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Siu's lemma: Generalizations and applications

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Abstract. In this survey paper, we present some generalizations of Siu's lemma related to multiplier ideal sheaves and discuss their applications in some problems related to optimal L^2 extension, comparison between singular metrics on exceptional fibers of twisted relative pluricanonical bundles, and subadditivity of Kodaira–Iitaka dimensions with multiplier ideal sheaves. We also discuss some ideas in the proofs.

1. Introduction

L^2 extensions with precise estimates and multiplier ideal sheaves related to plurisubharmonic (psh) functions and their singularities have been playing a useful role in recent progress in several complex variables and complex geometry. Siu's lemma deals with multiplier ideal sheaves which is an important invariant of the singularities of the psh functions and quite closely related to L^2 extensions. In the present paper, we will outline recent progress on the generalizations and applications of Siu's lemma. Before we present our main results, let us first recall some notions and notations (see [7–9, 18, 20, 25]), which will be used in this paper.

Let X be a complex manifold. A function $\varphi : X \to [-\infty, +\infty)$ is said to be *quasi-plurisubharmonic* (quasi-psh) if φ is locally the sum of a psh function and a smooth function.

A *singular (Hermitian) metric* h of a holomorphic line bundle L over X is simply a Hermitian metric which is expressed locally as $e^{-\varphi}$ with respect to local holomorphic trivialization of L such that $\varphi \in L^1_{\mathrm{loc}}$. The curvature current $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\partial\bar{\partial}\varphi$ is well defined on X. A holomorphic line bundle L is called pseudoeffective if it is endowed with a singular Hermitian metric h with positive or semipositive curvature current (i.e., φ is psh in the sense of distribution). In particular, L is called a positive line bundle if φ is smooth strictly psh; L is called a big line bundle if the

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curvature current is a Kähler current, i.e., $\Theta \geq \varepsilon\omega$ for some $\varepsilon > 0$, where ω is the $(1,1)$ form associated to a Kähler metric.

A quasi-psh function φ on X is said to have *(neat) analytic singularities* if every point $x \in X$ possesses an open neighborhood U on which φ can be written as

$$\varphi = c \log \sum_{1 \leq j \leq j_0} |g_j|^2 + u,$$

where c is a nonnegative number, $g_j \in \mathcal{O}_X(U)$, and u is bounded on U ($u \in C^\infty(U)$).

1.1. Multiplier ideal sheaf

For any quasi-psh function φ on X, the L^p *multiplier ideal sheaf* ($0 < p < +\infty$) is defined by

$$\mathcal{I}_{L^p}(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x};\ \exists U \ni x \text{ such that } \int_U |f|^p e^{-\varphi}\, d\lambda < +\infty \right\},$$

where $U \subset X$ is a coordinate chart, and $d\lambda$ is the Lebesgue measure. $\mathcal{I}_{L^2}(\varphi)$ is just the usual multiplier ideal sheaf $\mathcal{I}(\varphi)$.

We will write the $L^{\frac{2}{m}}$ multiplier ideal sheaf $\mathcal{I}_{L^{\frac{2}{m}}}(\varphi)$ (m is a positive integer) by $\mathcal{I}_m(\varphi)$ for simplicity.

We list some basic properties of the multiplier ideal sheaves as follows.

(1) Nadel's theorem: $\mathcal{I}(\varphi)$ is coherent.

Consequently, the nonlocally integrable point set of $e^{-\varphi}$ (= the zero set of $\mathcal{I}(\varphi) = \operatorname{supp} \mathcal{O}/\mathcal{I}(\varphi)$) is an analytic set, since the support of a coherent analytic sheaf is an analytic set.

(2) Theorem: A multiplier ideal sheaf is integrally closed, i.e., the integral closure of $\mathcal{I}(\varphi)$ is itself.

(3) Nadel's vanishing theorem: Let $(L, e^{-\varphi})$ be a big line bundle on a compact Kähler manifold X. Then

$$H^q\big(X, K_X \otimes L \otimes \mathcal{I}(\varphi)\big) = 0,$$

for any $q \geq 1$.

Recently, a new property of the multiplier ideal sheaves, i.e., the strong openness of the multiplier ideal sheaves, is established by the solution of Demailly's strong openness conjecture [13]. The solution and its applications are based on the above basic properties of the multiplier ideal sheaves.

Demailly's strong openness conjecture. For any psh function φ on X, one has

$$\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\varepsilon > 0} \mathcal{I}\big((1+\varepsilon)\varphi\big) = \mathcal{I}\big((1+\varepsilon_0)\varphi\big).$$

The last equality is well known by the Noetherian property of the coherent analytic sheaves. This conjecture was also stated by Y. T. Siu [25], Demailly–Kollár [10] and many others. For a reformulation of the conjecture, see Theorem 3.2.

It should be noted that $(1+\varepsilon)\varphi$ in the conjecture could be replaced by any increasing sequence of psh functions which converges to φ [13]; more generally, stability of the multiplier ideal sheaves holds by Guan–Li–Zhou [15] based on [12]. Strong openness also holds for L^p multiplier ideal sheaves for $0 < p < \infty$; it follows from the strong openness for L^2 multiplier ideal sheaves and Hölder's inequality; see Fornaess [11]. By the strong openness, it follows that L^p multiplier ideal sheaves are coherent [6, 25].

1.2. Optimal L^2 extension

In [21], Ohsawa and Takegoshi obtained the following L^2 extension theorem with psh weights.

Theorem 1.1 ([21]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, φ a psh function on Ω, and s a holomorphic function on Ω. Let*

$$H := \{x \in \Omega;\ s(x) = 0\}.$$

Assume that $|s| \leq 1$ on Ω and $ds \not\equiv 0$ on H. Then there exists an absolute constant C such that, for every holomorphic function f on H satisfying

$$\int_H \frac{|f|^2 e^{-\varphi}}{|ds|^2}\, dV_H < +\infty,$$

there exists a holomorphic function F on Ω satisfying $F = f$ on H and

$$\int_\Omega |F|^2 e^{-\varphi}\, d\lambda_n \leq C \int_H \frac{|f|^2 e^{-\varphi}}{|ds|^2}\, dV_H,$$

where $d\lambda_n$ is the $2n$-dimensional Lebesgue measure, and dV_H is the $2(n-1)$-dimensional Hausdorff measure on H.

Unifying various L^2 extension theorems, a general L^2 extension theorem with precise estimate for the almost Stein case and its geometric meaning was established and discovered in [14]. Later on, the optimal L^2 extension theorem with singular metrics for the Kähler case was obtained in [6, 30] by using Guan–Zhou's work on optimal L^2 extension and strong openness of multiplier ideal sheaves, and the generalized Siu's lemma stated below plays also a key role in [30].

