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DOI 10.4171/8ECM/37

From art and circuit design to geometry and combinatorics

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Abstract. These notes provide a detailed insight on the interplay between crossing numbers of graphs and random geodesic drawings, and try to explain a relationship with the main fundamental open questions about crossing numbers of graphs. A very general class of geodesic drawings on the sphere attaining the Hill bound is presented.

1. Introduction

Crossing number minimization in drawings of graphs on surfaces appears in diverse applications across disciplines. It came into mathematical research through problems in modern constructionist art. Crossing number problems have various applications within engineering (e.g., design of large electrical circuits [17]) and in computer science.

Later it became a useful concept in theoretical questions about graph drawing, algorithm design, and robotics, and became an important notion in discrete and computational geometry. The famous crossing lemma made a very surprising impact within pure mathematics, after it was discovered that it gives greatly simplified proofs for various (seemingly unrelated) hard geometric [32] and algebraic problems [29]. On the other hand, the rectilinear crossing number is related to the classical Sylvester four-point problem, which gave motivation for developments of geometric probability theory. We refer to [26, 27] for a more complete overview of this area of mathematics.

These notes are related to the public talk at the 8th ECM in Portorož, Slovenia, where the author presented some of the complex issues related to geodesic drawings of graphs on surfaces and crossing minimization in such drawings, with a special emphasis on random drawings.

The presentation herein includes historical remarks, it overviews the main fundamental open problems about crossing numbers of graphs, and through a generalization of Sylvester's four-point problem gives special emphasis on random geodesic

2020 Mathematics Subject Classification. 05C10.
Keywords. Crossing number, geodesic drawing, graph drawing.

drawings of graphs on surfaces. When dealing with random geodesic drawings on the sphere, we give a very general class of geodesic drawings attaining the conjectured minimum crossing number.

2. Hill conjecture and Hill drawings

English painter Anthony Hill[1] made an extraordinary conjecture in the 1950s that remained unanswered until today despite serious attacks using powerful machinery in trying to resolve his conjecture. Starting with a question underlying some of his painting projects, Hill tried to understand how to draw $\binom{n}{2}$ connections between n objects so that the painting would involve a minimum number of under or over-crossings. This lead to the formal notion of the *crossing number* of a graph, which he introduced in a mathematical paper jointly with Harary [13].

Given a graph G, one can consider its drawing in the plane (or in some other surface), where vertices are represented as distinct points and edges are drawn as rectifiable arcs joining the corresponding points. One may restrict attention to *good drawings*, where we request that any two edges intersect in at most one point, which is either their common endvertex or a (proper) crossing of two arcs, and no three arcs have their pairwise crossings in the same point. The *crossing number of a (good) drawing D* of a graph G is the number of crossings of pairs of edges in D, and the *crossing number of the graph G*, denoted by $\mathrm{cr}(G)$, is the minimum crossing number taken over all good drawings of G in the plane.

Hill found a general drawing for any complete graph K_n of order n that involves precisely

$$H(n)=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor=\begin{cases}\frac{1}{64}n(n-2)^2(n-4), & n \text{ is even;}\\ \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ is odd}\end{cases} \tag{2.1}$$

crossings. Based on these drawings and inability of producing any drawing with less than $H(n)$ crossings, Hill conjectured the following.

Conjecture 2.1 (Hill, 1959). *For any complete graph K_n of order n, we have*

$$\mathrm{cr}(K_n)=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

[1] Anthony Hill (1930–2020) was one of leading modern British painters. The following is an abstract from his obituary in *Guardian*: “Anthony Hill, who has died aged 90, was a singular, but not solitary, figure in the art world. An artist under two names, and a mathematician and writer under more than one alias, he was a member of the constructionist group of geometrical abstract artists that emerged in Britain in the mid-1950s, and was its leading theoretician.”

Hill's drawings of complete graphs are called *cylindrical drawings* because they can be realized on a cylinder in such a way that all vertices lie (evenly split) on the two circles forming the cylinder and no edge crosses those two circles. Soon after these drawings were published in [13], Blažek and Koman [8] found another kind of drawings of complete graphs involving precisely the same number of crossings. Their drawings correspond to 2-*page drawings* in which the vertices are drawn on the boundary of a unit disk in the plane and no edge crosses this boundary, so each edge is drawn entirely inside the disk or entirely outside. It has been proved quite recently that no cylindrical [2] and no 2-page drawing [1] of K_n has fewer than $H(n)$ crossings, thus giving the first real support to the conjecture of Hill.

