DOI 10.4171/8ECM/07

On primes, almost primes, and the Möbius function in short intervals

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Abstract. In this article, aimed at a general mathematical audience, we have three goals. First, we give a brief account of the classical theory connecting primes, the Riemann zeta function, and the Möbius function. Second, we discuss the state-of-art results concerning primes, almost primes, and the Möbius function in short intervals. Third, we outline the most fundamental concepts underlying the proofs of such results.

1. Introduction

Some of the most prominent topics in analytic number theory include the prime numbers, the Riemann zeta function, and the Möbius function. In this article, aimed at a general mathematical audience, we first introduce some classical results on the primes and their relation to the Riemann zeta function in Section 2. Then we go on to discuss primes and almost primes in short intervals in Section 3, starting with classical results and moving on to very recent works. In Section 4 we make a similar journey with the Möbius function. Finally, in Section 5 we discuss the proof strategies, mostly in rather general terms.

2. Primes and the Riemann zeta function

2.1. Primes

We write $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ for the set of primes, i.e., natural numbers > 1 that are only divisible by 1 and themselves. The letter p with or without subscripts will always denote a prime.

One of the first theorems concerning primes is that of Euclid (ca. 300 BC), stating that there are infinitely many prime numbers. This can be quickly proved in various ways. The most classical way is to make a counter assumption that only $p_1, \ldots, p_k$

2020 Mathematics Subject Classification. Primary 11N37; Secondary 11M06, 11N05, 11N25.
Keywords. Prime numbers, Riemann zeta function, Möbius function, short intervals.

are primes. Then $p_1 \cdots p_k + 1$ is either a new prime or divisible by a new prime which is a contradiction.

By the fundamental theorem of arithmetic every natural number can be uniquely written as a product of primes, e.g., $2021 = 43 \cdot 47$. In other words, primes are like the building blocks of the natural numbers.

By Euclid's theorem there are infinitely many primes, but we have much more precise information. Hadamard and de la Valleé Poussin showed independently in the end of the 19th century (see e.g. [12, Notes to Chapter 12]) that[1]

$$\#\{p \in \mathbb{P} : p \leq x\} = \big(1 + o(1)\big) \int_2^x \frac{dx}{\log x} = \big(1 + o(1)\big) \frac{x}{\log x}.$$

This is called the prime number theorem (PNT) and it sort of asserts that the "probability" that an integer n is prime is about $1/\log n$.

In light of this, it is convenient to normalize prime p by $\log p$. More precisely, we write $\Lambda(n)$ for the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Now the PNT is equivalent to the fact that

$$\sum_{n \leq x} \Lambda(n) = \big(1 + o(1)\big) x.$$

As for the $o(1)$ error term in the PNT, the best result (see e.g. [12, Theorem 12.2]) currently is that

$$\sum_{n \leq x} \Lambda(n) = x + O\bigg(x \exp\bigg(-\frac{c(\log x)^{3/5}}{(\log\log x)^{1/5}}\bigg)\bigg) \tag{2.1}$$

for some absolute constant $c > 0$.

2.2. The Riemann zeta function

Next we introduce some basic properties of the Riemann zeta function. For a reference to the results in this and the following subsection, and much more, see e.g. [12] or [25].

[1] We use, for $f\colon \mathbb{R} \to \mathbb{C}$ and $g\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$, the notation $f(x) = O(g(x))$ when there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x and the notation $f(x) = o(g(x))$ when $\lim_{x\to\infty} f(x)/g(x) = 0$. For instance $O(x^{1/2})$ denotes a quantity which is, for some constant $C > 0$, at most $Cx^{1/2}$ for all x and $o(1)$ denotes a quantity tending to 0 when $x \to \infty$.

Write, for $\Re s > 1$,

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2.2}$$

The function $\zeta(s)$ can be analytically continued to the whole complex plane except for a simple pole at $s = 1$ with residue 1. The function $\zeta(s)$ is called the Riemann zeta function and it satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{2.3}$$

where $\Gamma(s)$ is the gamma function.

The functional equation can be used to obtain some basic information about the zeros of the Riemann zeta function. Notice first that on the right-hand side of the functional equation (2.3) the function $\sin(\pi s/2)$ has a zero at each even integer. For $s = 0$ the zero is cancelled by the pole of $\zeta(1-s)$ whereas for positive even integers the poles of $\Gamma(1-s)$ cancel with the zeros. But for negative even integers there are no poles and hence also $\zeta(s)$ has a zero at each negative even integer $-2, -4, -6, \ldots$. These zeros are called the trivial zeros of $\zeta(s)$.

The remaining zeros of $\zeta(s)$ are called non-trivial. From the Euler product (2.2) one sees that there are no zeros with $\Re s > 1$ and thus, by the functional equation (2.3) there are no non-trivial zeros with $\Re s < 0$. Hence all the non-trivial zeros of the zeta function must lie in the critical strip $0 \leq \Re s \leq 1$.