1.3. Siu's lemma

In the study of algebraic geometry problems such as Fujita's conjecture [1, 24], Siu obtained the semi-continuity of multiplier ideal sheaves and the following lower limit property about integrals with psh weights having trivial multiplier ideal sheaves by using Theorem 1.1.

Theorem 1.2 (Siu's lemma; see [22]). *Let $\varphi(z', z'')$ be a nonpositive psh function on $\mathbb{B}^1_r \times \mathbb{B}^{n-1}_r$ such that*

$$\int_{z'' \in \mathbb{B}^{n-1}_r} e^{-\varphi(0,z'')}\, d\lambda_{n-1} < +\infty, \tag{1.1}$$

where $\mathbb{B}^{n-1}_r$ denotes the open ball in $\mathbb{C}^{n-1}$ centered at 0 with radius $r > 0$, and $d\lambda_{n-1}$ denotes the $2(n-1)$-dimensional Lebesgue measure. Assume that $r_1 \in (0, r)$. Then there exists a positive number C independent of φ, such that

$$\varliminf_{z' \to 0} \int_{z'' \in \mathbb{B}^{n-1}_{r_1}} e^{-\varphi(z',z'')}\, d\lambda_{n-1} \le C \int_{z'' \in \mathbb{B}^{n-1}_r} e^{-\varphi(0,z'')}\, d\lambda_{n-1}. \tag{1.2}$$

The inequality (1.1) means that φ restricted to the center fiber has a trivial multiplier ideal sheaf.

Siu's lemma was also used by Phong and Sturm [22] to obtain a holomorphic stability result for 1-parameter deformations; i.e., for any nonpositive psh function $\varphi(z', z'')$ which has neat analytic singularities and satisfies (1.1), the stronger equality

$$\lim_{z' \to 0} \int_{z'' \in \mathbb{B}^{n-1}_{r_1}} e^{-\varphi(z',z'')}\, d\lambda_{n-1} = \int_{z'' \in \mathbb{B}^{n-1}_{r_1}} e^{-\varphi(0,z'')}\, d\lambda_{n-1} \tag{1.3}$$

holds.

In general, one could not expect that (1.3) holds for a general nonpositive psh function φ which satisfies (1.1) but does not have neat analytic singularities (one can see [29] for a simple counterexample).

Theorem 1.3 (Lemma on the semi-continuity of multiplier ideal sheaves; [1, 24]). *The limit of the zero-sets of the multiplier ideal sheaves defined by a holomorphic family of multivalued holomorphic sections contains the zero-set of the limit.*

For a concrete explanation of the above result, the reader is referred to [1, Lemma 6.1] and [24, Section 3]. The above two lemmas follow from the L^2 extension theorem.

An equivalent version of the above semi-continuity is as follows. Let $\varphi(z', z'')$ be a psh function on $\Delta^n \times \Delta^m$. If $e^{-\varphi(z',z'')}$ is not integrable at $z' = 0$ for almost all $z'' \in \Delta^m \setminus 0$, then $e^{-\varphi(z',0)}$ is not integrable at $z' = 0$.

The generalized Siu lemma stated below implies both Siu's lemma and Siu's semi-continuity of multiplier ideal sheaves which seem to not imply each other, and could be regarded as a unified version of both properties.

1.4. Main content of the present paper

In [29], we generalized Siu's lemma by proving a limit property, which implies Siu's lemma with an optimal estimate. In [32], we further generalized Siu's lemma to the case that the multiplier ideal sheaf of φ is not necessarily trivial when restricted to the center fiber.

Moreover, we used in [32] the generalization of Siu's lemma with nontrivial multiplier ideal sheaves to prove a refined optimal L^2 extension theorem with singular metrics in the Kähler case. As another application, we gave in [32] a positive answer to a comparison question posed by Berndtsson–Păun [5] and Păun–Takayama [23] about singular metrics on exceptional fibers of twisted relative pluricanonical bundles.

By using optimal L^2 extension theorem with singular metrics in the Kähler case, one can prove the positivity or pseudoeffectivity of twisted relative pluricanonical bundles with singular metrics in the Kähler case (see [6, 30, 32] for the Kähler case, and see also [3, 5, 23] for the projective case). This positivity can be used to study the subadditivity of Kodaira–Iitaka dimensions for Kähler fibrations (see [27, 33]).

We also proved in [33] a more general version of Siu's lemma with nontrivial multiplier ideal sheaves near a subvariety, which generalizes the submanifold case obtained in [32]. This result gives a relation between two measures used in previous L^2 extension theorems in [9, 14, 19, 31, 34].

In the rest sections, we will discuss the above results explicitly.

2. A generalization of Siu's lemma with trivial multiplier ideal sheaves

Under similar assumptions as in Siu's lemma (Theorem 1.2), we proved the following limit property, which is a generalization of Siu's lemma.

Theorem 2.1 ([29]). *Let $\varphi(z', z'')$ be a psh function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$ $(1 \le k \le n)$ such that*

$$\int_{z'' \in \mathbb{B}_r^{n-k}} e^{-\varphi(0,z'')}\, d\lambda_{n-k} < +\infty.$$

Let P be a nonnegative continuous function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$. Assume that $r_1 \in (0, r)$. Then

$$\lim_{\varepsilon \to 0} \frac{1}{\lambda_k(\mathbb{B}_\varepsilon^k)} \int_{\mathbb{B}_\varepsilon^k \times \mathbb{B}_{r_1}^{n-k}} P(z', z'') e^{-\varphi(z',z'')}\, d\lambda_n = \int_{z'' \in \mathbb{B}_{r_1}^{n-k}} P(0, z'') e^{-\varphi(0,z'')}\, d\lambda_{n-k}, \tag{2.1}$$

where $\lambda_k(\mathbb{B}_\varepsilon^k) :=$ the $2k$-dimensional Lebesgue measure of $\mathbb{B}_\varepsilon^k$.

It is easy to see that (2.1) implies that (1.2) holds with $C = 1$.

The following two results are used in the proof of Theorem 2.1.

Lemma 2.2 ([22, 29]). *Let $\varphi(z', z'')$ be a negative psh function on $\mathbb{B}^k_\delta \times \mathbb{B}^{n-k}_\delta$ $(1 \le k \le n, \delta > 0)$ such that*

$$I_\varphi := \int_{z'' \in \mathbb{B}^{n-k}_\delta} e^{-\varphi(0,z'')}\, d\lambda_{n-k} < +\infty \quad (I_\varphi := e^{-\varphi(0)} \text{ if } k = n).$$

Assume that $r_1 \in (0, \delta)$. Then there exist two positive numbers C and $\varepsilon_\varphi \in (0, r_1]$ (C is independent of φ), such that

$$\frac{1}{\varepsilon^{2k}} \int_{\mathbb{B}^k_\varepsilon \times \mathbb{B}^{n-k}_{r_1}} e^{-\varphi(z',z'')}\, d\lambda_n \le C I_\varphi^{k+1}$$

for all $\varepsilon \in (0, \varepsilon_\varphi]$ (z'' and $\mathbb{B}^{n-k}_{r_1}$ will disappear if $k = n$).

