J. Serra, Quantitative stability of the free boundary in the obstacle problem. *Anal. PDE* **11** (2018), no. 7, 1803–1839 Zbl 1391.35432 MR 3810473

[50] A. Signorini, Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. e Appl. (5)* **18** (1959), 95–139 Zbl 0091.38006 MR 118021

[51] J. Simons, Minimal varieties in riemannian manifolds. *Ann. of Math. (2)* **88** (1968), 62–105 Zbl 0181.49702 MR 233295

[52] J. Stefan, Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Ann. Phys. Chem.* **42** (1891), 269–286

[53] J. E. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math. (2)* **103** (1976), no. 3, 489–539 Zbl 0335.49032 MR 428181

[54] G. S. Weiss, A homogeneity improvement approach to the obstacle problem. *Invent. Math.* **138** (1999), no. 1, 23–50 Zbl 0940.35102 MR 1714335

Joaquim Serra
Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland;
joaquim.serra@math.ethz.ch