DOI 10.4171/8ECM/09

The topology of dissipative systems

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Abstract. This expository article is dedicated to the study of some topological features of dissipative flows defined in locally compact metric spaces, especially in manifolds and in the Euclidean space. We show that they exhibit a host of interesting topological properties in areas as diverse as Conley's index theory, population dynamics, and the dynamics of planar systems.

1. Introduction

In the study of flows in non-compact spaces, the dissipative ones play an important role. Their interest lies in the fact that it is possible to reduce the fundamental part of the flow and its asymptotic behavior to a compact set. The concept of dissipativity was introduced by Levinson in 1944 [28] for flows in the Euclidean space in his study of the periodically forced van der Pol equation.

This expository article is dedicated to the study of some topological features of dissipative flows defined in locally compact metric spaces, especially in manifolds and in the Euclidean space. We show that they exhibit a host of interesting topological properties in areas as diverse as Conley's index theory, population dynamics, and the dynamics of planar systems.

Through the paper we shall consider continuous dynamical systems (or flows) $\varphi : M \times \mathbb{R} \to M$, where M is a locally compact metric space. If M is not compact, then every flow can be extended to the Alexandrov compactification $M \cup \{\infty\}$ of M by leaving fixed the point ∞.

The main reference for the elementary concepts of dynamical systems is [10] but we also recommend [34, 35, 37]. We use the notation $\gamma(x)$ for the *trajectory* of the point x, i.e.,

$$\gamma(x) = \{xt \mid t \in \mathbb{R}\}.$$

Similarly, for the *positive semi-trajectory* and the *negative semi-trajectory*

$$\gamma^+(x) = \{xt \mid t \in \mathbb{R}_+\}, \quad \gamma^-(x) = \{xt \mid t \in \mathbb{R}_-\}.$$

2020 Mathematics Subject Classification. Primary 34D23; Secondary 37G35.
Keywords. Dissipative flow, global attractor, Borsuk homotopy, Conley index, continuation.

By the *omega-limit* of a point x we understand the set

$$\omega(x) = \bigcap_{t>0} \overline{x[t, \infty)}.$$

In an analogous way, the *negative omega-limit* is the set

$$\omega^*(x) = \bigcap_{t<0} \overline{x(-\infty, t]}.$$

An invariant compactum K is *stable* if every neighborhood U of K contains a neighborhood V of K such that $V[0, \infty) \subset U$.

We recall that an *attractor* is a stable invariant compactum K satisfying that there exists a neighborhood U of K such that $\emptyset \neq \omega(x) \subset K$, for every $x \in U$. A *repeller* is just an attractor for the reverse flow given by $\overline{\varphi}(x, t) = \varphi(x, -t)$.

If K is an attractor, its region (or basin) of attraction of K is the set

$$\mathcal{A}(K) = \{x \in M \mid \emptyset \neq \omega(x) \subset K\}.$$

It is well known that $\mathcal{A}(K)$ is an open invariant set. If in particular $\mathcal{A}(K)$ is the whole phase space, we say that K is a *global attractor*.

We use some topological notions through this paper. We recommend the books of Hatcher and Spanier [24, 44] to cover this material. We use the notation H^* for the singular cohomology. We consider cohomology taking coefficients in $\mathbb{Z}$.

If a pair of spaces (X, A) satisfies that its cohomology $H^k(X, A)$ is finitely generated for each k and is non-zero only for a finite number of values of k (as it happens if (X, A) is a pair of compact manifolds), its *Poincaré polynomial* is defined as

$$P_t(X, A) = \sum_{k \geq 0} \operatorname{rk} H^k(X, A) t^k.$$

There is a form of homotopy theory which has proved to be the most convenient for the study of the global topological properties of the invariant spaces involved in dynamics, namely *Borsuk's homotopy theory* or *shape theory*, introduced and studied by Karol Borsuk. We present here a short introduction based on the presentation given by Kapitanski and Rodnianski in [27].

A metric space X is said to be an *absolute neighborhood retract* or, shortly, an *ANR* if it satisfies that whenever there exists an embedding $f : X \to Y$ of X into a metric space Y such that $f(X)$ is closed in Y, there exists a neighborhood U of $f(X)$ such that $f(X)$ is a retract of U. Some examples of ANRs are manifolds, CW complexes, and polyhedra. Besides, an open subset of an ANR is an ANR and a retract of ANR is also an ANR. For more information about ANRs we recommend [26]. Notice that by Kuratowski–Wojdyslawski theorem, every metric space can be embedded in an ANR as a closed subspace.

Let X be a closed subset of an ANR M and Y a closed subset of an ANR N. Denote by $\mathbb{U}(X; M)$ (resp. $\mathbb{U}(Y; N)$) the set of all open neighborhoods of X in M (resp. Y in N).