Writing $N(T)$ for the number of non-trivial zeros with $|\Im s| \leq T$, the Riemann-von Mangoldt formula states that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{2.4}$$

The famous Riemann hypothesis (RH) asserts that all these non-trivial zeros actually lie on the critical line $\Re s = 1/2$. This has been numerically verified in [21] for all zeros with $|\Im s| \leq 3 \cdot 10^{12}$. Furthermore we know that the exists a constant $c > 0$ such that, for any zero $s = \beta + it$ of $\zeta(s)$ with $|t| \geq 10$, one has

$$\beta \leq 1 - \frac{c}{\left(\log |t|\right)^{2/3} \left(\log \log |t|\right)^{1/3}}; \tag{2.5}$$

the complement of this region is called the Vinogradov–Korobov zero-free region.

2.3. The relation between primes and the Riemann zeta function

It turns out that the non-trivial zeros of the zeta function are closely related to the prime numbers. The relation between von Mangoldt function and the zeros of the zeta function stem from a Dirichlet series identity; for $\Re s > 1$, one has

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = \frac{d}{ds}\log\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)$$
$$= \sum_{p\in\mathbb{P}}\frac{d}{ds}\log\left(1-\frac{1}{p^s}\right) = \sum_{p\in\mathbb{P}}\frac{p^{-s}\log p}{1-\frac{1}{p^s}} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}. \tag{2.6}$$

This identity is one of the reasons why it is more convenient to study $\Lambda(n)$ than the characteristic function of the primes.

To utilize (2.6) to study primes in $[1, x]$, one uses the contour integration formula

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s}\,ds = \begin{cases} 0 & \text{if } y < 1; \\ 1 & \text{if } y > 1. \end{cases} \tag{2.7}$$

Combining these two observations we obtain (when $x \notin \mathbb{N}$)

$$\sum_{n\le x}\Lambda(n) = \sum_n \Lambda(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{(x/n)^s}{s}\,ds = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds.$$

Moving the integration to the left side of the line $\Re s = 1$, one picks up a pole at $s = 1$ with residue $-x$, so this gives the main term in the PNT. The zeros of the zeta function are also poles of the integrand and one can derive, for any $x \ge T \ge 2$, the explicit formula

$$\sum_{n\le x}\Lambda(n) = x - \sum_{\substack{\rho \\ \zeta(\rho)=0 \\ |\Im(\rho)|\le T}}\frac{x^\rho - 1}{\rho} + O\left(\frac{x}{T}\log^2 x\right).$$

One can now use this and (2.4) to relate the error term in the PNT to the zero-free region for the zeta function. In particular, one can show that

$$\text{PNT} \Leftrightarrow \sum_{n\le x}\Lambda(n) = \big(1+o(1)\big)x \Leftrightarrow \zeta(s)\neq 0 \quad \text{when } \Re s = 1.$$

Furthermore, one obtains (2.1) using the zero-free region (2.5). Finally, it is possible to show this way that

$$\text{RH} \Leftrightarrow \sum_{n\le x}\Lambda(n) = x + O(x^{1/2+\varepsilon}) \quad \text{for all } \varepsilon > 0, \tag{2.8}$$

where the implied constant is allowed to depend on ε.

2.4. Dirichlet L-functions

The Riemann zeta function is the simplest member of a large family of L-functions (for a lot of information about general L-functions, see [14, Chapter 5]). Let us introduce here also Dirichlet L-functions, which are L-functions of degree 1 like $\zeta(s)$.

Let $\chi\colon \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character of modulus q; i.e., a function that

(i) is periodic with period q (i.e., $\chi(a+q) = \chi(a)$ for all $a \in \mathbb{Z}$);

(ii) is completely multiplicative (i.e., $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$);

(iii) is such that $\chi(r) = 0$ whenever $(r, q) \neq 1$.

For every $q \in \mathbb{N}$, a trivial instance of Dirichlet character is the principal character $\chi_0(n) = 1_{(n,q)=1}$. For modulus 4 the only non-principal character is χ_4 defined at primes by

$$\chi_4(p) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4; \\ -1 & \text{if } p \equiv 3 \pmod 4; \\ 0 & \text{if } p = 2. \end{cases}$$

For a Dirichlet character χ, the corresponding Dirichlet L-function $L(s, \chi)$ is defined for $\Re s > 1$ by

$$L(s,\chi) = \sum_{n \in \mathbb{N}} \frac{\chi(n)}{n^s} = \prod_{p \in \mathbb{P}} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The Dirichlet L-functions play an important role when studying prime numbers in arithmetic progressions thanks to the orthogonality relation

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod q; \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet L-functions have a very similar theory as the Riemann zeta function, with a functional equation, the generalized RH, etc. The zero-free region for $L(s, \chi)$ is not as good as for $\zeta(s)$. In particular, one has not been able to rule out the possibility of a real exceptional character which has a real zero very close to $s = 1$.

