DOI 10.4171/8ECM/38

Global properties of some weight 3 variations of Hodge structure

Simion Filip

Abstract. We survey results on the global geometry of variations of Hodge structure with Hodge numbers $(1,1,1,1)$. Included are uniformization results of domains in flag manifolds, a strong Torelli theorem, as well as the formula for the sum of Lyapunov exponents conjectured by Eskin, Kontsevich, Möller, and Zorich. Additionally, we establish the Anosov property of the monodromy representation, using gradient estimates of certain functions derived from the Hodge structure.

1. Introduction

Consider the following family of algebraic 3-manifolds in $\mathbb{P}^4$:

$$\left\{t \cdot (x_1^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) - x_1x_2x_3x_4x_5 = 0\right\} =: Q_t.$$

This is the famous mirror quintic family, used by string theorists in [4] to make predictions about the number of rational curves on a generic quintic 3-fold. Every member of the family admits a nowhere vanishing 3-form Ω_t, which, integrated over an explicit cycle (near $t = 0$), yields the hypergeometric series

$$\psi_0(t) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} t^n. \tag{1.1}$$

This function satisfies (after rescaling t to $5^5 t$) the hypergeometric differential equation

$$\left[D^4 - t\left(D + \frac{1}{5}\right)\left(D + \frac{2}{5}\right)\left(D + \frac{3}{5}\right)\left(D + \frac{4}{5}\right)\right]\psi_0 = 0, \quad \text{where } D := t\frac{d}{dt}.$$

All the results below are of interest already in this particular example, although they apply to a much larger class of variations of Hodge structure.

2020 Mathematics Subject Classification. Primary 14D07; Secondary 34D08, 37D20, 14L24.
Keywords. Variations of Hodge structure, Calabi–Yau, Anosov representation, GIT, Lyapunov exponents.

Monodromy. Let $X := X(\infty, \infty, 5)$ denote the orbifold Riemann surface $\mathbb{P}^1 \setminus \{0, 1\}$ with an orbifold point of order 5 at ∞; the notation is meant to suggest that it comes from the hyperbolic triangle group with angles $(\pi/\infty, \pi/\infty, \pi/5)$. The cohomology $H^3(Q_t; \mathbb{Z})$ has rank 204, but of interest to us is a piece invariant by a natural finite abelian group of roots of unity. This invariant subspace has rank 4, and in fact gives a local system over X (the monodromy around ∞ has order 5).

The explicit matrices are, in an appropriate choice of basis, around 1 and ∞:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{bmatrix}.$$

It can be checked that the last matrix has order 5, and that both preserve the standard symplectic form on $\mathbb{R}^4$. Here is the first result:

Theorem 1.1 (Log-Anosov monodromy). *The monodromy representation*

$$\rho\colon \pi_1(X) \to \mathbf{Sp}_4(\mathbb{R})$$

is log-Anosov. There exists a continuous, dynamics-preserving, ρ-equivariant map

$$\xi\colon \partial\widetilde{X} \to \mathbb{P}(\mathbb{R}^4).$$

Let us explain the terms. The universal cover of X, denoted by $\widetilde{X}$, is isometric to the hyperbolic plane and as such has a boundary, isomorphic to $\mathbb{P}^1(\mathbb{R})$, and denoted by $\partial\widetilde{X}$. The notion of Anosov representation, introduced by Labourie [15], requires a quantitative divergence of the singular values of the monodromy matrices; see Definition 3.1 for the details. In the present context, we need to take into account the presence of unipotent elements in the group, hence the term "log-Anosov". Under the name relatively Anosov, or relatively dominated, such representations were studied by Kapovich–Leeb [12] and Zhu [21], respectively.

Integral vectors. The "limit set" curve, i.e., the image of ξ, provided by Theorem 1.1 is a fractal curve (in fact it is possible and not hard to prove that because of the rank 1 unipotent element, the limit curve cannot be rectifiable). See Figure 1 for some illustrations. Nonetheless, we can classify the rational points on the curve, and similarly an analogous limit set in the Lagrangian Grassmannian.

Theorem 1.2 (Rational directions on limit curve). *Let $\Gamma \subset \mathbf{Sp}_4(\mathbb{Z})$ denote the image of the monodromy representation.*

(1) *A line $[v] \in \xi(\partial\widetilde{X}) \subset \mathbb{P}(\mathbb{Q}^4)$ has rational coordinates if and only if there exists a unipotent transformation $\gamma \in \Gamma$ such that v is both in the kernel and*

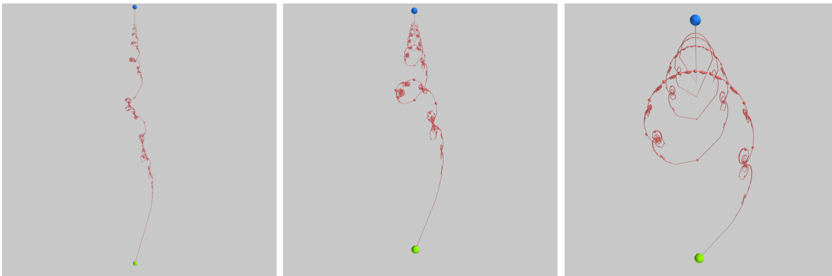

Figure 1. Sample images of the map ξ from Theorem 1.1 for several families of Calabi–Yau manifolds. The middle curve corresponds to the mirror quintic.