Let $\mathbf{f} = \{f : U \to V\}$ be a collection of continuous maps from the neighborhoods $U \in \mathbb{U}(X; M)$ to $V \in \mathbb{U}(Y; N)$. We say that $\mathbf{f}$ is a *mutation* from X to Y if it satisfies

(1) for every $V \in \mathbb{U}(Y; N)$ there exists at least a map $f : U \to V$ in $\mathbf{f}$;

(2) if $f : U \to V$ is in $\mathbf{f}$, then the restriction $f|_{U_1} : U_1 \to V_1$ is also in $\mathbf{f}$ for every neighborhood $U_1 \subset U$ and every neighborhood $V_1 \supset V$;

(3) if two maps $f, f' : U \to V$ are in $\mathbf{f}$, there exists a neighborhood $U_1 \subset U$ such that the restrictions $f|_{U_1}$ and $f'|_{U_1}$ are homotopic.

An example of mutation is the *identity mutation* $\mathrm{id}_{\mathbb{U}(X;M)}$ consisting of the identity maps $\mathrm{id} : U \to U$.

Composition of mutations $\mathbf{f} = \{f : U \to V\}$, $\mathbf{g} = \{g : V \to W\}$ from X to Y and from Y to Z, respectively, is defined in the straightforward way. Two mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{f}' = \{f' : U' \to V'\}$ (both from X to Y) are said to be *homotopic* if for every pair of maps $f : U \to V$ and $f' : U' \to V$ belonging to $\mathbf{f}$ and $\mathbf{f}'$, respectively, there exists a neighborhood $U_0 \in \mathbb{U}(X; M)$, $U_0 \subset U \cap U'$ such that $f|_{U_0}$ is homotopic to $f'|_{U_0}$. It is easy to see that homotopy of mutations is an equivalence relation.

Two metric spaces X and Y have the same *Borsuk homotopy type* or *shape*, denoted by $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, if they can be embedded as closed sets in ANRs M and N in such a way that there exist mutations $\mathbf{f} = \{f : U \to V\}$ and $\mathbf{g} = \{g : V \to U\}$ such that the compositions $\mathbf{gf}$ and $\mathbf{fg}$ are homotopic to the identity mutations $\mathrm{id}_{\mathbb{U}(X;M)}$ and $\mathrm{id}_{\mathbb{U}(Y;N)}$, respectively. In this case, the mutation $\mathbf{f}$ (resp. $\mathbf{g}$) is said to be a *shape equivalence*.

We stress the following basic features whose proofs can be found in [11].

(1) The notion of shape of sets depends neither on the ANRs they are embedded in nor on the particular embeddings.

(2) Spaces belonging to the same homotopy type have the same shape.

(3) ANRs have the same shape if and only if they have the same homtopy type.

In the case of plane continua, the relation of having the same Borsuk homotopy type has an easy visualization as it establishes the following result.

Theorem 1.1 (Borsuk [11]). *Two continua K and L contained in $\mathbb{R}^2$ have the same Borsuk homotopy type if and only if they disconnect $\mathbb{R}^2$ in the same number of connected components. In particular, a continuum has the Borsuk homotopy type of a point if and only if it does not disconnect $\mathbb{R}^2$. A continuum has the Borsuk homotopy type of a circle if and only if it disconnects $\mathbb{R}^2$ into two connected components.*

Every continuum has the Borsuk homotopy type of a wedge of circles, finite or infinite (Hawaiian earring).

For more information about Borsuk homotopy theory we recommend the books [11, 17, 32]. The papers [3, 9, 19–21, 23, 27, 39, 40, 42] illustrate some applications of this theory to the study of dynamical systems.

An important class of invariant compacta is the so-called *isolated invariant sets* (see [15, 16, 18] for details). These are compact invariant sets K which possess an *isolating neighborhood*, i.e., a compact neighborhood N such that K is the maximal invariant set in N.

To introduce the Conley index, that plays an essential role in this paper, we use a special kind of isolating neighborhoods, the so-called *isolating blocks*. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and the exit sets, satisfying

(1) $\partial N = N^i \cup N^o$;

(2) for each $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M \setminus N$ and for each $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M \setminus N$;

(3) for each $x \in \partial N \setminus N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \mathring{N}$ and for every $x \in \partial N \setminus N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \mathring{N}$.

These blocks form a neighborhood basis of K in M.

Let K be an isolated invariant set. Its *Conley index* $h(K)$ is defined as the pointed homotopy type of the topological space $(N/N^o, [N^o])$, where N is an isolating block of K. A weak version of the Conley index which will be useful for us is the *cohomological index* defined as $CH^*(K) = H^*(h(K))$. It can be proved that $CH^*(K) \cong H^*(N, N^o)$. Our main references for the Conley index theory are [15, 38]. An exhaustive study of the Conley index in the case of two-dimensional flows can be found in [2, 4] and some applications of this theory to the evolution of the Lorenz strange set are contained in [8]. In addition, the Conley index has recently been used to find counterexamples to the triangulation conjecture (see [30, 31]).

The Conley index allows us to establish some conections between local and global dynamics via Morse decompositions. We recall that if K is a compact invariant set, a finite collection $\{M_1, \ldots, M_n\}$ of pairwise disjoint invariant subcompacta of K is a *Morse decomposition* if it satisfies that

$$\text{for each } x \in \left(K \setminus \bigcup_{i=1}^{n} M_i\right), \quad \omega(x) \subset M_j \text{ and } \omega^*(x) \subset M_k \text{ with } j < k.$$

Each set M_i is said to be a *Morse set*.