We will say that a drawing D of the complete graph K_n is a *Hill drawing* if it has precisely $H(n)$ crossings.

No other Hill drawings of complete graphs have been discovered until 2014 when Ábrego et al. [3] described modifications of cylindrical drawings of K_{2n} yielding Hill drawings of K_{2n+1}, K_{2n+2}, and K_{2n+3} that are different in the sense that they are not "shellable". In Section 4.3, we describe a much more general class of Hill drawings that include in particular all known examples of Hill drawings. These drawings have not appeared previously in the mathematical literature.[2]

After 60+ years, Conjecture 2.1 is still widely open. It has been confirmed for every $n \leq 12$ (with K_{11} and K_{12} confirmed in [23]), but it is still unresolved for $n = 13$ and beyond. In fact, the weaker, asymptotic version of Conjecture 2.1 is also open.

Conjecture 2.2 (Asymptotic Hill conjecture).

$$\mathrm{cr}(K_n) = \frac{1}{64} n^4 \big(1 - o(1)\big) = \frac{3}{8} \binom{n}{4} \big(1 - o(1)\big).$$

Of course, the main problem is to show the lower bound – that there are no better drawings than those with precisely $H(n)$ crossings. To that effect, there are several exciting new results that were obtained through elaborate analysis of drawings and use of semidefinite programming and Razborov's flag algebra calculus [18, 25].

Theorem 2.3 (Balogh, Lidický, and Salazar [7]). *For every sufficiently large n,*

$$\mathrm{cr}(K_n) \geq 0.985\, H(n).$$

In the same paper [7], the authors also proved that the spherical geodesic crossing number of K_n (see Section 4.3 for the definition) is asymptotically at least $0.996 H(n)$.

[2]After the author put a preprint of this construction on the arXiv [20] in 2018, he was informed that almost the same construction was mentioned by Kyncl on `math`*overflow* [16].

3. Turán's brick factory problem

Turán's brick factory problem asks for the minimum number of crossings in a drawing of a complete bipartite graph $K_{m,n}$. During World War II, Turán was forced to work in a brick factory, pushing wagon loads of bricks from kilns to storage sites, and the corresponding rail network with m kilns and n storage barracks was the same as a special drawing of the complete bipartite graph $K_{m,n}$. Crossings of rail tracks made the transport challenging, and Turán, inspired by this situation, asked himself how the rail network might be redesigned to minimize the number of crossings between the railway tracks [33].

Paul Turán discussed his brick factory problem during 1950s in his talks, and Zarankiewicz and Urbanik, who attended some of his presentations, independently found drawings of complete bipartite graphs for which they claimed that they are optimal [34, 35]. Unfortunately, proofs in both published papers were flawed. This was discovered only a couple of years later, and the claimed minimum number of crossings was turned into the following conjecture, which remains widely open even today.

Conjecture 3.1. *For any positive integers m and n, the crossing number* $\mathrm{cr}(K_{m,n})$ *of the complete bipartite graph with $m+n$ vertices is equal to*

$$Z(m,n) = \frac{1}{4}\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The conjecture has since been confirmed for the cases where one of the parameters is at most 6 and also for $K_{7,7}$ and $K_{7,8}$, but it remains open even for such small graphs as $K_{7,9}$ and $K_{9,9}$.[3]

4. Geodesic drawings

When we consider drawings of graphs in the plane, where each edge is drawn as a straight-line segment, we come to the notion of the rectilinear crossing number $\overline{\mathrm{cr}}(G)$. It turns out that for most small graphs the rectilinear crossing number is equal to the usual crossing number. However, it was discovered early that $\mathrm{cr}(K_8) = 18$ and $\overline{\mathrm{cr}}(K_8) = 19$ and that this difference extends to larger graphs. However, it was open for a long time how large can be the difference $\overline{\mathrm{cr}}(K_n) - \mathrm{cr}(K_n)$. A breakthrough was made in 2004 by Lovász, Vesztergombi, Wagner, and Welzl [19], who proved that the normalized rectilinear crossing number of K_n is strictly greater than the corresponding limit for the usual crossing number. Building on the work in [19], Ábrego, Cetina,

[3] The cases with even parameters were not mentioned since they would follow from odd cases by known parity arguments; see, e.g. [27].