Theorem 2.3 ([2]). *Let φ be a psh function on the ball $\mathbb{B}^n_r$ of $\mathbb{C}^n$ centered at 0 with radius r. Assume that*

$$\int_{\mathbb{B}^n_r} e^{-\varphi}\, d\lambda_n < +\infty.$$

Let $\delta \in (0, r)$. Then there exists $\beta > 0$ such that

$$\int_{\mathbb{B}^n_\delta} e^{-(1+\beta)\varphi}\, d\lambda_n < +\infty.$$

The idea in our proof of Theorem 2.1 consists of two steps.

The first step is to control the integral near the set $\{\varphi = -\infty\}$, which can be completed by using the openness property of multiplier ideal sheaves (Theorem 2.3) and a variation of Siu's lemma (Lemma 2.2).

The second step is to prove that Theorem 2.1 holds for φ which is bounded, which can be completed by mainly using Lebesgue's dominated convergence theorem.

To be more precise, let r, r_1 be as in Theorem 2.1 and denote $\frac{r+r_1}{2}$ by δ. Then $r_1 < \delta < r$ and we obtain from Theorem 2.3 that

$$\int_{z'' \in \mathbb{B}^{n-k}_\delta} e^{-(1+\beta)\varphi(0,z'')}\, d\lambda_{n-k} < +\infty$$

for some positive number β.

Then applying Lemma 2.2 to the psh function $(1 + \beta)\varphi$, we have

$$\frac{1}{\varepsilon^{2k}} \int_{\mathbb{B}^k_\varepsilon \times \mathbb{B}^{n-k}_{r_1}} e^{-(1+\beta)\varphi(z',z'')}\, d\lambda_n \le C \tag{2.2}$$

for all small enough ε, where C is a positive constant independent of ε.

Let v be a positive integer. Then (2.2) implies that

$$\frac{1}{\lambda(\mathbb{B}_\varepsilon^k)} \int_{\{\varphi \le -v\} \cap (\mathbb{B}_\varepsilon^k \times \mathbb{B}_{r_1}^{n-k})} P(z', z'') e^{-\varphi(z', z'')} \, d\lambda_n \le C_1 e^{-\beta v}$$

for all small enough ε, where C_1 is a positive constant independent of ε.

Hence the integral near $\{\varphi = -\infty\}$ is uniformly small if v is sufficiently large, and we complete the first step.

Set $\varphi_v = \max\{\varphi, -v\}$. The second step is to prove

$$\lim_{\varepsilon \to 0} \frac{1}{\lambda(\mathbb{B}_\varepsilon^k)} \int_{\mathbb{B}_\varepsilon^k \times \mathbb{B}_{r_1}^{n-k}} P(z', z'') e^{-\varphi_v(z', z'')} \, d\lambda_n = \int_{z'' \in \mathbb{B}_{r_1}^{n-k}} P(0, z'') e^{-\varphi_v(0, z'')} \, d\lambda_{n-k},$$

which can be obtained by mainly using Lebesgue's dominated convergence theorem (see [29] for the details).

3. A generalization of Siu's lemma with nontrivial multiplier ideal sheaves

In both Theorem 1.2 and Theorem 2.1, the multiplier ideal sheaf of φ is trivial when restricted to the center fiber. It is natural to consider the case when the multiplier ideal sheaf is nontrivial.

In [32], we obtained the following generalization of Siu's lemma for psh functions having nontrivial multiplier ideal sheaves when restricted to the center fiber.

Theorem 3.1 ([32,35]). *Let $p \in (0, 2]$. Let $\varphi(z', z'')$ be a psh function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$ $(1 \le k \le n)$, let $P(z', z'')$ be a nonnegative continuous function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$, let $M(z')$ be a bounded nonnegative measurable function on $\mathbb{C}^k$ with compact support, and let $f(z'')$ be a holomorphic function on $\mathbb{B}_r^{n-k}$ satisfying*

$$\int_{z'' \in \mathbb{B}_r^{n-k}} |f(z'')|^p e^{-\varphi(0, z'')} \, d\lambda_{n-k} < +\infty.$$

Assume that $r_1, r_2 \in (0, r)$ and $r_1 < r_2$. Let β be a positive number such that

$$I_\beta := \int_{z'' \in \mathbb{B}_{r_2}^{n-k}} |f(z'')|^p e^{-(1+\beta)\varphi(0, z'')} \, d\lambda_{n-k} < +\infty \tag{3.1}$$

and $\alpha \in (1 - \frac{p}{2k}\beta, 1) \cap [0, 1)$. Then there exists a holomorphic function $F(z', z'')$ on

$\mathbb{B}_r^k \times \mathbb{B}_{r_2}^{n-k}$ *such that* $F(0, z'') = f(z'')$ *on* $\mathbb{B}_{r_2}^{n-k}$,

$$\int_{(z',z'')\in\mathbb{B}_r^k\times\mathbb{B}_{r_2}^{n-k}} \frac{\left|F(z',z'')\right|^p e^{-(1+\beta)\varphi(z',z'')}}{|z'|^{2k\alpha}}\, d\lambda_n < +\infty, \tag{3.2}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0+} &\int_{(z',z'')\in\mathbb{C}^k\times\mathbb{B}_{r_1}^{n-k}} \frac{1}{\varepsilon^{2k}} M\left(\frac{z'}{\varepsilon}\right) P(z',z'')\left|F(z',z'')\right|^p e^{-\varphi(z',z'')}\, d\lambda_n \\ &= \int_{z'\in\mathbb{C}^k} M(z')\, d\lambda_k \int_{z''\in\mathbb{B}_{r_1}^{n-k}} P(0,z'')\left|f(z'')\right|^p e^{-\varphi(0,z'')}\, d\lambda_{n-k}.\end{aligned} \tag{3.3}$$

Moreover, any holomorphic extension F *of* f *satisfying* (3.2) *has the property* (3.3).

The existence of β in Theorem 3.1 is guaranteed by the strong openness property of multiplier ideal sheaves, i.e., Theorem 3.2 below.

Theorem 3.2 ([13]). *Let* $p \in (0, +\infty)$. *Let* φ *be a psh function on the unit ball* $\mathbb{B}_1^n$ *of* $\mathbb{C}^n$. *Assume that* F *is a holomorphic function on* $\mathbb{B}_1^n$ *satisfying*

$$\int_{\mathbb{B}_1^n} |F|^p e^{-\varphi}\, d\lambda_n < +\infty.$$

Then there exists $r \in (0, 1)$ *and* $\beta \in (0, +\infty)$ *such that*

$$\int_{\mathbb{B}_r^n} |F|^p e^{-(1+\beta)\varphi}\, d\lambda_n < +\infty.$$

In [32], we proved Theorem 3.1 by developing the method established in [30] and using the iteration method in [4] or [5].