Given a Morse decomposition $\{M_1, M_2, \ldots, M_k\}$ of an isolated invariant set K, there exists a polynomial $Q(t)$ whose coefficients are non-negative integers such that

$$\sum_{i=1}^{n} P_t\big(h(M_i)\big) = P_t\big(h(K)\big) + (1+t)Q(t).$$

This formula, which relates the Conley indices of the Morse sets with the Conley index of the isolated invariant set, is known as the *Morse equation* of the Morse decomposition and it generalizes the classical Morse inequalities.

Another central concept of the Conley index theory that plays a crucial role in this paper is that of continuation of isolated invariant sets. Let M be a locally compact metric space, and let $\varphi_\lambda : M \times \mathbb{R} \to M$ be a parametrized family of flows (parametrized by $\lambda \in [0,1]$, the unit interval). The family $(K_\lambda)_{\lambda \in J}$, where $J \subset [0,1]$ is a closed (non-degenerate) subinterval and, for each $\lambda \in J$, K_λ is an isolated invariant set for φ_λ, is said to be a *continuation* if for each $\lambda_0 \in J$ and each N_{λ_0} isolating neighborhood for K_{λ_0}, there exists $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$. We say that the family $(K_\lambda)_{\lambda \in J}$ is a continuation of K_{λ_0} for each $\lambda_0 \in J$.

Notice that [38, Lemma 6.1] ensures that if K_{λ_0} is an isolated invariant set for φ_{λ_0}, there always exists a continuation $(K_\lambda)_{\lambda \in J_{\lambda_0}}$ of K_{λ_0} for some closed (non-degenerate) subinterval $\lambda_0 \in J_{\lambda_0} \subset [0,1]$.

There is a simpler definition of continuation based on [38, Lemma 6.2]. There, it is proved that if $\varphi_\lambda : M \times \mathbb{R} \to M$ is a parametrized family of flows and if N_1 and N_2 are isolating neighborhoods of the same isolated invariant set for φ_{λ_0}, then there exists $\delta > 0$ such that N_1 and N_2 are isolating neighborhoods for φ_λ, for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$, with the property that, for every λ, the isolated invariant subsets in N_1 and N_2 which have N_1 and N_2 as isolating neighborhoods coincide.

Therefore, the family $(K_\lambda)_{\lambda \in J}$, with K_λ an isolated invariant set for φ_λ, is a continuation if for every $\lambda_0 \in J$ there are an isolating neighborhood N_{λ_0} for K_{λ_0} and a $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ, for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$.

Notice that, since this should not lead to any confusion, sometimes we will only say that K_λ is a continuation of K_{λ_0} without specifying the subinterval $J \subset [0,1]$ to which the parameters belong.

In the particular case that K_{λ_0} is an attractor for $\lambda_0 \in J$, there exists $\delta > 0$ such that K_λ is attractor with $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K_{\lambda_0})$ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$ (see [41, Theorem 4]).

The paper is structured as follows. In Section 2 the concept of dissipative flow is introduced and some of the basic properties of this class of flows are presented. In particular, we see that dissipative flows coincide with those that have a global attractor.

We also present some characterizations of the global attractor of a dissipative flow in the Euclidean space. Section 3 is devoted to study parametrized families of dissipative flows. We see that the property of being a global attractor is not robust and introduce a characterization of continuations that consist entirely of global attractors. We also survey some results regarding the bifurcation global to non-global. In Section 4 we study connections between dissipative flows and populations dynamics and present some results about uniform persistence, a central concept in population dynamics. Finally, in Section 5, we present some results that ensure that the global attractor of a dissipative flow defined on the non-negative orthant of the plane is contained in the boundary.

2. Dissipative flows

We start by recalling the definition of dissipative flow and some of its basic properties. We assume that M is a locally compact, non-compact metric space.

Definition 2.1 (Levinson 1944). A flow $\varphi : M \times \mathbb{R} \to M$ is said to be *dissipative* provided that, for each $x \in M$, the omega limit $\omega(x) \neq \emptyset$ and the closure of the set

$$\Omega(\varphi) = \bigcup_{x \in M} \omega(x)$$

is compact.

The following characterization of dissipative flows, which gives a very clear interpretation of their dynamics, was provided by Pliss.

Proposition 2.2 (Pliss 1966 [36]). *A flow $\varphi : M \times \mathbb{R} \to M$ is dissipative if and only it has a global attractor.*

It should be noted that, in general, the global attractor does not necessarily coincide with the closure of $\Omega(\varphi)$. On the other hand, it can be seen that the flow φ is dissipative if and only if $\{\infty\}$ is a repeller.

The following result gives a characterization of the global attractor of a dissipative flow. It relies heavily on the non-existence of bounded orbits outside the attractor.

Proposition 2.3. *Let φ be a dissipative flow in $\mathbb{R}^n$ and K a compact invariant set. Then K is the global attractor if and only if $\mathbb{R}^n \setminus K$ does not contain bounded orbits.*

In the case of flows on the two-dimensional Euclidean space it is possible to obtain a simpler characterization of global attractors of dissipative flows.