Probability	Shape	Author
$3/4$		Cayley and Sylvester
$1/2$		DeMorgan
$2/3$	Triangle	Wilson
$> 1/2$		Ingleby
$5/8$		(No name given)
$1 - 35/(12\pi^2)$	Disk	Woolhouse
$25/36$	Rectangle	[10]
$2(18 - \sqrt{5})/45$	Regular pentagon	[9]
$683/972$	Regular hexagon	[14, p. 46]

Table 1. Answers to Sylvester's question. The upper part of the table is taken from [24], where the complementary probabilities for non-convex 4-gon are shown.

Fernández-Merchant, Leaños, and Salazar [5] improved the bounds from [19]. They also found today's best upper bounds [4]. Their results are summarized in the following inequalities:

$$0.379972 < \frac{277}{729} \le \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4} \le \frac{83247328}{218791125} < 0.380488. \tag{4.1}$$

Unlike for the Hill conjecture, we are lacking understanding of the rectilinear crossing number and there is no good evidence about whether the lower or the upper bound in (4.1) is closer to the normalized limit.

The rectilinear crossing number of complete graphs is tightly related to an old problem in geometric probability that was originally proposed by Julius Sylvester in 1864, and which we will discuss next.

4.1. Sylvester's four-point problem

In 1864, Sylvester asked [30] "what is the probability that four randomly chosen points in the plane form a convex 4-gon?". As it turned out, the problem was ill-posed since by 1865, at least six solutions were received, all with different answers (see the first six entries in Table 1). Depending on the method chosen to pick points from the infinite plane, a number of different solutions are possible, and Sylvester concluded [31] that his problem does not admit a determinate solution (see also [24]).

The reason for so many distinct answers was that it was not clear what "randomly chosen points" in the plane would be. Sylvester himself changed the question a year later [31]. The revised four-point problem asks for the probability $q(R)$ that four points chosen at random in a bounded planar region R have a convex hull which is a quadrilateral.

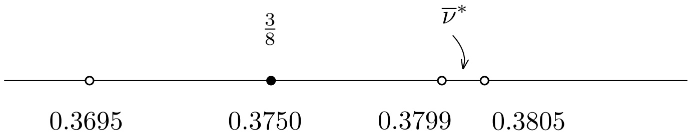

Figure 1. Currently best bounds on the asymptotic values of normalized crossing numbers of large complete graphs: $0.3695 \le \mathrm{cr}(K_n)/\binom{n}{4} \le \frac{3}{8}$ and $0.3799 \le \overline{\mathrm{cr}}(K_n)/\binom{n}{4} \le 0.3805$.

Scheinerman and Wilf linked the Sylvester problem to the rectilinear crossing number [28]. Let $\overline{\nu}^*$ be the limit of $\overline{\mathrm{cr}}(K_n)/\binom{n}{4}$ as $n \to \infty$, and let $q(R)$ be as defined above. Scheinerman and Wilf proved that $\overline{\nu}^* = \inf q(R)$, where the infimum is taken over all open planar sets R whose area is 1 (equivalently, over all unions of finitely many disjoint circles). Let us recall that today's best estimates for the rectilinear crossing number of complete graphs given in (4.1) yield that $0.3799 < \overline{\nu}^* < 0.3805$; see Figure 1.

Note that, in the plane, four points form a convex quadrilateral if and only if the six line segments joining pairs of these points make a crossing. We will use this interpretation in the sequel.

One can pose similar questions when considering randomly chosen points on any surface. Suppose that $\mathbb{S}$ is a compact Riemannian surface and that μ is a probability measure[4] on $\mathbb{S}$. Then we define $q(\mu)$ as the probability that for four μ-randomly chosen points, two of the six geodesics joining pairs of these points cross each other. Then $q(\mu)$ is called the *geometric crossing probability* of μ.