The existence of a holomorphic extension F satisfying (3.2) can be obtained by using L^2 extension theorems. The main property that needs to be proved is (3.3).

The key step in our proof of (3.3) is to construct holomorphic functions F_ε on $\mathbb{B}_{r_2}^k \times \mathbb{B}_{r_2}^{n-k}$ such that $F_\varepsilon = f$ on $\{0\} \times \mathbb{B}_{r_2}^{n-k}$,

$$\int_{\mathbb{B}_\varepsilon^k\times\mathbb{B}_{r_2}^{n-k}} |F_\varepsilon|^p e^{-(1+\beta)\varphi}\, d\lambda_n \le C_1\varepsilon^{2k}, \tag{3.4}$$

and

$$\int_{\mathbb{B}_{r_2}^k\times\mathbb{B}_{r_2}^{n-k}} |F_\varepsilon|^p e^{-(1+\beta)\varphi}\, d\lambda_n \le C_2\varepsilon^{-2\beta_1} \tag{3.5}$$

for any $\varepsilon \in (0, r_2)$, where $\beta_1 \in (0, \frac{p}{2})$ is a small enough positive number.

Then by some calculation, we can get

$$\int_{\mathbb{B}_\varepsilon^k\times\mathbb{B}_{r_2}^{n-k}} \frac{|F-F_\varepsilon|^p e^{-(1+\beta_2)\varphi}}{\varepsilon^{2k}}\,d\lambda_n \le C_3 \tag{3.6}$$

for all ε small enough, where $\beta_2 \in (0,\beta)$ is a small enough positive number, and C_3 is a positive number independent of ε.

Then we get the desired property (3.3) by using (3.4), (3.6), and the following proposition.

Proposition 3.3 ([30]). *Let $\varphi(z',z'')$ be a psh function on $\mathbb{B}_r^k\times\mathbb{B}_r^{n-k}$, $f(z'')$ a holomorphic function on $\mathbb{B}_r^{n-k}$, $P(z',z'')$ a nonnegative continuous function on $\mathbb{B}_r^k\times\mathbb{B}_r^{n-k}$, and $M(z')$ a bounded nonnegative measurable function on $\mathbb{C}^k$ with compact support. Let C, β_2, r_1, and r_2 be positive numbers. Assume that $r_1<r_2<r$. Suppose that F is a holomorphic function on $\mathbb{B}_{r_2}^k\times\mathbb{B}_{r_2}^{n-k}$ satisfying $F(0,z'')=f(z'')$ $(\forall z''\in\mathbb{B}_{r_2}^{n-k})$,*

$$\sup_{\mathbb{B}_{r_2}^k\times\mathbb{B}_{r_2}^{n-k}} |F| \le C,$$

and

$$\overline{\lim_{\varepsilon\to 0+}}\ \frac{1}{\varepsilon^{2k}}\int_{(z',z'')\in\mathbb{B}_\varepsilon^k\times\mathbb{B}_{r_2}^{n-k}} \left|F(z',z'')\right|^p e^{-(1+\beta_2)\varphi(z',z'')}\,d\lambda_n \le C.$$

Then

$$\lim_{\varepsilon\to 0+}\int_{(z',z'')\in\mathbb{C}^k\times\mathbb{B}_{r_1}^{n-k}} \frac{1}{\varepsilon^{2k}} M\left(\frac{z'}{\varepsilon}\right) P(z',z'')\left|F(z',z'')\right|^p e^{-\varphi(z',z'')}\,d\lambda_n$$
$$= \int_{z'\in\mathbb{C}^k} M(z')\,d\lambda_k \int_{z''\in\mathbb{B}_{r_1}^{n-k}} P(0,z'')\left|f(z'')\right|^p e^{-\varphi(0,z'')}\,d\lambda_{n-k}.$$

4. A refined optimal L^2 extension theorem with singular metrics on Kähler manifolds

Now we regard Theorem 3.1 as a local property on coordinate charts, and discuss a global version of it on complex manifolds.

Let $\mathfrak{R}$ be the class of functions defined by

$$\Big\{R\in C^\infty(-\infty,0];\ R>0,\ R \text{ is decreasing},\ C_R := \int_{-\infty}^0 \frac{1}{R(t)}\,dt < +\infty$$
$$\text{and } e^t R(t) \text{ is bounded above on } (-\infty,0]\Big\}.$$

By using Theorems 3.1 and 3.2, we obtained the following refined optimal L^2 extension theorem with singular metrics on Kähler manifolds as a global version of Theorem 3.1.

Theorem 4.1 ([32]). *Let $R \in \mathfrak{R}$ and let $\Pi : X \to S$ be a surjective proper holomorphic map from a Kähler manifold (X, ω) of dimension n to a Stein domain S contained in the unit ball $\mathbb{B}^k \subset \mathbb{C}^k$ $(1 \le k \le n)$, where ω is a Kähler metric on X. With respect to the coordinate functions on $\mathbb{C}^k$, we regard Π as a k-dimensional vector of holomorphic functions $s = (s_1, \ldots, s_k)$ on X. Assume that $0 \in S$ and $ds := ds_1 \wedge \cdots \wedge ds_k$ is nonvanishing on*

$$X_0 := \{x \in X;\ s(x) = 0\}.$$

Let (L, h) be a holomorphic line bundle over X equipped with a singular Hermitian metric h such that the curvature current $\sqrt{-1}\Theta_{L,h} \ge 0$. Write h as $h_1 e^{-\phi}$, where h_1 is any fixed smooth metric of L and ϕ is a global quasi-psh function on X. Assume that $f \in H^0(X_0, K_X|_{X_0} + L|_{X_0})$ satisfies

$$\int_{X_0} \frac{|f|^2_{\omega,h}}{|ds|^2_\omega} \, dV_{X_0,\omega_0} < +\infty,$$

where $dV_{X_0,\omega_0} := \frac{\omega_0^{n-k}}{(n-k)!}$ and ω_0 is the Kähler metric on X_0 induced from ω. Let β be a positive number such that

$$\mathcal{I}\big((1+\beta)\phi + \log|s|^{2k}\big)_x = \mathcal{I}\big(\phi + \log|s|^{2k}\big)_x$$

and

$$f \in \mathcal{I}\big((1+\beta)\phi|_{X_0}\big)_x$$

hold for any $x \in X_0$ (the existence of β is guaranteed by Theorem 3.2). Then there exists $F \in H^0(X, K_X + L)$ such that $F = f$ on X_0,

$$F \in \mathcal{I}\big((1+\beta)\phi + \alpha \log|s|^{2k}\big)_x \quad \big(\forall \alpha \in [0,1),\ \forall x \in X_0\big), \tag{4.1}$$

$$\int_X \frac{|F|^2_{\omega,h}}{|s|^{2k} R\big(\log|s|^{2k}\big)} \, dV_{X,\omega} \le C_R \frac{(2\pi)^k}{k!} \int_{X_0} \frac{|f|^2_{\omega,h}}{|ds|^2_\omega} \, dV_{X_0,\omega_0}, \tag{4.2}$$

and

$$\lim_{\varepsilon \to 0+} \int_X \frac{1}{\varepsilon^{2k}} M\Big(\frac{s}{\varepsilon}\Big) P |F|^2_{\omega,h} \, dV_{X,\omega} = 2^k \int_{z' \in \mathbb{C}^k} M(z') \, d\lambda_k \int_{X_0} \frac{P|f|^2_{\omega,h}}{|ds|^2_\omega} \, dV_{X_0,\omega_0}, \tag{4.3}$$

where M is as in Theorem 3.1, and P is any nonnegative continuous function on X.