Theorem 2.4 (Barge–Sanjurjo [5]). *Let K be an isolated invariant continuum of a dissipative flow φ in $\mathbb{R}^2$. The following conditions are equivalent:*

(i) *K is a global attractor;*

(ii) *there are no fixed points in* $\mathbb{R}^2 \setminus K$ *and there exists an orbit* γ *connecting* ∞ *and* K *(i.e., such that* $\|\gamma(t)\| \to \infty$ *when* $t \to -\infty$ *and* $\omega(\gamma) \subset K$*).*

This result is inspired by the following result that gives a relation between global asymptotic stability of a fixed point and the non-existence of additional fixed points in the case of dissipative discrete dynamical systems.

Theorem 2.5 (Alarcón–Guíñez–Gutiérrez [1], Ortega–Ruiz del Portal [33]). *Assume that* $h \in \mathcal{H}_+$ *(orientation preserving homeomorphisms of* $\mathbb{R}^2$*) is dissipative and* p *is an asymptotically stable fixed point of* h*. The following conditions are equivalent:*

(i) p *is globally asymptotically stable;*

(ii) $\mathrm{fix}(h) = p$ *and there exists an arc* $\gamma \subset S^2$ *with end points at* p *and* ∞ *such that* $h(\gamma) = \gamma$.

The proof in [1] is based on Brouwer's theory of fixed point free homeomorphisms of the plane. Ortega and Ruiz del Portal give in [33] an alternative proof based on the theory of prime ends.

The previous results suggest that if K is an attractor of a dissipative flow, then $\mathcal{A}(K)$ being bounded is in the sharpest contrast to K being global.

3. Robustness of global attractors

This section is dedicated to the presentation of some results related to properties of dissipative systems that concern Conley's index theory.

We give a simple example which shows that the property of being global is not a robust property for an attractor since small perturbations of the flow can create bounded orbits in its region of attraction.

Example 3.1. Consider the family of ordinary differential equations defined on the plane in polar coordinates:

$$\begin{cases} r' = -r^3\left(\frac{1}{r} - \lambda\right)^2, \\ \theta' = 1, \end{cases} \qquad \lambda \in [0, 1].$$

The phase portraits of this family of differential equations are depicted in Figure 1. The picture on the left describes the phase portrait for the parameter $\lambda = 0$. We see that in this case the origin is a globally attracting fixed point and the orbit of any other point spirals towards it. The picture on the right describes the phase portrait when $\lambda > 0$. In this case we see that the origin is still an asymptotically stable fixed point but it is not a global attractor anymore since, for each $\lambda > 0$, the circle centered at the origin and radius $1/\lambda$ is a periodic trajectory which attracts uniformly all the points of the unbounded component of its complement and repels all the points of the bounded one except the origin.

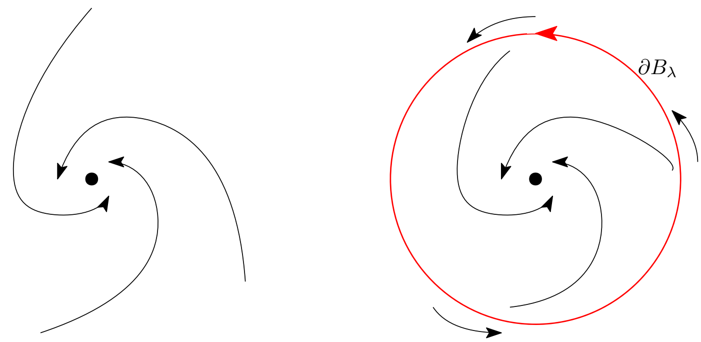

Figure 1. Phase portraits of the family of ordinary differential equations from Example 3.1 for $\lambda = 0$ (left) and $\lambda > 0$ (right).

Example 3.1 motivates the following definition.

Definition 3.2. A parametrized family of dissipative flows $\varphi_\lambda : M \times \mathbb{R} \to M$ is said to be *coercive* if for any continuation K_λ of the global attractor K_0 of φ_0 there exists a λ_0 such that $\mathcal{A}_\lambda(K_\lambda)$ is bounded for every λ with $0 < \lambda < \lambda_0$.

However, note that in this situation, since all the flows are dissipative, then each φ_λ still has a global attractor $\widehat{K}_\lambda$ but the family of global attractors is not a continuation of K_0.

The following definition introduces a notion which is, in some sense, the opposite of the previous one.

Definition 3.3. A parametrized family of dissipative flows $\varphi_\lambda : M \times \mathbb{R} \to M$ is said to be *uniformly dissipative* provided that for each $x \in M$ and $\lambda \in [0, 1]$ we have that $\omega_\lambda(x) \neq \emptyset$ and the closure of the set

$$\Omega = \bigcup_{\lambda \in [0,1]} \Omega(\varphi_\lambda)$$

is compact.

The importance of the above definition is that it can be used to provide a characterization of continuations that consist entirely of global attractors.

Theorem 3.4 (Barge–Sanjurjo [7]). *Let $\varphi_\lambda : M \times \mathbb{R} \to M$ be a parametrized family of dissipative flows with $\lambda \in [0, 1]$. Let K_λ denote the global attractor of φ_λ. Then the family $(K_\lambda)_{\lambda \in [0,1]}$ is a continuation of K_0 if and only if the family $(\varphi_\lambda)_{\lambda \in [0,1]}$ is uniformly dissipative.*

We see a nice application of the previous result.