image of $\gamma - \mathbf{1}$*. In particular, the rational vectors on* $\xi(\partial\widetilde{X})$ *fall into finitely many orbits under the action of* Γ*, corresponding to the cusps of* X*.*

(2) *Suppose that a Lagrangian* $L \subset \mathbb{Q}^4$ *contains* $\xi(p)$*, for some point* $p \in \partial\widetilde{X}$*. Then there exists a unipotent* $\gamma \in \Gamma$ *and* $[v] = \xi(p)$ *as in the first part, in the kernel and image of* γ*, such that* $[v] \subset L$*.*

Note that the property of being both in the kernel, and in the image, of $\gamma - \mathbf{1}$ is equivalent (in $\mathbf{Sp}_4$) to v belonging to the deepest part of the monodromy weight filtration.

Torelli theorems. Our methods also allow us to gain some insight into the global structure of some moduli spaces of Calabi–Yau 3-folds. Before discussing it, let us introduce some notation. Let $\mathbb{V} \to X$ denote the rank 4 local system of the cohomology of interest of Q_t. It admits a variation of Hodge structure, i.e., a decomposition of the complexification:

$$\mathbb{V}_{\mathbb{C}} = \mathcal{V}^{3,0} \oplus \mathcal{V}^{2,1} \oplus \mathcal{V}^{1,2} \oplus \mathcal{V}^{0,3}$$

which depends on the point on X (with additional properties). The symplectic pairing on $\mathbb{V}$ induces an indefinite Hermitian pairing on $\mathbb{V}_{\mathbb{C}}$, for which $F^2 = \mathcal{V}^{3,0} \oplus \mathcal{V}^{2,1}$ is of signature $(1,1)$. Note that it is a Lagrangian subspace for the symplectic form, and we denote by $\mathrm{LGr}^{1,1}(V_{\mathbb{C}})$ the space of all such Lagrangians in a fixed vector space $V_{\mathbb{C}}$.

Theorem 1.3 (Strong Torelli using Lagrangians). (1) *The maps induced by the Hodge filtration:*

$$\widetilde{X} \xrightarrow{F^2} \mathrm{LGr}^{1,1}(V_{\mathbb{C}})$$

and taking the quotient by π_1*:*

$$X \xrightarrow{F^2} \Gamma \backslash \mathrm{LGr}^{1,1}(V_{\mathbb{C}})$$

are injective.

(2) *Furthermore, for* $x, y \in \widetilde{X}$*, if* L *is a real Lagrangian such that* $(F^2(x) \cap L_{\mathbb{C}}) \neq 0 \neq (F^2(y) \cap L_{\mathbb{C}})$*, then* $x = y$*.*

A further dichotomy, related to rational Lagrangian subspaces, is contained in Corollary 1.4. For the mirror quintic family, a generic Torelli theorem using the full Griffiths period domain was proved by Usui [19].

Remark (On Griffiths' intermediate Jacobians). When the real weight 3 Hodge structure has an underlying integral structure, one can associate to it the Griffiths intermediate Jacobian $V_{\mathbb{Z}} \backslash V_{\mathbb{C}} / F^2$. In general, this is not an abelian variety, and the period domain of such objects is $\mathrm{LGr}^{1,1}(V_{\mathbb{C}})$, a pseudo-Hermitian homogeneous space, in contrast to Siegel spaces parametrizing marked abelian varieties.

It is not possible to take a Hausdorff quotient of $\mathrm{LGr}^{1,1}(V_{\mathbb{C}})$ by $\mathbf{Sp}_4(\mathbb{Z})$, or any lattice in $\mathbf{Sp}_4(\mathbb{R})$. However, Theorem 4.3 implies that for the monodromy Γ of a VHS satisfying assumption A, *it is* possible to take the quotient. Theorem 1.3 then says that the period map to this quotient is injective. So it can be viewed a Torelli theorem in the classical sense.

Theorem 1.2 above, combined with the strong Torelli theorem provides an interesting property of rational Lagrangian subspaces:

Corollary 1.4 (Dichotomy for rational Lagrangian subspaces). *With assumptions as in Theorem 1.2, for every rational Lagrangian subspaces* $L \subset \mathbb{Q}^4$*, precisely one of the following holds:*

- *either there exists a unipotent transformation* $\gamma \in \Gamma$ *and vector* $v \in L$ *such that* v *is both in the kernel and in the image of* $\gamma - \mathbf{1}$*,*
- *or there exists a unique* $x \in \widetilde{X}$ *such that* $L_{\mathbb{C}} \cap F^2(x) \neq \{0\}$*.*

Lyapunov exponents. The original reason that prompted the above results was a conjecture of Eskin, Kontsevich, Möller, and Zorich from [6] relating Lyapunov exponents, which are invariants coming from dynamical systems, with the degrees of certain line bundles. Using the above results, we can establish their conjecture:

Theorem 1.5 (Formula for the sum of Lyapunov exponents). *Let* $\lambda_1 \geq \lambda_2 \geq 0$ *be the nonnegative Lyapunov exponents of the geodesic flow on the unit tangent, for the cocycle induced by* $\mathbb{V}$*. Then*

$$\lambda_1 + \lambda_2 = \frac{\deg \mathcal{V}^{0,3}_{\mathrm{ext}} + \deg \mathcal{V}^{1,2}_{\mathrm{ext}}}{\chi(X)} = \frac{6}{5},$$

where $\chi(X)$ *is the (orbifold) Euler characteristic of* X*,* deg *denotes the (orbifold) degree of a complex line bundle, and the subscript* ext *denotes the Deligne extension of a bundle across punctures.*

As we shall explain below, in fact the stronger conjecture, Conjecture 6.4 of [6], of a number-theoretic flavor, also holds. Note that the degrees of the bundles were computed in loc. cit. and come from the parameters of the corresponding hypergeometric differential equation.

Explicit nonvanishing. The conjecture alluded to above is formulated in terms of the subspace invariant by the monodromy near the singularity at 0. It has an explicit statement in terms of power series, which we now explain.

Recall from equation (1.1) that we defined one solution of the hypergeometric equation, and consider the second one

$$\psi_1(t) := \sum_{n\geq 0} \frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n} \frac{1}{k}\right)\cdot t^n.$$

There are two more (with further logarithmic terms), but we are interested in the following Wronskian determinant (which is a 2×2 minor of the full matrix of solutions)

$$Wr(t) := \psi_0(t)\psi_1'(t) - \psi_0'(t)\psi_1(t).$$

To describe the uniformization of the orbifold X, we make use again of classical hypergeometric functions:

$$F_0(t) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(n!)^2} t^n,$$

$$F_1(t) = F_0(t)\log t + \sum_{n\geq 0} \frac{(a)_n(b)_n}{(n!)^2}\left(\sum_{k=1}^{n} \frac{1}{a+k-1} + \frac{1}{b+k-1} - \frac{2}{k}\right)t^n,$$

where $a = \frac{2}{5}$ and $b = \frac{3}{5}$.

Define now the map

$$q := \exp\left(\frac{F_1(t)}{F_0(t)}\right) = t\cdot\exp\left(\frac{P_1(t)}{P_0(t)}\right),$$

where $P_1(t)$, $P_0(t)$ are the power series appearing under the summation sign in the definition of F_0, F_1. Finally, define the inverse power series $\lambda_5(q) = \sum_{n\geq 0} \Lambda_n q^n$ such that

$$\lambda_5\left(\exp\left(\frac{F_1(t)}{F_0(t)}\right)\right) = t.$$

This yields the uniformization cover

$$\left\{0 < |q| < e^{D(1)}\right\} \xrightarrow{\lambda_5} X,$$

where $D(1)$ denotes a sum of values of the logarithmic derivative of the gamma-function.

The explicit nonvanishing, conjectured in [6, Conj. 6.4], then reads

$$Wr\left(\frac{1}{5^5}\lambda_5(q)\right) \text{ never vanishes.}$$

Domains of discontinuity. A consequence of the Anosov property of the monodromy representation is that the image group $\Gamma \subset \mathbf{Sp}_4(\mathbb{R})$ has a large class of domains of discontinuity in real and complex flag manifolds associated to $\mathbf{Sp}_4$. In Theorem 4.3, we list some of them.

Thin groups. No lattice in $\mathbf{Sp}_4(\mathbb{R})$ can have a domain of discontinuity as in Theorem 4.3. It follows that the monodromy group Γ, when it has an integral structure, is necessarily a "thin group" in the sense of Sarnak [17]. Let us note that the proofs in the present paper offer an alternate route to some of the results from [2, 9], where thinness is established using ping-pong and an explicit construction of cones. These explicit cones have, nonetheless, other applications to a more detailed understanding of the monodromy groups.

Below we outline the main notions that go into the proof of the above results, in greater generality. A detailed account is in [8].

2. Variations of Hodge structure and hypergeometric equations

2.1. Variations of Hodge structure

Let X be a complex manifold and $\mathbb{V} \to X$ a local system of real vector spaces. Equivalently, this is a bundle with flat connection ∇, also called the Gauss–Manin connection.

Definition 2.1 (Variation of Hodge structure). A *variation of Hodge structure* (or VHS) on $\mathbb{V}$ of weight n is a decomposition of the complexification

$$\mathbb{V}_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{V}^{p,q}(x), \quad x \in X$$

such that the following hold.

- The *Hodge filtration* $\mathcal{F}^p := \oplus_{s \geq p} \mathcal{V}^{s,n-s}$ varies holomorphically.
- The *Griffiths transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$$

  is satisfied.
- Under complex conjugation, we have that $\mathcal{V}^{p,q} = \overline{\mathcal{V}^{q,p}}$.

Additionally, the variation is *polarized* if there exists a $(-1)^n$-symmetric nondegenerate bilinear form on $\mathbb{V}$, parallel for the Gauss–Manin connection, and such that the induced Hermitian pairing on $V_{\mathbb{C}}$ has signature $(-1)^q$ on $\mathcal{V}^{p,q}$.