One can see [6, 9, 28, 30] for many L^2 extension theorems with singular metrics on Kähler manifolds without the properties (4.1) and (4.3).

The properties (4.1) and (4.3) refine the optimal L^2 extension theorem with singular metrics on Kähler manifolds discussed in [30]. In our proof of Theorem 4.1 (even without (4.1) and (4.3)), a key step is to use some construction essentially used in the proof of Theorem 3.1 in the case $p = 2$ (see (3.4), (3.5), and [30]).

Furthermore, applying the iteration method in [4] or [5], the following refined optimal $L^{\frac{2}{m}}$ extension theorem with singular metrics on Kähler manifolds can be obtained from Theorem 4.1.

Theorem 4.2 ([32]). *Let R, Π, (X,ω), S, (L,h), (X_0,ω_0), s, ds, h_1, ϕ, M, and P be the same as in Theorem 4.1. Let $h_\omega := (dV_{X,\omega})^{-1}$ (it defines a smooth Hermitian metric on K_X). Assume that $f \in H^0(X_0, mK_X|_{X_0} + L|_{X_0})$ (m is a positive integer) satisfies*

$$C_f := \int_{X_0} \frac{|f|^{\frac{2}{m}}_{h_\omega^{\otimes m}\otimes h}}{|ds|^2_\omega}\, dV_{X_0,\omega_0} < +\infty,$$

and assume that there exists a holomorphic extension F_1 of f to an open neighborhood of X_0 in X. Then there exists a positive number β and a holomorphic section $F \in H^0(X, mK_X + L)$ such that $F = f$ on X_0,

$$F \in \mathcal{I}_m\Big((1+\beta)\frac{\phi}{m} + \alpha \log |s|^{2k}\Big)_x \quad \big(\forall \alpha \in [0,1),\ \forall x \in X_0\big), \tag{4.4}$$

$$\int_X \frac{|F|^{\frac{2}{m}}_{h_\omega^{\otimes m}\otimes h}}{|s|^{2k} R\big(\log |s|^{2k}\big)}\, dV_{X,\omega} \le C_R \frac{(2\pi)^k}{k!} C_f, \tag{4.5}$$

and

$$\lim_{\varepsilon\to 0+} \int_X \frac{1}{\varepsilon^{2k}} M\Big(\frac{s}{\varepsilon}\Big) P |F|^{\frac{2}{m}}_{h_\omega^{\otimes m}\otimes h}\, dV_{X,\omega}$$
$$= 2^k \int_{z'\in\mathbb{C}^k} M(z')\, d\lambda_k \int_{X_0} \frac{P|f|^{\frac{2}{m}}_{h_\omega^{\otimes m}\otimes h}}{|ds|^2_\omega}\, dV_{X_0,\omega_0}. \tag{4.6}$$

5. Comparison of singular metrics on exceptional fibers of twisted relative pluricanonical bundles

Theorem 4.2 can be used to prove the positivity of the twisted relative pluricanonical bundles (Theorem 5.1), and it can also be used to obtain a comparison result of singular metrics on exceptional fibers of twisted relative pluricanonical bundles (Theorem 5.2).

Let X be an n-dimensional Kähler manifold, Y a k-dimensional connected complex manifold ($1 \le k \le n$), and (L, h) a pseudoeffective holomorphic line bundle over X.

Let $\Pi : X \to Y$ be a surjective proper holomorphic map. Denote by Y_0 the set of all points in Y which are regular values of Π. Let $X_y := \Pi^{-1}(y)$, $L_y := L|_{X_y}$, $h_y := h|_{X_y}$, $Y_h := \{y \in Y_0;\ h_y \not\equiv +\infty\}$ and

$$Y_{m,\text{ext}} := \left\{y \in Y_0;\ \dim H^0(X_y, mK_{X_y} + L_y) = \operatorname{rank} \Pi_*(mK_{X/Y} + L)\right\}.$$

Then $Y_{m,\text{ext}}$ is the Zariski open subset of Y consisting of all $y \in Y_0$ such that every section in $H^0(X_y, mK_{X_y} + L_y)$ has a holomorphic extension to some open neighborhood of X_y in X. Denote $\Pi^{-1}(Y_{m,\text{ext}})$ by $X_{m,\text{ext}}$.

Denote by ω_y the Kähler metric on X_y induced by ω. Let $dV_{X_y,\omega_y} := \frac{\omega_y^{n-k}}{(n-k)!}$ and let $h_{\omega_y} := (dV_{X_y,\omega_y})^{-1}$ (it defines a smooth metric on K_{X_y}).

For every $y \in Y_{m,\text{ext}}$ and every $x \in X_y$, by the isomorphism

$$(mK_{X/Y} + L)|_{X_y} \simeq mK_{X_y} + L_y,$$

the *relative m-Bergman kernel* $B^o_{m,X/Y}$ of the line bundles $(mK_{X/Y} + L)|_{X_{m,\text{ext}}}$ is defined as

$$B^o_{m,X/Y}(x) := \sup\left\{u(x) \otimes \overline{u(x)};\ u \in H^0(X_y, mK_{X_y} + L_y)\right.$$
$$\left.\text{and} \int_{X_y} |u|^{\frac{2}{m}}_{h_{\omega_y}^{\otimes m} \otimes h_y}\, dV_{X_y,\omega_y} \le 1\right\}.$$

Assume that $B^o_{m,X/Y} \not\equiv 0$. Then the following positivity of the twisted relative pluricanonical bundles $mK_{X/Y} + L$ holds. The projective case was proved in [3, 5, 23]. We give a proof of the Kähler case in [32] by using Theorem 4.2 (see also [6, 30] for the Kähler case).