The geometric crossing probabilities are related to the *geodesic crossing number* of the complete graph on $\mathbb{S}$, for which we consider all drawings of the graph in which all edges are drawn as shortest geodesic segments on $\mathbb{S}$. This is not hard to see and we prove it as Lemma 4.1 below. But let us first discuss random drawings.

Let μ be a probability measure on $\mathbb{S}$. We say that μ is *geodesically non-degenerate* if the probability that two μ-random points x, y are distinct and that they are joined with a unique geodesic is equal to 1, and the probability that a third random point lies on this geodesic is 0. If this is the case, then, with probability 1, n randomly selected points define a unique good geodesic drawing of K_n. Such a drawing will be referred to as a *μ-random drawing*.

Lemma 4.1. *Let $\overline{\nu}^*(\mathbb{S})$ be the limit of $\overline{\mathrm{cr}}_{\mathbb{S}}(K_n)/\binom{n}{4}$ (where n tends to infinity) and let $q(\mu)$ be as defined above. Then $\overline{\nu}^*(\mathbb{S}) = \inf q(\mu)$, where the infimum is taken over all geodesically non-degenerate probability, measures μ. The same holds when the infimum is taken over all uniform probability measures whose support is the union of finitely many disjoint disks on $\mathbb{S}$.*

[4]The measure μ has to fulfill some simple non-degeneracy conditions, which will be discussed later.

Proof. Every μ-random drawing D_n of K_n gives an upper bound on the geodesic crossing number of K_n in $\mathbb{S}$. For any four points $a, b, c, d \in \mathbb{S}$, let Q_{abcd} be equal to 1 if two of the geodesics between points a, b, c, d in $\mathbb{S}$ cross each other. Note that $\mathbb{E}(Q_{abcd}) = q(\mu)$ if a, b, c, d are chosen at random with respect to μ, and that

$$\mathrm{cr}(D_n) = \sum \left\{ Q_{abcd} \mid \{a, b, c, d\} \in \binom{V}{4} \right\}.$$

By linearity of expectations, we have

$$\mathbb{E}(\mathrm{cr}(D_n)) = \mathbb{E}\left(\sum Q_{abcd}\right) = \sum \mathbb{E}(Q_{abcd}) = \binom{n}{4} q(\mu).$$

This implies that $\overline{\nu}^*(\mathbb{S}) \le q(\mu)$ for every μ.

To establish equality, let $\delta > 0$ and consider an optimal geodesic drawing D of K_n in $\mathbb{S}$, such that $\mathrm{cr}(D)/\binom{n}{4} - \overline{\nu}^* < \delta$. Let $x_1, \dots, x_n \in \mathbb{S}$ be the vertices of D. There are $\varepsilon > 0$ and balls $B_1, \dots, B_n$ centered at these vertices, each of area ε, such that for any choice of points $x_i' \in B_i$ $(1 \le i \le n)$, the geodesic drawing on these points has exactly the same crossings as D.

Let μ_n be the uniform measure on $B_1 \cup \cdots \cup B_n$. We claim that $q(\mu_n)$ is close to $\mathrm{cr}(D)/\binom{n}{4}$. Let us consider four μ_n-random points. With probability at least $1 - O(1/n)$, the four points are in distinct balls B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4} and the four indices i_1, i_2, i_3, i_4 are chosen uniformly at random from $[n]$. Thus, the probability that the geodesics on these four points induces a crossing is at most $(1-O(1/n))^{-1}\,\mathrm{cr}(D)/\binom{n}{4}$. This shows that

$$q(\mu_n) \le \big(1 + o(1)\big)\,\mathrm{cr}(D)/\binom{n}{4} \le \big(1 + o(1)\big)(\overline{\nu}^* + \delta).$$

By letting $\delta \to 0$ and $n \to \infty$, we conclude that

$$\overline{\nu}^* \le \inf_{\mu} q(\mu) \le \lim_{n\to\infty} q(\mu_n) \le \overline{\nu}^*.$$

This completes the proof. ∎

4.2. Sylvester's problem on the sphere

Moon [22] proved that the expected number of crossings in random drawings of K_n on the unit sphere in $\mathbb{R}^3$ is asymptotically the same as the conjectured crossing number of K_n. His result can be expressed as follows.