The classical case is that of a weight 2 variation, when there are two bundles $\mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$. Such variations come from holomorphic families of abelian varieties or Riemann surfaces.

Second fundamental form. The Griffiths transversality condition allows us to define a second fundamental form, by taking $\nabla(\mathcal{F}^p)/\mathcal{F}^p$ to obtain a well-defined *linear* map of bundles

$$\sigma_\bullet = \oplus \sigma_{p,q} \quad \text{with } \sigma_{p,q}\colon \mathcal{V}^{p,q} \to \mathcal{V}^{p-1,q+1} \otimes \Omega^1_X.$$

Tensor constructions. We will be interested in variations with $\dim \mathcal{V}^{p,q} = 1$ and weight 3, or said differently with Hodge numbers $(1, 1, 1, 1)$. These admit an invariant symplectic form and the monodromy of the local system is valued in $\mathbf{Sp}_4(\mathbb{R})$. We can also perform natural tensor constructions, of which the most useful is the (reduced) second exterior power $\mathbb{W} := \Lambda^2_\circ \mathbb{V}$, where reduced means that we remove a 1-dimensional invariant subspace generated by the symplectic form. Therefore, $\mathbb{W}$ has rank 5 and Hodge numbers $(1, 1, 1, 1, 1)$ (we Tate-twist the construction so that it has weight 4, not 6). In fact, up to passing to a finite cover, it is possible to recover $\mathbb{V}$ from $\mathbb{W}$.

We assume from now on that X is a finite volume complete hyperbolic Riemann surface, and denote by the subscript ext the Deligne extension of bundles across the punctures (the reader unfamiliar with these terms can just assume that X is compact).

Definition 2.2 (Assumption A). We will say that $\mathbb{V}$ satisfies *assumption A* if the component of the second fundamental form

$$\sigma_{2,1}\colon \mathcal{V}^{2,1}_{\text{ext}} \to \mathcal{V}^{1,2}_{\text{ext}} \otimes \Omega^1_X$$

is an isomorphism.

Equivalently on $\mathbb{W}$, the requirement is that $\sigma_{4,0}$ is an isomorphism.

All the theorems described below apply as soon as the VHS satisfies assumption A.

2.2. Hypergeometric equations

For general information and the results below on hypergeometric equations, we refer to the texts of Beukers–Heckman [1] or Yoshida [20].

Let $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ be a list of $2n$ numbers (while we will always take them real later, at this stage they can be complex). Define the differential operator

$$D_{\alpha,\beta} := \prod_{i=1}^{n}(D - \beta_i) - z\prod_{i=1}^{n}(D + \alpha_i), \quad \text{where } D = z\partial_z.$$

$\alpha_\bullet$	$\beta_\bullet$	Conditions
$\left(\mu, \frac{1}{2}, \frac{1}{2}, 1-\mu\right)$	$(0,0,0,0)$	$\mu \in \left(0, \frac{1}{2}\right]$
	$(0,0,\nu,1-\nu)$	$0 < \nu < \mu \leq \frac{1}{2}$

Table 1. Parameters μ, ν are real.

$\beta_\bullet$	Conditions
$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	
$(0,0,0,0)$	
$(0,0,\mu,1-\mu)$	$0 < \mu < \frac{2k-1}{2N}$
$\left(\mu, \frac{1}{2}, \frac{1}{2}, 1-\mu\right)$	$\frac{N-1}{2N} < \mu < \frac{1}{2}$
$\left(\frac{M-(2k_M+1)}{2M}, \frac{M-1}{2M}, \frac{M+1}{2M}, \frac{M+(2k_M+1)}{2M}\right)$	$\frac{2k_M+1}{M} < \frac{1}{N}$

Table 2. Parameters M, k_M are integers; μ is real. Throughout, $\alpha_\bullet = (\frac{N-(2k+1)}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+(2k+1)}{2N})$ with arbitrary integers $k \geq 1$, $N > 2k+1$.

With these conventions, the exponents (or Riemann scheme) of the operator are:

- at 0: $\beta_1, \ldots, \beta_n$,
- at ∞: $\alpha_1, \ldots, \alpha_n$,
- at 1: $0, 1, \ldots, n-2, \gamma := (n-1) - \sum_{i=1}^{n}(\alpha_i + \beta_i)$.

The parameters are slightly different from those of [1], specifically $\beta_i^{BK} = 1 - \beta_i$, and there are plenty of other variations in the literature.

The local system of solutions of the hypergeometric equations is *rigid*, meaning that it has no nontrivial deformations when we require the conjugacy classes at the cusps to be fixed. By a theorem of Simpson [18, Cor. 8.1], it follows that the local system underlies a variation of Hodge structure. Fedorov [7] provided a recipe for computing the Hodge numbers of the corresponding VHS.

This allows us to tabulate the values of the hypergeometric parameters which satisfy assumption A, listed in Tables 1 and 2.

Assumption B/maximal representations. It is natural to consider also a variant of assumption A, asking instead that the second fundamental form $\sigma_{3,0}$ is an isomorphism. This leads to the class of *maximal monodromy representations*, in the sense introduced in [3]. Following an analogous algorithm as in the case of assumption A, it is possible to tabulate the hypergeometric rank 5 parameters such that the local system $\mathbb{W}$, with monodromy in $\mathbf{SO}_{2,3}(\mathbb{R})$, is maximal. The results are listed in Table 3.