Example 3.5. An important example of global attractor is provided by the Lorenz equations

$$\begin{cases} x' = \sigma(y - x), \\ y' = rx - y - xz, \\ z' = xy - bz, \end{cases}$$

where σ, r, and b are three real positive parameters. If we fix σ and b, we obtain a family of flows

$$\varphi_r : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$$

corresponding to the Lorenz equations for the different values of r.

E. N. Lorenz proved that for every value of r there exists a global attractor of zero volume for the flow associated to these equations. This attractor should not be confused with the famous Lorenz attractor, which is a proper subset of the global attractor.

The family φ_r is uniformly dissipative and, as a consequence, it defines a continuation of global attractors K_r. The proof of this fact uses the function

$$V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2.$$

C. Sparrow studied in [45] this function and showed that it is a Lyapunov function for the flow φ_r. By using this function he was able to prove that K_r lies in a ball B_r centered at 0 and with radius $O(r)$, such that $O(r)$ depends continuously on r. Hence, if we consider an arbitrary r_0 and an interval $[c, d]$ containing r_0, we have that the set $C = \overline{\bigcup_{c \le r \le d} B_r}$ is compact and that $\emptyset \neq \omega_r(x) \subset C$ for every $x \in \mathbb{R}^3$ and every $r \in [c, d]$. Therefore, the family of Lorenz flows φ_r is uniformly dissipative and the corresponding family K_r of global attractors is a continuation.

The coercive families of flows are in sharp contrast with the uniformly dissipative families. For coercive families, the continuations of global attractors are never global. The study of coercive families of flows has some topological interest. The following result provides a graphic characterization of this kind of families.

Theorem 3.6 (Barge–Sanjurjo [7]). *Let φ_λ, with $\lambda \in [0, 1]$, be a coercive family of flows in $\mathbb{R}^n$. We denote by K_0 the global attractor of φ_0 and by K_λ, with $\lambda \in [0, 1]$, a continuation of K_0. Then there exists $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ there is an isolated invariant compactum C_λ in $\mathbb{R}^n \setminus K_\lambda$ such that*

(i) *C_λ separates $\mathbb{R}^n$ into two components and K_λ lies in the bounded component;*

(ii) *C_λ has the Borsuk homotopy type (shape) of S^{n-1};*

(iii) *C_λ attracts uniformly all the points of the unbounded component and repels all the points of the bounded one which are not in K_λ;*

(iv) $\operatorname{diam} C_\lambda \to \infty$ *when* $\lambda \to 0$, *where* $\operatorname{diam} C_\lambda$ *denotes the diameter of* C_λ.

Moreover, the existence of such a C_λ *for* $0 < \lambda < \lambda_0$ *is sufficient for the family to be coercive.*

In view of the previous results, it is interesting to study in all its generality the mechanism which produces the global to non-global bifurcation in families of dissipative flows. With this objective we introduce the following definition.

Definition 3.7. Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, with $\lambda \in [0, 1]$, be a parametrized family of dissipative flows. The family is said to be *polar* if it has arbitrarily large bounded trajectories. More precisely, for every $L > 0$ (arbitrarily large) there is a $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ there is a bounded trajectory γ_λ of φ_λ and a $t_\lambda < 0$ such that $\|\gamma_\lambda(t)\| > L$ for every t with $-\infty < t < t_\lambda$.

Obviously, if K_λ is a continuation of the global attractor K_0 of φ_0, then for L sufficiently large, γ_λ lies in $\mathbb{R}^n \setminus K_\lambda$.

The following proposition makes it clear that polarity is a key notion regarding the transition from global to non-global.

Proposition 3.8 (Barge–Sanjurjo [7]). *Let* $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, *with* $\lambda \in [0, 1]$, *be a parametrized family of dissipative flows. Then the family is polar if and only if for every continuation* K_λ *of the global attractor* K_0 *of* φ_0 *there exists a* $\lambda_0 > 0$ *such that* K_λ *is a non-global attractor for every* λ *with* $0 < \lambda < \lambda_0$.

The following result describes the general picture of the polar families of dissipative flows.

Theorem 3.9 (Barge–Sanjurjo [7]). *If* $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, *with* $\lambda \in [0, 1]$, *is a polar family of dissipative flows, then there exists a* $\lambda_0 > 0$ *such that for every* λ *with* $0 < \lambda < \lambda_0$ *the maximal invariant compactum lying in* $\mathbb{R}^n \setminus K_\lambda$ *for the flow* φ_λ, *which we denote by* C_λ, *is non-empty and isolated, and its cohomological Conley index is trivial in every dimension. Moreover, the family is coercive if and only if* C_λ *has the Borsuk homotopy type of* S^{n-1}.

The isolated invariant compactum C_λ can be seen as the obstruction for the existence of a continuation of global attractors. An interesting feature of the above proposition is that it provides an equivalence between a topological property (having the Borsuk homotopy type of S^{n-1}) and a dynamical property (coercivity).

4. Dissipative flows and populations dynamics

Another area in which dissipative systems play a fundamental role is population dynamics. We shall suppose here that M is a closed subset of a larger locally compact metric space X and denote by ∂M the boundary of M in X.