Theorem 4.2 (Moon [22]). *Let μ be the uniform probability distribution on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Then $q(\mu) = 3/8$.*

Guy, Jenkyns, and Schaer [12] considered the crossing number of K_n on the flat torus (obtained from the unit square by identifying opposite sides). Their computation shows the following.

Theorem 4.3 (Guy, Jenkyns, and Schaer [12]). *Let $\mathbb{T}$ be the flat torus obtained from a rectangle in the plane by identifying opposite sides. Let μ be the uniform probability distribution on $\mathbb{T}$. Then $q(\mu) = 5/18$.*

As noted in [12], the Sylvester crossing probability is the same for every rectangle model of the flat torus. However, they neglected the possibility of other parallelogram representations of the flat torus. Interestingly, they give smaller crossing probabilities.

Theorem 4.4 (Elkies [11]). *Let $\mathbb{T}_\alpha$ be the flat torus obtained from a rhombus with side length 1 and angle α ($0 < \alpha \le \pi/2$) by identifying opposite sides. If μ_α is the uniform distribution on $\mathbb{T}_\alpha$, then*

$$q(\mu_\alpha) \ge \frac{22}{81}.$$

The smallest value occurs at $\alpha = \pi/3$, where $q(\mu_{\pi/3}) = \frac{22}{81}$.

In [15], Koman bounded the crossing number of K_n in the projective plane:

$$\frac{41}{273}\binom{n}{4} \le \mathrm{cr}_{\mathbb{N}_1}(K_n) \le \frac{39}{128}\binom{n-1}{4}, \tag{4.2}$$

where the left inequality holds only when $n \ge 11$.

Below we give an improvement of Koman's upper bound by using the model of the projective plane as the surface endowed with constant curvature 1 and considering random drawings.

Let $\mathbb{P}^2$ be the projective plane obtained from the unit sphere $\mathbb{S}^2$ by identifying all antipodal pairs of points. This defines the projective plane as a surface of constant curvature 1. Its total area is one half of the area of the unit sphere, $A(\mathbb{P}^2) = 2\pi$. The geodesics in $\mathbb{P}^2$ are the *great semicircles*, each of which has length equal to π.

Theorem 4.5. *The uniform distribution μ on $\mathbb{P}^2$ has crossing probability $q(\mu) = 3\pi^{-2}$. Consequently,*

$$\overline{\mathrm{cr}}_{\mathbb{P}^2}(K_n) \le 3\pi^{-2}\binom{n}{4}.$$

Proof. Let us consider two random points in $\mathbb{P}^2$ and let ℓ denote the length of the geodesic joining them. We claim that $\mathbb{E}(\ell) = 1$. To see this, we may assume that the first point is the North pole of $\mathbb{S}^2$. Then $\ell = \alpha$, where $0 \le \alpha \le \pi/2$ is the angle between the lines through the origin in $\mathbb{R}^3$ and the two points. Now,

$$\mathbb{E}(\ell) = \iint_S \alpha \, dS = \int_0^{2\pi}\int_0^{\pi/2} \alpha \sin\alpha \, d\alpha \, ds = 2\pi \int_0^{\pi/2} \alpha \sin\alpha \, d\alpha = 1.$$

Next, we consider the conditional probability of a crossing of two random segments S_1, S_2, conditioned on their lengths ℓ_1 and ℓ_2. The two great semicircles containing S_1 and S_2 cross each other at a point p. The segments are positioned randomly on these two segments, so the probability that they both contain p (which is the only way they would cross) is equal to $(\ell_1/\pi) \cdot (\ell_2/\pi)$. Thus the conditional probability of the event X that S_1 and S_2 cross is

$$\Pr[X \mid \ell_1, \ell_2] = \frac{\ell_1}{\pi} \cdot \frac{\ell_2}{\pi}.$$

Since ℓ_1 and ℓ_2 are independent and $\mathbb{E}(\ell_1) = \mathbb{E}(\ell_2) = 1$, we get

$$\mathbb{E}(X) = \pi^{-2}\,\mathbb{E}(\ell_1\ell_2) = \pi^{-2}\,\mathbb{E}(\ell_1)\,\mathbb{E}(\ell_2) = \pi^{-2}.$$

Finally, we have that $q(\mu) = 3\,\mathbb{E}(X) = 3\pi^{-2}$. ∎

Elkies [11] realized that $3\pi^{-2} < 39/128$ and concluded that the bound of Theorem 4.5 asymptotically beats Koman's upper bound (4.2). In comparison with the Hill conjecture, it was conjectured in [11] that the bound of the theorem is best possible. However, Arroyo, McQuillan, Richter, Salazar, and Sullivan [6] recently found better drawings of complete graphs in the projective plane. Their drawings can also be approximated with random drawings, but the probability measure is not uniform.