Schwarz reflection. Hypergeometric equations with real parameters satisfy a complex conjugation symmetry and their monodromy groups can be embedded with

$\alpha_\bullet$	$\beta_\bullet$
$\left(\mu, \frac{1}{2}, \frac{1}{2}, 1-\mu, \frac{1}{2}\right)$	$\left(0, 0, 0, \frac{M}{2M+1}, \frac{M+1}{2M+1}\right)$
or	or
$\left(\frac{N-k_N}{2N}, \frac{N-1}{2N}, \frac{1}{2}, \frac{N+1}{2N}, \frac{N+k_N}{2N}\right)$	$\left(0, \frac{k_M}{M}, \frac{k_M+1}{M}, \frac{M-(k_M+1)}{M}, \frac{M-k_M}{M}\right)$

Table 3. Any set choice from the first column is compatible with any choice from the second, subject to the condition $\alpha_{\min} > \beta_{\text{med}}$, where $\alpha_{\min} := \mu$ or $\frac{N-k_N}{2N}$, and $\beta_{\text{med}} := \frac{M}{2M+1}$ or $\frac{k_M+1}{M}$ depending on the choices. The parameter μ is real, while M, N, k_M, k_N are positive integers with $1 < k_N < N$ and $2(k_M + 1) < M$.

index 2 into a group generated by three order 2 involutions. Geometrically, this corresponds to the following construction. Take a basis of solutions in the upper half plane. Analytically continue it into the lower half-plane by choosing one of the three segments formed by removing 0, 1 from $\mathbb{R}$. Then, apply the Schwarz reflection, i.e., map $f(z)$ to $\overline{f(\bar{z})}$, to obtain another basis of solutions in the upper half-plane.

The operation of the Schwarz reflection is just complex-conjugating the coefficients of a Taylor expansion of $f(z)$. Because the hypergeometric equation has real coefficients, the resulting functions are still solutions of the hypergeometric equation. Note that on each of the three segments of the real axis, there is a basis of solutions with real coefficients, but on each segment the basis is different. For instance, $\log z$ is real-valued on $(0, \infty)$ but has also an imaginary component on $(-\infty, 0)$.

It is possible to choose a basis and explicitly give the matrices of the three reflections generating the above construction. Let $\mathcal{R}_A$, $\mathcal{R}_B$, $\mathcal{R}_C$ be the transformations corresponding to crossing along $(1, \infty)$, $(0, 1)$, and $(\infty, 0)$ on the real axis. Then the matrices giving the transformations are listed in equation (2.1). To obtain the action of $\mathcal{R}_X$ on the space of solutions, one must apply complex conjugation to the coordinates after applying the matrix R_X:

$$R_A := \begin{bmatrix} 0 & \cdots & 0 & 1 & -A_1 \\ 0 & \cdots & 1 & 0 & -A_2 \\ & \iddots & & & \\ 1 & \cdots & 0 & 0 & -A_{n-2} \\ 0 & \cdots & 0 & 0 & -A_n \end{bmatrix} \quad R_B := \begin{bmatrix} 0 & \cdots & 0 & 1 & -\overline{B_1} \\ 0 & \cdots & 1 & 0 & -\overline{B_2} \\ & \iddots & & & \\ 1 & \cdots & 0 & 0 & -\overline{B_{n-2}} \\ 0 & \cdots & 0 & 0 & -\overline{B_n} \end{bmatrix}$$

$$\text{as well as } R_C := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.1}$$

Recall that the hypergeometric equation has parameters α_i, β_j and we set $a_j := \exp(2\pi\sqrt{-1}\alpha_j)$ and $b_j := \exp(2\pi\sqrt{-1}\beta_j)$ as well as

$$p_A(t) := \prod_{j=1}^{n}(t-a_j) = t^n + A_1t^{n-1} + \cdots + A_n,$$
$$p_B(t) := \prod_{j=1}^{n}(t-b_j) = t^n + B_1t^{n-1} + \cdots + B_n$$

to obtain the entries of the matrices. This should be compared to the Levelt presentation of the monodromy matrices of the hypergeometric equation from [1, Thm. 3.5].

Tiling by polyhedra. For the domains of discontinuity constructed in Theorem 4.3, it is possible to give a detailed description of a tiling by polyhedra, in direct analogy with the tiling of the hyperbolic plane by hyperbolic triangles. Crucially, the variation of Hodge structure underlying the hypergeometric local system provides the data needed to construct the edges of the polyhedron.

3. Log-Anosov representations

3.1. Lie theory and Anosov representations

The notion of Anosov representation was introduced by Labourie in [15]. We refer to the works of Guichard–Wienhard [11, 13] for extensive references and a general introduction to the subject. Some general background in Lie theory can be gathered from [14].

It is worth keeping in mind that there is both an extrinsic and an intrinsic approach to Anosov representations, and that both are useful. Fix a semisimple real algebraic group G. One can look at a representation $\rho\colon \pi_1(X) \to G$, or consider a (possibly reducible) algebraic representation $\phi\colon G \to \mathbf{GL}(V)$. It is possible to go back and forth between the properties of ρ and the properties of $\phi \circ \rho$, and it is useful to do so.