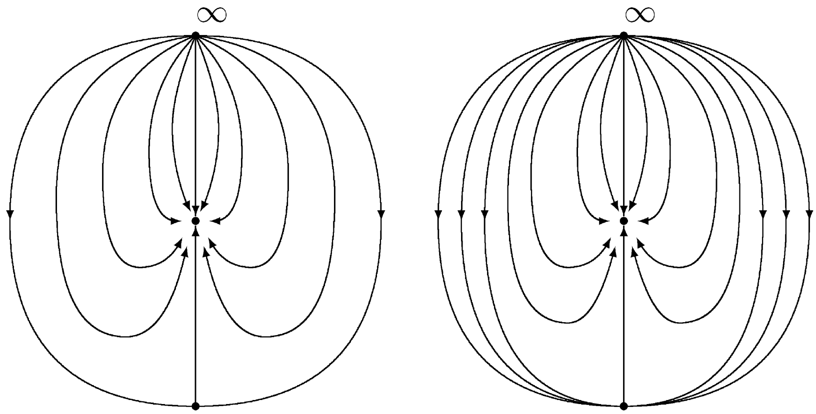

Figure 2. A small perturbation of a uniformly persistent flow that is not uniformly persistent.

Definition 4.1. We will say that the dissipative flow $\varphi : M \times \mathbb{R} \to M$ is *uniformly persistent* if there exists $\beta > 0$ such that for every $x \in \overset{\circ}{M}$

$$\liminf \left\{ d\left(\varphi(x,t), \partial M\right) \mid t \to \infty \right\} \geq \beta.$$

If M is compact, then φ is persistent if and only if ∂M is a repeller of φ. If M is not compact, then φ is persistent if and only if $\partial M \cup \{\infty\}$ is a repeller for the flow extended to $M \cup \{\infty\}$. As a consequence, there exists a compactum K which attracts all points $x \notin \partial M$. This compactum is called the *global attractor* of the system and represents a state of coexistence of all the species that make up the population. The most significant case is when $X = \mathbb{R}^n$, $M = \mathbb{R}^n_+$ (the non-negative orthant). The boundary $\partial \mathbb{R}^n_+$ represents populations some of whose components have become extinct.

It is easy to see that, in a general context, uniform persistence is not a robust property. For instance, the illustration in Figure 2 shows that small perturbations of a uniformly persistent flow can destroy this property.

Despite this fact, we see in our next result that all uniformly persistent flows have *weak continuation properties*, meaning by this that small perturbations of the flow never drive to extinction populations within a certain range (which can be arbitrarily chosen).

Theorem 4.2 (Weak continuation of uniform persistence, Sanjurjo [43]). *Let X be a locally compact metric space and let M be a closed subset of X. Suppose we are given a (continuous) parametrized family of dissipative dynamical systems φ_λ, with $\lambda \in I$, on M, for which ∂M is invariant. Further, assume that φ_0 is uniformly persistent. Then there exists $\beta > 0$ such that for every compact set $K \subset \overset{\circ}{M}$ there exists $\lambda_0 > 0$*

such that

$$\liminf \left\{ d\left(\varphi_\lambda(x,t), \partial M\right) \mid t \to \infty \right\} \geq \beta$$

for every $\lambda \leq \lambda_0$ *and for every* $x \in K$.

When M is the non-negative orthant, some nice topological conclusions can be reached about special regions of the flow. In particular, there is a contractible region where populations are guaranteed their survival and another region of spherical shape where populations have their survival compromised.

Corollary 4.3 (Sanjurjo [43]). *Let* φ_λ, *with* $\lambda \in I$, *be a (continuous) parametrized family of dissipative flows on the non-negative orthant* $\mathbb{R}^n_+$. *Further, assume that* φ_0 *is uniformly persistent. Then there exists* $\alpha > 0$ *such that for every* ε *and every* L *with* $0 < \varepsilon < L$ *there exists* $\lambda_0 > 0$ *such that*

(i) $\liminf\{d(\varphi_\lambda(x,t), \partial\mathbb{R}^n_+)|t \to \infty\} > \alpha$ *for every* x *with* $d(x, \partial\mathbb{R}^n_+) \geq \varepsilon$ *and* $\|x\| \leq L$ *and for every* $\lambda \leq \lambda_0$,

(ii) *the set*

$$W_\lambda = \left\{x \in \mathbb{R}^n_+ \mid \liminf \left\{ d\left(\varphi_\lambda(x,t), \partial\mathbb{R}^n_+\right) \mid t \to \infty \right\} > \alpha \right\}$$

is contractible and the set

$$R_\lambda = \left\{x \in \mathbb{R}^n_+ \mid \liminf \left\{ d\left(\varphi_\lambda(x,t), \partial\mathbb{R}^n_+\right) \mid t \to \infty \right\} \leq \alpha \right\} \cup \{\infty\}$$

has the Borsuk homotopy type (shape) of S^{n-1} *for every* $\lambda \leq \lambda_0$.