Theorem 4.6 ([6]). $\overline{\mathrm{cr}}_{\mathbb{P}^2}(K_n) < 0.3024\binom{n}{4}$.

Note that $0.3024 < 3\pi^{-2} \approx 0.304$.

4.3. Antipodal drawings on the sphere

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. For any two points $p, q \in \mathbb{S}^2$, consider the great circle through p and q (the great circle is unique unless q is antipodal to p in which case there are many). The shorter of the two segments on this circle from p to q is called a *geodesic arc* (or just a *geodesic*). Any geodesic arc joining two antipodal points in $\mathbb{S}^2$ is a half of a great circle and will be referred to as a *half-circle*.

A *geodesic drawing* of a graph G on $\mathbb{S}^2$ is a drawing in which all edges are drawn as geodesic arcs. We define the *geodesic crossing number* of the graph G on the sphere as the minimum number of crossings of edges of G in a geodesic drawing of the graph, and denote it by $\mathrm{cr}_{\mathbb{S}^2}(G)$.

A set P of points in $\mathbb{S}^2$ is *in general position* if no three points in P lie on a common great circle in the sphere.

Let $k \geq 3$ be a positive integer and let $n = 2k$. The graph M_n obtained from the complete graph K_n by removing edges of a perfect matching in K_n is isomorphic to the complete k-partite graph $K_{2,2,\ldots,2}$ with k parts of size 2 each. The edge-set of this graph consists of $\binom{k}{2}$ 4-cycles, each of which joins two parts of size 2 and is called a *basic 4-cycle* in M_n.

We will consider some special drawings of M_n. Let P be a set of k points in general position in $\mathbb{S}^2$. Let $\mathcal{S}$ be obtained from P by adding, for each $p \in P$, its antipodal point $\bar{p}$ into $\mathcal{S}$. The geodesic drawing of M_n on these points, where each antipodal pair represents a pair of nonadjacent vertices in M_n, is said to be an *antipodal geodesic drawing of* M_n. We will denote by $D_n(P)$ the antipodal drawing of M_n determined by P.

Lemma 4.7 ([20]). *For every $k \geq 3$, every antipodal drawing $D_n(P)$ of M_n has precisely $\frac{1}{4}k(k-1)(k-2)(k-3)$ crossings, and by adding any geodesic half-circle between a pair of antipodal points $p, \bar{p}$ ($p \in P$), we obtain precisely $\frac{1}{2}(k-1)(k-2)$ additional crossings.*

Proof. Note that every pair of points $p, q \in P$ together with their antipodes $\bar{p}$, $\bar{q}$ determines a great circle Q_{pq} that consists of four edges forming the basic 4-cycle between $\{p, \bar{p}\}$ and $\{q, \bar{q}\}$. Any two such great circles Q_{pq} and Q_{rs} cross twice and make two crossings if $\{p, q\} \cap \{r, s\} = \emptyset$. If $|\{p, q\} \cap \{r, s\}| = 1$, then they do not cross. Thus, the edges in each Q_{pq} participate in precisely $2\binom{k-2}{2} = (k-2)(k-3)$ crossings. By summing up these numbers over all $\binom{k}{2}$ possibilities for the pair $\{p, q\}$, we count each crossing twice, so

$$\mathrm{cr}(D_n) = \frac{1}{2}\binom{k}{2}(k-2)(k-3) = \frac{1}{4}k(k-1)(k-2)(k-3).$$

By adding any great circle through two antipodal points $p, \bar{p}$, $p \in P$, we separate $k-1$ of the points in $P \cup \overline{P}$ from their antipodal pairs. There are precisely $(k-1)(k-2)$ edges joining them. Because of the antipodal symmetry of the drawing D_n, precisely half of these edges cross each half-circle. Thus, each half-circle is crossed $\frac{1}{2}(k-1)(k-2)$ many times. ∎

We say that a set P of points in $\mathbb{S}^2$ has *strength s* if there is a choice of half-circles joining each point in P with its antipodal point $\bar{p}$ such that these half-circles cross each other s times.