Theorem 5.1 ([3, 5, 6, 23, 30, 32]). *The metric $(B^o_{m,X/Y})^{-1}$ is a singular metric on $(mK_{X/Y} + L)|_{X_{m,\text{ext}}}$ with semipositive curvature current. Moreover, $(B^o_{m,X/Y})^{-1}$ extends across $X \setminus X_{m,\text{ext}}$ uniquely to a singular metric $(B_{m,X/Y})^{-1}$ on $mK_{X/Y} + L$ with semipositive curvature current on all of X.*

Theorems 4.1 and 5.1 can be used to prove the positivity of the direct images of twisted relative pluricanonical bundles with singular metrics for Kähler fibrations (see [32], and see also [3, 16, 23] for the projective case).

Now we discuss comparison of singular metrics on exceptional fibers of twisted relative pluricanonical bundles.

For $y \in Y_0 \setminus Y_{m,\mathrm{ext}}$, the fiber X_y is called an exceptional fiber. The metric $(B_{m,X/Y})^{-1}$ over the exceptional fibers X_y is defined as the unique extension of $(B^o_{m,X/Y})^{-1}$.

There is an extremal metric $(B_{m,y})^{-1}$ on the bundle $mK_{X_y} + L_y$ over the exceptional fibers X_y defined as

$$B_{m,y}(x) := \sup \Big\{ u(x) \otimes \overline{u(x)};\ u \in H^0(X_y, mK_{X_y} + L_y), \int_{X_y} |u|^{\frac{2}{m}}_{h_{\omega_y}^{\otimes m} \otimes h_y} \, dV_{X_y,\omega_y} \leq 1 \text{ and } u \text{ has a holomorphic extension to some open neighborhood of } X_y \text{ in } X \Big\}.$$

When the metric h on L is continuous, the inequality

$$(B_{m,X/Y})^{-1}|_{X_y} \leq (B_{m,y})^{-1} \quad (\forall y \in Y_0 \setminus Y_{m,\mathrm{ext}}) \tag{5.1}$$

was obtained in [5,23] in the projective case. When h has arbitrary singularities, (5.1) was guessed in [5,23].

By using Theorem 4.2, we obtained the following result, which shows that (5.1) is actually an equality in the Kähler case for those h with arbitrary singularities.

Theorem 5.2 ([32]). *$(B_{m,X/Y})^{-1}|_{X_y} = (B_{m,y})^{-1}$ holds for any $y \in Y_0 \setminus Y_{m,\mathrm{ext}}$.*

The key point in the proof of Theorem 5.2 is to obtain a holomorphic extension whose $L^{\frac{2}{m}}$ integral with singular metrics on nearby fibers has a lower limit property with an optimal estimate. This property can be implied by (4.6).

6. Subadditivity of generalized Kodaira–Iitaka dimensions

Theorem 5.1 can be used to prove the subadditivity of the generalized Kodaira–Iitaka dimensions with multiplier ideal sheaves for certain Kähler fibrations (Theorem 6.2).

Let X be a connected compact complex manifold, and (L, h_L) a holomorphic $\mathbb{Q}$-line bundle on X with a singular metric h_L. Let k_0 be the smallest positive integer such that $k_0 L$ is a holomorphic line bundle.

In terms of the singular metric h_L of L, we will denote the $L^{\frac{2}{m}}$ multiplier ideal sheaf $\mathcal{I}_{L^{\frac{2}{m}}}(\varphi)$ (m is a positive integer) by the global notation $\mathcal{I}_m(h_L)$, where φ is the local weight of h_L. We also write $\mathcal{I}_1(h_L)$ as $\mathcal{I}(h_L)$ for simplicity.

The notion of the generalized Kodaira–Iitaka dimension with multiplier ideal sheaves is defined below.

Definition 6.1 ([33]). The generalized Kodaira–Iitaka dimension $\kappa(X, K_X + L, h_L)$ is defined to be

$$\sup\left\{v \in \mathbb{Z};\ \overline{\lim_{k\to+\infty}} \frac{h^0\big(X, (kk_0K_X + kk_0L) \otimes \mathcal{I}_{kk_0}(h_L)\big)}{k^v} > 0\right\}$$

if $\overline{\lim}_{k\to+\infty} h^0(X, (kk_0K_X + kk_0L) \otimes \mathcal{I}_{kk_0}(h_L)) \neq 0$. Otherwise, $\kappa(X, K_X + L, h_L)$ is defined to be $-\infty$.

Then the following subadditivity of the generalized Kodaira–Iitaka dimensions for certain Kähler fibrations holds.

Theorem 6.2 ([33]). *Let $\Pi : X \to Y$ be a surjective holomorphic map with connected fibers from a compact Kähler manifold X to a compact connected complex manifold Y; $\dim X = n$ and $\dim Y = m$. Let L be a holomorphic $\mathbb{Q}$-line bundle on X possessing a singular metric h_L such that the curvature current $\sqrt{-1}\Theta_{L,h_L} \geq 0$ on X. Assume that the canonical bundle K_Y of Y possesses a singular metric h such that*

(a) $\sqrt{-1}\Theta_{K_Y,h} \geq 0$ *on Y in the sense of currents,*

(b) *there exists an open subset U of Y and a continuous positive $(1,1)$-form γ on U such that $\sqrt{-1}\Theta_{K_Y,h} \geq \gamma$ on U in the sense of currents.*

Then

$$\kappa(X, K_X + L, h_L) \geq \kappa(Z, K_Z + L|_Z, h_L|_Z) + m, \tag{6.1}$$

where Z denotes a general fiber of Π.

If X and Y are projective, and (L, h_L) is trivial, Theorem 6.2 is just Kawamata–Viehweg's result; that is,

$$\kappa(X) \geq \kappa(Z) + \dim Y$$

holds for Y of general type [17, 26].

If $\mathcal{I}(h_L) = \mathcal{O}_X$ (for example, (X, L) is Kawamata log terminal), (6.1) becomes

$$\kappa(X, K_X + L) \geq \kappa(Z, K_Z + L|_Z) + \dim Y.$$

One can also see [27] for some related results in the case when (X, L) is Kawamata log terminal.

Our proof of Theorem 6.2 is analytic and relies on Theorem 5.1 and a general L^2 extension theorem with singular metrics on weakly pseudoconvex Kähler manifolds [9, Theorem 2.8 and Remark 2.9 (b)]. They are used to prove the following $L^{\frac{2}{k}}$ extension theorem (Theorem 6.3 below) for twisted pluricanonical sections on compact Kähler manifolds, which is the crucial step in the proof of Theorem 6.2.

For $y \in Y$, denote $\Pi^{-1}(y)$ by X_y. Denote $L|_{X_y}$ simply by L_y. Denote by Y_0 the set of all points in Y which are regular values of Π.