It would be of interest to study the implications of these results in some particular situations. Theorem 4.2 suggests that permanence does not vanish completely in an abrupt way. Even if it does not continue, permanence still remains when we limit ourselves to populations within a certain range. As an interesting case, S. Cano-Casanova and J. López-Gómez prove in [14] (see also [29]) that permanence of two species is possible under strong mutual aggression. In other words, they prove that if the birth rates are high enough, then the species are permanent irrespective of the competition strength in the regions where competition occurs. They actually measure how large the birth rate must be.

An interesting problem would be to study to what extent permanence remains for populations within a certain range despite their reproduction rate being below the limit threshold.

As we said before, uniformly persistent flows have a global attractor towards which all the states of the interior evolve. The following results concern the fine structure of this global attractor of the flow and some of its topological properties. We recall that a continuum K is *point-like* in $\mathbb{R}^n$ provided $\mathbb{R}^n \setminus K$ is homeomorphic to $\mathbb{R}^n \setminus \{p\}$, where p is a point.

Theorem 4.4 (Sanjurjo [43]). *Let $\varphi : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n_+$ be a dissipative flow. If φ is uniformly persistent then:*

(i) *Suppose L is a point-like repeller (in particular a repelling point) in the interior of $\mathbb{R}^n_+$, then there exists an attractor K_0 with the Borsuk homotopy type (shape) of S^{n-1} contained in the global attractor K and whose basin of attraction is int $\mathbb{R}^n_+ \setminus L$.*

(ii) *Suppose L is a repeller with the Borsuk homotopy type (shape) of S^{n-1} in the interior of $\mathbb{R}^n_+$. Then L decomposes $\mathring{\mathbb{R}}^n_+$ into two connected components. Moreover, if the bounded component is simply connected, then there exists an attractor with the Borsuk homotopy type (shape) of a point contained (together with its basin of attraction) in the interior of the global attractor K.*

In our next result we see that the Morse theory of uniformly persistent flows with an attracting cycle can be described in a simple way, irrespective of the complexity of the flow in the boundary. Suppose $\varphi : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n_+$ is a uniformly persistent flow. We say that $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is a natural Morse decomposition of the flow if

(a) $\{M_1, M_2\}$ is an attractor-repeller decomposition of the global attractor K,

(b) $M_i \subset \partial\mathbb{R}^n_+$ for $i \geq 3$, and

(c) $\{M_1, M_2, \ldots, M_k, \infty\}$ is a Morse decomposition of $\mathbb{R}^n_+ \cup \{\infty\}$.

By the Morse equation of $\mathcal{M}$ we mean the Morse equation of $\{M_1, M_2, \ldots, M_k, \infty\}$. The next theorem shows that if M_1 is an attracting cycle or, more generally, an attractor with the Borsuk homotopy type (shape) of S^1, then the Morse equation of $\mathcal{M}$ takes a simple form. On the opposite direction we see that using this equation we can recognize the existence of attractors with the Borsuk homotopy type (shape) of S^1 in the plane or attractors whose suspension has the Borsuk homotopy type (shape) of S^2 for higher dimensions.

Theorem 4.5 (Sanjurjo [43]). *Let $\varphi : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n_+$ be a dissipative flow. Suppose φ is uniformly persistent and $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is a natural Morse decomposition of $\mathbb{R}^n_+$ for φ. Then:*

(i) *If M_1 has the Borsuk homotopy type of S^1, then the Morse equation of the decomposition $\mathcal{M}$ with coefficients in $\mathbb{Z}$ or a field is*

$$1 + t + t^2 = 1 + (1 + t)t. \tag{4.1}$$

(ii) *Conversely, if the Morse equation of $\mathcal{M}$ is (4.1), then* $\mathrm{Sh}(M_1) = \mathrm{Sh}(S^1)$ *for $n = 2$ and* $\mathrm{Sh}(\Sigma M_1) = \mathrm{Sh}(S^2)$ *for $n \geq 2$, where ΣM_1 is the suspension of M_1.*

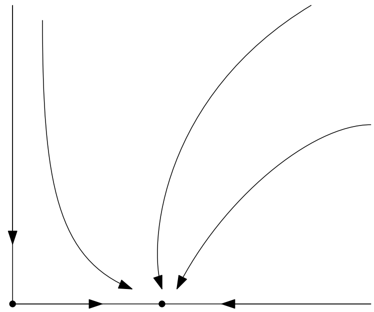

Figure 3. Phase portrait of the Lotka–Volterra system for $a/\lambda \leq c/d$.

5. Planar dissipative flows

In this section, we present some results regarding dissipative flows defined on the non-negative orthant of the plane.

Example 5.1. Consider the Lotka–Volterra equation in $\mathbb{R}^2_+$ (see [25] for more information):

$$\begin{cases} \dot{x} = x(a - by - \lambda x), \\ \dot{y} = y(-c + dx - \mu y), \end{cases} \qquad a, b, c, d, \lambda > 0 \text{ and } \mu \geq 0.$$

This equation, which plays a central role in population dynamics, induces a family of dissipative flows depending on the parameters. The point $(a/\lambda, 0) \in \partial\mathbb{R}^2_+$ is a fixed point (regardless of the parameter value) which is a sink for $a/\lambda \leq c/d$ (Figure 3). In this case, there are no fixed points in $\mathring{\mathbb{R}}^2_+$ and the global attractor of the flow is the closed interval $[0, a/\lambda] \times \{0\}$ contained in $\partial\mathbb{R}^2_+$. As a consequence, the extinction of one of the populations takes place. This situation is, in a certain sense, the opposite of that described for uniformly persistent flows.