Corollary 4.8 ([20]). *If a set P of k points in general position on $\mathbb{S}^2$ has strength s, then the drawing $D_n(P)$ can be extended to a geodesic drawing of the complete graph K_n with $H(n) + s$ crossings.*

Proof. We extend the drawing D_n by adding half-circles joining the antipodal pairs $p, \bar{p}$ for $p \in P'$ so that these half-circles make s crossings among each other. By Lemma 4.7, the number of crossings is $\frac{1}{4}k(k-1)(k-2)(k-3) + \frac{1}{2}(k-1)(k-2)|P| + s$, which is equal to $H(n) + s$. ∎

It is easy to see that there are many sets of strength 0. They give rise to antipodal Hill drawings.

Corollary 4.9. *Let $P \subset \mathbb{S}^2$ be a set of k points in general position in $\mathbb{S}^2$, whose strength is* 0. *Then the geodesic drawing $D_n(P)$ $(n = 2k)$ can be extended to a Hill drawing of K_n. This drawing has the following additional properties:*

(a) *the drawing is antipodally symmetric except for the drawing of the half-circles joining antipodal pairs;*

(b) *for every vertex v of K_n, the edges incident with v participate in precisely $\frac{1}{16}(n-2)^2(n-4)$ crossings;*

(c) *by deleting any point from $P \cup \overline{P}$, we obtain a drawing of K_{n-1} with precisely $H(n-1)$ crossings;*

(d) *by adding any new point (in general position with respect to P) and adding geodesics from that point to $P \cup \overline{P}$, we obtain a geodesic drawing of K_{n+1} with precisely $H(n+1)$ crossings.*

Proof. Statements (a)–(c) are easy observations and their proof is left for the reader. To prove (d), let $Q = P \cup \{q\}$, where q is the added point. Consider the corresponding drawing of K_{n+2} for $\widehat{Q}$. Note that Q may no longer have strength 0, but since P has strength 0, there is a drawing where the only half-circle intersecting other half-circles is the half-circle joining q and $\bar{q}$. All these added crossings disappear after removing $\bar{q}$, and thus by (b), the extended drawing of K_{n+1} has $H(n+2) - \frac{1}{16}n^2(n-2) = H(n+1)$ crossings. ∎

The last corollary implies that $\mathrm{cr}_{\mathbb{S}^2}(K_n) \leq H(n)$ for every positive integer n. This result is surprising in two ways. Firstly, it is known that the rectilinear crossing number (geodesic version in the Euclidean plane) of complete graphs is strictly larger than the usual crossing number. So, assuming the Hill conjecture, it is surprising that the geodesic crossing number in the sphere is not different. Secondly, the abundance of obtained Hill drawings is also quite unexpected.

4.4. Moon's result revisited

A probability measure μ on the sphere $\mathbb{S}^2$ is *non-degenerate* if $\mu(C) = 0$ for each great circle C. This is equivalent to saying that the probability that n μ-random points on the sphere lie in general position is equal to 1 (with probability 1, they are all distinct and no three are on the same great circle). Further, we say that μ is *antipodally symmetric* if for any μ-measurable set $A \subseteq \mathbb{S}^2$, its antipodal set $\overline{A}$ has the same measure, $\mu(\overline{A}) = \mu(A)$.

As mentioned before (see Theorem 4.2), Moon proved that random geodesic drawings of complete graphs on the sphere have asymptotically about the same number of crossings as the conjectured best drawings. Corollary 4.8 gives a simple explanation of this phenomenon. Indeed, in a forthcoming work [21] the following result with several interesting consequences is derived.

Theorem 4.10 (Mohar and Wesolek [21]). *Let μ be a non-degenerate antipodally symmetric probability distribution on the unit sphere $\mathbb{S}^2$. Then a μ-random set of n points on $\mathbb{S}^2$ joined by geodesics gives rise to a drawing D_n of the complete graph K_n such that* $\operatorname{cr}(D_n)/H(n) = 1 + o(1)$ *asymptotically almost surely.*

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