Lie theory preliminaries. Continuing with the semisimple Lie group G as above, let $K \subset G$ be maximal compact and let $\mathfrak{a} = \operatorname{Lie} A$ be the split part of a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subset \mathfrak{a}^\vee$ be the roots and $\Delta \subset \Phi$ the subset of simple roots, for a choice of ordering, which also yields the Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Finally, let $\theta \subset \Delta$ be a (nonempty) set of roots.

KAK, or polar decomposition. One way to approach the coarse geometry of the group G is via its KAK decomposition. Namely, any $g \in G$ can be written as

$$g = k_-(g) \cdot e^{\mu(g)} k_+(g) \quad \text{with } k_\pm(g) \in K \text{ and } \mu(g) \in \mathfrak{a}^+.$$

The K-components are not necessarily unique, but $\mu(g)$ is.

Let also $\|-\|\colon \pi_1(X) \to \mathbb{R}_{\geq 1}$ be the matrix norm induced by some Fuchsian representation $\pi_1(X) \to \mathbf{SL}_2(\mathbb{R})$.

Definition 3.1 (Log-Anosov representation). A representation $\rho\colon \pi_1(X) \to G$ is called *log-Anosov* if there exist $C, \varepsilon > 0$ such that

$$\alpha\big(\mu\big(\rho(\gamma)\big)\big) \geq \varepsilon \cdot \log \|\gamma\| - C \quad \forall \alpha \in \theta,\ \forall \gamma \in \pi_1(X),$$

where μ denotes the element in $\mathfrak{a}^+$ from the KAK decomposition.

In the more general setting of relatively hyperbolic groups, this notion was studied under the name relatively Anosov, or relatively dominated, by Kapovich–Leeb [12] and Zhu [21].

Boundary map. A consequence of the (log)-Anosov property is the existence of boundary maps; see for instance [10, Thm. 1.1] and [21, Thm. 1.2]. Specifically, recall that the universal cover $\widetilde{X}$ is isometric to the hyperbolic plane, and it has a visual boundary $\partial\widetilde{X}$. Then there exists a continuous, ρ-equivariant map

$$\xi\colon \partial\widetilde{X} \to \mathcal{F}_\theta,$$

where $\mathcal{F}_\theta$ is the manifold of flags associated to θ, so $\mathcal{F}_\theta \cong G/P_\theta$, where P_θ is a parabolic subgroup corresponding to θ.

3.2. Proper discontinuity, stability, and GIT

Kapovich, Leeb, and Porti in [13] have emphasized the analogy between the action of discrete subgroups of Lie groups, and those of algebraic groups in linear representations, viewed from the lens of Mumford's Geometric Invariant Theory [16]. We make some further definitions and take it as a viewpoint to perform some of the constructions appearing later.

Suppose for this section that $\Gamma \subset G$ is a closed subgroup of G. Associated to it are various notions of limit sets, similarly to how there are various notions of boundaries for G, or G/K. The one of interest to us, which we will denote by $\mathcal{L}_+ \subset K \times \mathbb{P}\mathfrak{a}^+$, consists of all possible accumulation points of the coordinates $k_+(\gamma) \in K$ and $[\mu(\gamma)] \in \mathbb{P}\mathfrak{a}^+$ (the projectivized cone) as $\mu(\gamma) \to +\infty$. Suppose now that V is a G-representation.

Definition 3.2 (Stable and semistable points). Let $[v] \in \mathbb{P}(V)$ be a point and $v \in V$ a lift of it to the vector space. Then $[v]$ is *stable* if

$$\|e^{t\mu} k_+ v\| \to +\infty \quad \forall\big(k, [\mu]\big) \in \mathcal{L}_+$$

and $[v]$ is *semistable* if

$$\liminf \|e^{t\mu} k_+ v\| > 0 \quad \forall\big(k, [\mu]\big) \in \mathcal{L}_+.$$

This definition is the direct analogue of the Hilbert–Mumford numerical criterion. It is then possible to show that the following theorem holds.

Theorem 3.3 (Proper discontinuity). (1) *The set of stable points is open in* $\mathbb{P}(V)$.

(2) *The action of* Γ *on the set of stable points is properly discontinuous.*

(3) *A stable point and a semistable point cannot be dynamically related.*

Two points x, y are said to be dynamically related if there exist sequences x_i and $\gamma_i \in \Gamma$ such that $x_i \to x$ and $\gamma_i x_i \to y$. Note that this is an equivalence relation (take $y_i = \gamma_i x_i$ and γ_i^{-1} as the sequence).

If the group Γ is the image of a log-Anosov representation, then the boundary map ξ gives useful control on the limit set $\mathcal{L}_+$, and hence on the set of stable points in various linear representations of G.

4. Variations of Hodge structure and log-Anosov representations

In this section, we combine some ideas from Hodge theory with those coming from Anosov representations.