Motivated by the situation just described, we present some results that ensure that the global attractor of a dissipative flow defined on $\mathbb{R}^2_+$ is contained in $\partial\mathbb{R}^2_+$.

Theorem 5.2 (Barge–Sanjurjo [6]). *Suppose that $\varphi : \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}^2_+$ is a flow without equilibria in $\mathring{\mathbb{R}}^2_+$. Then, the ω-limit (resp. the ω^*-limit) of any point, when non-empty, is entirely composed of fixed points and, hence, it is contained in $\partial\mathbb{R}^2_+$. If, in addition, the fixed point set is bounded and totally disconnected, then the ω-limit (resp. the ω^*-limit) of each trajectory, when non-empty, is a singleton. Moreover, if the flow is dissipative, the following hold.*

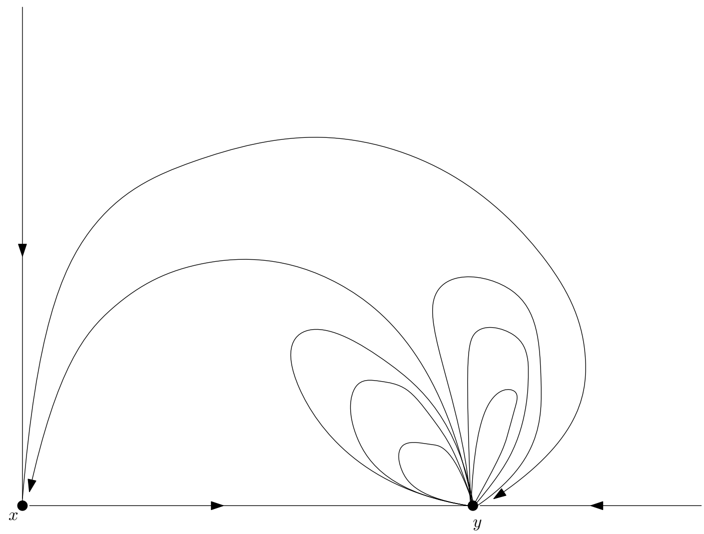

Figure 4. Dissipative flow in $\mathbb{R}^2_+$ with only two fixed points x and y, both contained in $\partial\mathbb{R}^2_+$ and such that $I(x) = 4$ and $I(y) = +\infty$.

(i) *Given $x \in \mathbb{R}^2_+$, if $\gamma^-(x)$ is bounded so is $\gamma(x)$. Hence, both $\omega(x)$ and $\omega^*(x)$ are non-empty and entirely composed of fixed points.*

(ii) *Let K be the global attractor of $\varphi|_{\partial\mathbb{R}^2_+}$. Then, K is the global attractor of φ if and only if K is isolated for φ or, equivalently, $\gamma^-(x)$ is unbounded for each $x \in \mathring{\mathbb{R}}^2_+$.*

Remark. The fact that $\omega(x)$ is composed of fixed points for discrete systems of the disc having all the fixed points in the boundary was proved by Campos, Ortega, and Tineo in [13] by using some ideas of Brown [12] and a classical result of Brouwer (see [22]) on homeomorphisms of the plane. The proof of the previous result, that can be seen in [6], makes use of the Poincaré–Bendixson theorem.

Let $\Gamma_B(\varphi)$ be the set of bounded trajectories of φ and let x be an equilibrium point. We define

$$\Gamma(x) := \{\gamma \in \Gamma_B \mid x \in \omega(\gamma) \cup \omega^*(\gamma)\}.$$

Definition 5.3. Let x be an equilibrium point. We define the index $I(x) \in \mathbb{N} \cup \{+\infty\}$ to be $k \in \mathbb{N}$ if the cardinal of $\Gamma(x)$ is k and $I(x) = +\infty$ if the cardinal of $\Gamma(x)$ is not finite.

Remark. For each $k \in \mathbb{N} \cup \{+\infty\}$ there exists a flow on $\mathbb{R}^2_+$ with all its equilibria contained in $\partial\mathbb{R}^2_+$ and having a fixed point x such that $I(x) = k$. In Figure 4, a flow having a fixed point of index 4 is depicted.

Theorem 5.4 (Barge–Sanjurjo [6]). *Suppose that $\varphi : \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}^2_+$ is a dissipative flow having a countable amount of fixed points, all of them contained in $\partial\mathbb{R}^2_+$. Then, the global attractor of the flow is in the boundary if and only if all the fixed points have finite index. In such a case, for each fixed point x, $I(x)$ is either 1, 2 or 3. Moreover, if the index of an isolated fixed point x takes the value 1, then $\{x\}$ is the global attractor.*

Acknowledgments. The authors thank the referee for their careful reading of the manuscript and helpful remarks.

Funding. Research supported by the Spanish Ministerio de Ciencia e Innovación grants PGC2018-098321-B-I00 and PID2021-126124NB-I00.

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