Let k_0 be the smallest positive integer such that $k_0 L$ is a holomorphic line bundle, and let k be a positive integer such that $k_0|k$. Let

$$\begin{aligned} Y_{k,\text{ext}} &:= \{y \in Y_0;\ h^0(X_y, kK_{X_y} + kL_y) = \operatorname{rank} \Pi_*(kK_{X/Y} + kL)\}, \\ \widetilde{Y}_{k,h_L,\text{ext}} &:= \{y \in Y_0;\ h^0\big(X_y, (kK_{X_y} + kL_y) \otimes \mathcal{I}_k(h_L)|_{X_y}\big) \\ &\qquad\qquad = \operatorname{rank} \Pi_*\big((kK_{X/Y} + kL) \otimes \mathcal{I}_k(h_L)\big)\}, \end{aligned}$$

and

$$Y_{k,h_L,\text{ext}} := \{y \in Y_{k,\text{ext}} \cap \widetilde{Y}_{k,h_L,\text{ext}};\ \mathcal{I}_k(h_L|_{X_y}) = \mathcal{I}_k(h_L)|_{X_y}\}.$$

Denote $\bigcap_{k \in \mathbb{Z}^+,\, k_0|k} Y_{k,h_L,\text{ext}}$ simply by $Y_{h_L,\text{ext}}$. It is not hard to see that the $2m$-dimensional Lebesgue measure of $Y \setminus Y_{h_L,\text{ext}}$ is zero and $\kappa(X_y, K_{X_y} + L_y, h_L|_{X_y})$ is independent of y when $y \in Y_{h_L,\text{ext}}$.

Theorem 6.3 ([33]). *Let Π, X, Y, (L, h_L) be the same as in Theorem 6.2. Let X_y, L_y, k, Y_0, $Y_{k,\text{ext}}$, and $Y_{k,h_L,\text{ext}}$ be the notations defined above. Let $\varphi \le 0$ be a quasi-psh function on Y, which is smooth outside q distinct points $\{y_j\}_{j=1}^q$. For each y_j $(1 \le j \le q)$, assume that there exists a coordinate ball around y_j with coordinate functions $z = (z_1, z_2, \ldots, z_m)$ such that $z(y_j) = 0$ and $\varphi(z) - \log|z|^{2m}$ is smooth. Moreover, assume that the canonical bundle K_Y of Y possesses a singular Hermitian metric h such that*

$$(k-1)\sqrt{-1}\Theta_{K_Y,h} + \alpha\sqrt{-1}\partial\bar{\partial}\varphi \ge 0 \quad \text{on } Y \text{ for all } \alpha \in [1, 1+\varepsilon], \tag{6.2}$$

where ε is a positive number. Denote the pluripolar set $\{h = +\infty\}$ by Σ_h. Assume that $\{y_j\}_{j=1}^q \subset Y_{k,\text{ext}} \setminus \Sigma_h$. Then there exists a positive constant C such that, for any

$$f \in H^0\Big(\bigcup_{j=1}^q X_{y_j}, (kK_X + kL)|_{\bigcup_{j=1}^q X_{y_j}} \otimes \mathcal{I}_k(h_L|_{\bigcup_{j=1}^q X_{y_j}})\Big),$$

there exists $F \in H^0(X, (kK_X + kL) \otimes \mathcal{I}_k(h_L))$ such that

$$F|_{\bigcup_{j=1}^q X_{y_j}} = f$$

and

$$\int_X \big(|F|^2_{h_\omega^k \otimes h_L^k}\big)^{\frac{1}{k}}\, dV_{X,\omega} \le C \sum_{j=1}^q \int_{X_{y_j}} \big(|f|^2_{h_\omega^k \otimes h_L^k}\big)^{\frac{1}{k}}\, dV_{X_{y_j},\omega_{y_j}}, \tag{6.3}$$

where ω is the Kähler metric on X, ω_{y_j} is the Kähler metric on X_{y_j} induced from ω, $dV_{X,\omega} := \frac{\omega^n}{n!}$ is the volume form on X, and $h_\omega := (dV_{X,\omega})^{-1}$ (it defines a smooth Hermitian metric on K_X).

The idea in our proof of Theorem 6.2 is sketched below.

Let k be a positive integer sufficiently divisible, and let

$$p := h^0\big(X_y, (kK_{X_y} + kL_y) \otimes \mathcal{I}_k(h_L|_{X_y})\big).$$

Then

$$p \geq \alpha k^{\kappa(Z, K_Z + L|_Z, h_L|_Z)}$$

for some positive number α independent of k.

Let $q := \beta k^m$ and let $\{y_j\}_{j=1}^q$ be q distinct points in U in general position, where β is some positive number independent of k. We will make some explanation on the degree m $(= \dim Y)$ in the end of this section.

Assume U is small enough and let $\eta \in H^0(U, K_Y)$ be a holomorphic frame of $K_Y|_U$. For each $j = 1, 2, \ldots, q$, let

$$\{e_{ij}\}_{i=1}^p \subset H^0\big(X_{y_j}, (kK_{X_{y_j}} + kL_{y_j}) \otimes \mathcal{I}_k(h_L|_{X_{y_j}})\big)$$

be a basis.

Let δ_{lj} be the Kronecker delta function, where $1 \leq l \leq q$ and $1 \leq j \leq q$.

For each j,

$$f_{i,l,j} := \delta_{lj} e_{ij} \otimes (\Pi^*\eta)^k|_{X_{y_j}} \quad (1 \leq i \leq p,\ 1 \leq l \leq q)$$

belongs to $H^0(X_{y_j}, (kK_X|_{X_{y_j}} + kL|_{X_{y_j}}) \otimes \mathcal{I}_k(h_L|_{X_{y_j}}))$. Let

$$f_{i,l} \in H^0\Big(\bigcup_{j=1}^q X_{y_j}, (kK_X + kL)|_{\bigcup_{j=1}^q X_{y_j}} \otimes \mathcal{I}_k(h_L|_{\bigcup_{j=1}^q X_{y_j}})\Big)$$

be defined by $f_{i,l}|_{X_{y_j}} = f_{i,l,j}$ for all $1 \leq i \leq p$, $1 \leq l \leq q$, and $1 \leq j \leq q$.

Then by using Theorem 6.3, we can obtain that there exist holomorphic sections

$$\{F_{i,l}\}_{1 \leq i \leq p,\, 1 \leq l \leq q} \subset H^0\big(X, (kK_X + kL) \otimes \mathcal{I}_k(h_L)\big)$$

such that

$$F_{i,l}|_{\bigcup_{j=1}^q X_{y_j}} = f_{i,l} \quad \text{for all } i \text{ and } l.$$

It is obvious that $\{F_{i,l}\}_{1 \leq i \leq p,\, 1 \leq l \leq q}$ is linearly independent. Therefore,

$$h^0\big(X, (kK_X + kL) \otimes \mathcal{I}_k(h_L)\big) \geq pq \geq \alpha\beta k^{\kappa(Z, K_Z + L|_Z, h_L|_Z) + m}.$$

Hence

$$\kappa(X, K_X + L, h_L) \geq \kappa(Z, K_Z + L|_Z, h_L|_Z) + m.$$

Now we make some explanation on the degree m ($= \dim Y$) in the definition of q. In the above proof, we need to construct a quasi-psh function φ on Y such that (6.2) holds on Y when we use Theorem 6.3. In fact, we construct the function φ by splicing the polar functions $\log|z(y) - z(y_j)|^{2m}$ around the points $\{y_j\}_{j=1}^{q}$, and we need to control the negative part of $\sqrt{-1}\partial\bar{\partial}\varphi$ such that (6.2) holds on Y. In this way, $\sqrt{-1}\partial\bar{\partial}\varphi$ may get more negativity when q becomes larger. The integer m is just the largest degree in the definition of q such that (6.2) holds on Y.