4.1. Growth of vectors

Let $\mathbb{V} \to X$ be a VHS satisfying assumption A from Definition 2.2. Let $\mathbb{W} := \Lambda^2_\circ \mathbb{V}$ be the corresponding reduced second exterior power. We will work on a universal cover $\widetilde{X}$, where both local systems become trivial and can be identified with fixed vector spaces V, W, and the VHS gives a Hodge decomposition of these fixed vector spaces. Recall also that V is symplectic whereas W carries an indefinite pairing of signature $(2, 3)$.

Pick a nonzero vector $w \in W_{\mathbb{R}}$ and write its Hodge decomposition:

$$w = w^{4,0} \oplus w^{3,1} \oplus w^{2,2} \oplus w^{1,3} \oplus w^{0,4}$$

Theorem 4.1 (Growth of vectors). *Let* $f_w(x) := \|w^{0,4}\|^2$, *where* $\| - \|$ *is computed with respect to the Hodge norm at* $x \in \widetilde{X}$.

(1) *Suppose that* w *is isotropic. Then* f_w *has at most one critical point, which can only be a local minimum.*

(2) *Suppose that* w *is positive-definite, for the indefinite metric. Then* f_w *has precisely one critical point, which is a local minimum.*

The proof of this result is based on a gradient estimate for f_w, combined with an argument using Palais–Smale sequences to exclude multiple local minima. In fact, the gradient estimate can be strengthened and used to show that when w is positive-definite, it has exponential growth:

Theorem 4.2 (Exponential growth). *Suppose that w is positive-definite and f_w has a minimum at x_0. Then there exist $C, \varepsilon > 0$ such that*

$$f_w(x) \geq \frac{1}{C} e^{\varepsilon \cdot \mathrm{dist}(x, x_0)} - C.$$

With this information in hand, it is not hard to obtain the log-Anosov condition on the monodromy representation from Definition 3.1.

4.2. Uniformization results

With the log-Anosov property in hand, the formalism of stable vectors in representations from Section 3.2 provides a wealth of domains of discontinuity for the monodromy.

Theorem 4.3 (Domains of discontinuity). *Let $\Gamma \subset \mathbf{Sp}_4(V_{\mathbb{R}})$ be the image of the monodromy group.*

1. *In the real Lagrangian Grassmannian $\mathrm{LGr}(V_{\mathbb{R}})$, there exists an open nonempty set Ω_L on which Γ acts properly discontinuously.*
2. *The pseudo-sphere $\mathbb{S}^{1,3}$ of unit vectors in $W_{\mathbb{R}}$ is a domain of discontinuity for Γ.*
3. *The complex Grassmannian of Lagrangians of signature $(1, 1)$ for the indefinite Hermitian metric on $V_{\mathbb{C}}$, denoted by $\mathrm{LGr}^{1,1}(V_{\mathbb{C}})$, is also a domain of discontinuity for Γ.*
4. *In the complex projective space $\mathbb{P}(V_{\mathbb{C}})$, there exists an open, nonempty set Ω_P on which Γ acts properly discontinuously.*

More interestingly, it is possible to obtain a uniformization result for the domain of discontinuity in the Lagrangian Grassmannian. For an element $F^2 = \mathcal{V}^{3,0} \oplus \mathcal{V}^{2,1}$ of the Hodge filtration, set

$$\beta(F^2) := \big\{ \text{real Lagrangians } L \text{ s.t. } L_{\mathbb{C}} \cap F^2 \neq \{0\} \big\}.$$

In other words, we consider the real Lagrangians which are not transverse to F^2, after complexification. It can be directly checked that for a fixed F^2, these form a circle inside the real 3-dimensional manifold $\mathrm{LGr}(V_{\mathbb{R}})$. Let $\widetilde{\mathrm{Bad}} \to \widetilde{X}$ denote this circle bundle over the universal cover, and $\mathrm{Bad} \to X$ its quotient by $\pi_1(X)$. The reason for the name "Bad" will be explained in the section on Lyapunov exponents below. For now, observe that there is a tautological developing map

$$\widetilde{\mathrm{Bad}} \xrightarrow{\mathrm{Dev}} \mathrm{LGr}(V_{\mathbb{R}})$$

since each fiber of the bundle is a circle in that Lagrangian Grassmannian.

Theorem 4.4 (Uniformization). *The developing map* Dev *is a bijection between* $\widetilde{\text{Bad}}$ *and the domain of discontinuity* $\Omega_L \subset \text{LGr}(V_{\mathbb{R}})$ *from Theorem 4.3.*

4.3. Formula for Lyapunov exponents

Theorem 4.4 implies, in a stronger form, Conjecture 6.4 from [6] that the MUM Lagrangian is never "bad". Recall that in that context, to every vector $w \in W_{\mathbb{R}}$ one can associate a "bad locus" corresponding to points in the universal cover where $w^{0,4} = 0$. The emptiness of the bad locus, for at least one vector w, implies the expected formula for the sum of Lyapunov exponents. This is precisely the formula stated in the introduction in Theorem 1.5.

Maximal representations. We end with an observation regarding what we called "assumption B", or equivalently the condition that the VHS is maximal described in Section 2.1. A uniformization result analogous to Theorem 4.4 holds in this case, and was established by Collier, Tholozan, and Toulisse [5, Thm. 1]. It implies the following formula for the *top* Lyapunov exponent of $\mathbb{V}$:

$$\lambda_1(\mathbb{V}) = \frac{\deg \mathcal{V}^{0,3}_{\text{ext}}}{\chi(X)}.$$

Note that the only representations which satisfy both assumption A and assumption B are those which are a symmetric power of the standard Fuchsian representation. In that case, all the above theorems, including the domains of discontinuity and the formula for Lyapunov exponents, are immediate verifications in linear algebra.