7. A generalization of Siu's lemma with nontrivial multiplier ideal sheaves near a subvariety

Let (X, ω) be an n-dimensional Kähler manifold with a Kähler metric ω, let (L, h) be a holomorphic line bundle over X equipped with a singular metric h, and let ψ be a quasi-psh function on X.

Let $S := V(\mathcal{I}(\psi))$ be the zero variety of the multiplier ideal sheaf $\mathcal{I}(\psi)$. Then ψ is said to have *log canonical singularities* along S if $\mathcal{I}((1-\varepsilon)\psi)|_S = \mathcal{O}_X|_S$ for every $\varepsilon > 0$.

Denote by S^0 the set of regular points of S.

Definition 7.1 (see [9,14,19]). Assume that ψ has neat analytic singularities and has log canonical singularities along $S = V(\mathcal{I}(\psi))$. The positive measure $dV_{X,\omega}[\psi]$ on S^0 (the set of regular points of S) is defined by

$$\int_{S^0} g\, dV_{X,\omega}[\psi] = \lim_{t\to-\infty} \int_{\{x\in X: t<\psi(x)<t+1\}} \tilde{g} e^{-\psi}\, dV_{X,\omega} \tag{7.1}$$

for any compactly supported nonnegative continuous function g on S^0, where $\tilde{g}$ is a compactly supported nonnegative continuous extension of g to X such that $(\operatorname{supp}\tilde{g}) \cap S \subset S^0$.

If $f \in H^0(S^0, (K_X \otimes L)|_{S^0})$, then $|f|^2_{\omega,h} dV_{X,\omega}[\psi]$ is a positive measure on S^0 which depends on the property of h on S^0. There is another way to define a positive measure which is defined not only on the property of h on S^0 but also on the property of h near $\{\psi = -\infty\}$ (see Definition 7.3).

Note that ψ is neither assumed to have neat analytic singularities nor assumed to have log canonical singularities along S in Definition 7.2 and Definition 7.3.

Definition 7.2 ([9]). The *restricted multiplier ideal sheaf* $\mathcal{I}'_\psi(h)$ is defined to be the set of germs $f \in \mathcal{I}(h)_x \subset \mathcal{O}_{X,x}$ such that there exists a coordinate neighborhood U of x satisfying

$$\overline{\lim_{t\to-\infty}} \int_{\{y\in U: t<\psi(y)<t+1\}} |f|^2 e^{-\varphi-\psi}\, d\lambda < +\infty,$$

where U is small enough such that h can be written as $e^{-\varphi}$ with respect to a local holomorphic trivialization of L on a neighborhood of $\overline{U}$, and $d\lambda$ is the $2n$-dimensional Lebesgue measure on U.

Denote by S' the zero set of the ideal sheaf

$$\mathcal{J} := \{g \in \mathcal{O}_X;\ g \cdot \mathcal{I}(h) \subset \mathcal{I}(he^{-\psi})\}.$$

Let f be an element in

$$H^0\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_{\psi}(h)/\mathcal{I}(he^{-\psi})\big).$$

Then f is actually supported on S'.

Definition 7.3 ([9]). The positive measure $|f|^2_{\omega,h} dV'_{X,\omega}[\psi]$ (a purely formal notation) on S' is defined as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S'} g d\mu \geq \overline{\lim_{t\to-\infty}} \int_{\{x\in X: t<\psi(x)<t+1\}} g|\tilde{f}|^2_{\omega,h} e^{-\psi}\, dV_{X,\omega}$$

for any nonnegative continuous function g on X with $\operatorname{supp} g \subset\subset X$, where $\tilde{f}$ is a smooth extension of f to X such that

$$\tilde{f} - \hat{f} \in \mathcal{O}_X(K_X \otimes L) \otimes_{\mathcal{O}_X} \mathcal{I}(he^{-\psi}) \otimes_{\mathcal{O}_X} \mathcal{C}^{\infty}$$

locally for any local holomorphic representation $\hat{f}$ of f.

The following generalization of Siu's lemma with nontrivial multiplier ideal sheaves near a subvariety gives a relation between the two measures in Definitions 7.1 and 7.3. It can be proved by using Hironaka's desingularization theorem and the method in the proof of Theorem 3.1.

Theorem 7.4 ([33]). *Let $\Omega_0 \subset \mathbb{C}^n$ be a bounded domain, ψ a negative quasi-psh function on Ω_0 with neat analytic singularities, and φ a negative psh function on Ω_0. Denote by S the zero variety of $\mathcal{I}(\psi)$ and assume that ψ has log canonical singularities along S. Let Ω be a pseudoconvex domain such that $\Omega \subset\subset \Omega_0$. Suppose that f is a holomorphic function on S^0 satisfying*

$$\int_{S^0} |f|^2 e^{-\varphi}\, d\lambda[\psi] < +\infty,$$

where S^0 is the set of regular points of S, $d\lambda$ is the Lebesgue measure on Ω_0, and $d\lambda[\psi]$ is the positive measure on S^0 defined as in Definition 7.1. Then there exists a positive number β such that

$$\int_{\Omega\cap S^0} |f|^2 e^{-(1+\beta)\varphi}\, d\lambda[\psi] < +\infty,$$

and there exists a holomorphic function F *on* Ω *such that*

$$F = f \quad \text{on } \Omega \cap S^0 \tag{7.2}$$

and

$$\int_{\Omega} \frac{|F|^2 e^{-(1+\beta)\varphi}}{e^{\psi}(\psi^2+1)}\, d\lambda < +\infty. \tag{7.3}$$

Moreover, any holomorphic function F *on* Ω *satisfying* (7.2) *and* (7.3) *has the property*

$$\lim_{t\to-\infty} \int_{\Omega\cap\{t<\psi<t+1\}} v|F|^2 e^{-\varphi-\psi}\, d\lambda = \int_{\Omega\cap S^0} v|f|^2 e^{-\varphi}\, d\lambda[\psi]$$

for any compactly supported nonnegative continuous function v *on* Ω.

Theorem 7.4 shows that the two ways to define the measures are the same when ψ has neat analytic singularities and has log canonical singularities along S.

By using Theorem 7.4, we proved in [33] that the L^2 extension theorem in [31] can be regarded as a corollary of the L^2 extension theorem in [34].

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