Acknowledgments. Most of this work was completed when I visited the Institute for Advanced Study during the academic year 2018-2019. I am sincerely grateful for the excellent working conditions. This research was partially conducted during the period the author served as a Clay Research Fellow.

Funding. This material is based upon work supported by the US National Science Foundation under Grants No. DMS-2005470 (SF), No. DMS-1638352 (IAS), as well as DMS-1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation Varieties" (the GEAR Network).

References

[1] F. Beukers and G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Invent. Math.* **95** (1989), no. 2, 325–354 Zbl 0663.30044 MR 974906

[2] C. Brav and H. Thomas, Thin monodromy in Sp(4). *Compos. Math.* **150** (2014), no. 3, 333–343 Zbl 1311.14010 MR 3187621

[3] M. Burger, A. Iozzi, F. Labourie, and A. Wienhard, Maximal representations of surface groups: symplectic Anosov structures. *Pure Appl. Math. Q.* **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590 Zbl 1157.53025 MR 2201327

[4] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B* **359** (1991), no. 1, 21–74 Zbl 1098.32506 MR 1115626

[5] B. Collier, N. Tholozan, and J. Toulisse, The geometry of maximal representations of surface groups into $\mathrm{SO}_0(2,n)$. *Duke Math. J.* **168** (2019), no. 15, 2873–2949 Zbl 07145323 MR 4017517

[6] A. Eskin, M. Kontsevich, M. Möller, and A. Zorich, Lower bounds for Lyapunov exponents of flat bundles on curves. *Geom. Topol.* **22** (2018), no. 4, 2299–2338 Zbl 1386.37036 MR 3784522

[7] R. Fedorov, Variations of Hodge structures for hypergeometric differential operators and parabolic Higgs bundles. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 18, 5583–5608 Zbl 1408.32017 MR 3862114

[8] S. Filip, Uniformization of some weight 3 variations of Hodge structure, Anosov representations, and Lyapunov exponents. 2021, arXiv:2110.07533

[9] S. Filip and C. Fougeron, A cyclotomic family of thin hypergeometric monodromy groups in $\mathrm{Sp}_4(\mathbb{R})$. 2021, arXiv:2106.09181

[10] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, Anosov representations and proper actions. *Geom. Topol.* **21** (2017), no. 1, 485–584 Zbl 1373.37095 MR 3608719

[11] O. Guichard and A. Wienhard, Anosov representations: domains of discontinuity and applications. *Invent. Math.* **190** (2012), no. 2, 357–438 Zbl 1270.20049 MR 2981818

[12] M. Kapovich and B. Leeb, Relativizing characterizations of Anosov subgroups, I. 2018, arXiv:1807.00160

[13] M. Kapovich, B. Leeb, and J. Porti, Dynamics on flag manifolds: domains of proper discontinuity and cocompactness. *Geom. Topol.* **22** (2018), no. 1, 157–234 Zbl 1381.53090 MR 3720343

[14] A. W. Knapp, Structure theory of semisimple Lie groups. In *Representation Theory and Automorphic Forms (Edinburgh, 1996)*, pp. 1–27, Proc. Sympos. Pure Math. 61, Amer. Math. Soc., Providence, RI, 1997 Zbl 0902.17003 MR 1476489

[15] F. Labourie, Anosov flows, surface groups and curves in projective space. *Invent. Math.* **165** (2006), no. 1, 51–114 Zbl 1103.32007 MR 2221137

[16] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*. 3rd edn., Ergeb. Math. Grenzgeb. (2) 34, Springer, Berlin, 1994 Zbl 0797.14004 MR 1304906

[17] P. Sarnak, Notes on thin matrix groups. In *Thin Groups and Superstrong Approximation*, pp. 343–362, Math. Sci. Res. Inst. Publ. 61, Cambridge Univ. Press, Cambridge, 2014 Zbl 1365.11039 MR 3220897

[18] C. T. Simpson, Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.* **3** (1990), no. 3, 713–770 Zbl 0713.58012 MR 1040197

[19] S. Usui, Generic Torelli theorem for quintic-mirror family. *Proc. Japan Acad. Ser. A Math. Sci.* **84** (2008), no. 8, 143–146 Zbl 1164.14003 MR 2457802

[20] M. Yoshida, *Fuchsian Differential Equations. With Special Emphasis on the Gauss-Schwarz Theory*. Aspects of Mathematics, E11, Friedr. Vieweg & Sohn, Braunschweig, 1987 Zbl 0618.35001 MR 986252

[21] F. Zhu, Relatively dominated representations. *Ann. Inst. Fourier (Grenoble)* **71** (2021), no. 5, 2169–2235 Zbl 07492562 MR 4398259

Simion Filip
Department of Mathematics, University of Chicago, 5734 S University Ave, Chicago, IL 60637, USA; sfilip@math.uchicago.edu