3. Primes and almost primes in short intervals

3.1. Primes in short intervals

The PNT tells us about the behavior of primes in $[1, x]$ but even the best known quantitative result (2.1) is so weak that it does not imply that there exists $\varepsilon > 0$ such that, for sufficiently large x, the interval $(x, x + x^{1-\varepsilon}]$ contains primes. However, Hoheisel [10] showed such a statement already in 1930 and Huxley's [11] PNT from 1972 gives, for any $\varepsilon > 0$,

$$\sum_{x<n\leq x+H} \Lambda(n) = \big(1 + o(1)\big)H \quad \text{for } H \geq x^{7/12+\varepsilon}. \tag{3.1}$$

This is based on Huxley's [11] zero-density estimate

$$N(\sigma, T) = O\big(T^{(\frac{12}{5}+\varepsilon)(1-\sigma)}(\log T)^{O(1)}\big) \quad \text{for all } T \geq 2 \text{ and } \sigma \in [1/2, 1], \tag{3.2}$$

where $N(\sigma, T)$ is the number of zeros of the Riemann zeta function in the rectangle $\Re(s) \geq \sigma$, $|\Im(s)| \leq T$. Huxley's result has resisted improvements, except that Heath-Brown [9] has shown (3.2) for $H \geq x^{7/12-o(1)}$.

The so-called density hypothesis asserts that

$$N(\sigma, T) = O(T^{2-2\sigma+\varepsilon}) \quad \text{for all } T \geq 2 \text{ and } \sigma \in [1/2, 1], \tag{3.3}$$

and this would imply that (3.2) holds for $H \geq x^{1/2+\varepsilon}$ for any $\varepsilon > 0$ (see e.g. [12, Theorem 12.8]). Note that the density hypothesis is a consequence of the Lindelöf hypothesis (see e.g. [12, Section 1.9]) asserting that $|\zeta(1/2 + it)| \ll |t|^{\varepsilon}$ for every $\varepsilon > 0$.

If one does not require an asymptotic formula for the number of primes in a short interval but contends with a lower bound of correct order of magnitude, then shorter intervals can be reached. In particular, following the initial breakthrough of Iwaniec and Jutila [13] and a succession of further improvements, Baker–Harman–Pintz [1] showed that, for large enough x and some $\varepsilon > 0$,

$$\sum_{x<n\leq x+H} \Lambda(n) \geq \varepsilon H \quad \text{for } H \geq x^{0.525}. \tag{3.4}$$

For shorter intervals one does not even know existence of primes. However, assuming the RH one knows that, for large enough x, the interval $(x, x + x^{1/2}\log x]$ always contains primes (see e.g. [12, Theorem 12.10]). This barely falls short of one of the four famous problems of Landau, asserting that there is always a prime between two consecutive squares, which would follow if one could show that $(x, x + x^{1/2}]$ always contains primes.

Cramer made a probabilistic model based on "probability of n being prime is $1/\log n$". Based on this, one expects that, for a large enough C, the interval $(x, x + C\log^2 x]$ contains primes for all large x; for more precise conjectures, see [5,6]. Here we see a large gap between what is known and what is expected.

3.2. Primes in almost all short intervals

As even under the RH it is not known that $(x, x + x^{1/2}]$ always contains primes, it is natural to ask what if one only requires that almost all intervals contain primes.

A variant of Huxley's PNT says that, for almost all $x \in (X, 2X]$,

$$\sum_{x<n\leq x+H} \Lambda(n) = \big(1 + o(1)\big)H \quad \text{for } H \geq x^{1/6+\varepsilon},$$

see e.g. [7, Theorem 9.1]. This can be proved using a technique due to Selberg [23] and Huxley's zero-density estimate (3.2). Furthermore also this result has resisted improvements.

Again if one only wants a lower bound for the number of primes, one can do better. By a sieve method Jia [15] has shown that, for some $\varepsilon > 0$,

$$\sum_{x<n\leq x+H} \Lambda(n) \geq \varepsilon H \quad \text{for } H \geq x^{1/20}. \tag{3.5}$$

Assuming the density hypothesis (3.3) (or the Lindelöf hypothesis) one can show that, for every $\varepsilon > 0$, almost all intervals $(x, x + x^{\varepsilon}]$ contain asymptotically the expected number of primes (see [12, Theorem 12.9]).

Based on probabilistic models, one expects that, for any $h \to \infty$ with $X \to \infty$, the interval $(x, x + h \log x]$ contains primes for almost all $x \in (X, 2X]$, so again we are far from the expected truth. Heath-Brown [8] has established this conjecture assuming both the RH and the pair correlation conjecture for zeros of $\zeta(s)$ which concerns the distribution of the gaps between the imaginary parts of the zeros.

The author and Jori Merikoski [17] have worked on studying the distribution of primes under the very unlikely assumption that there exist so-called exceptional characters for which the corresponding L-function has a zero extremely close to $s = 1$. If such an exceptional character existed, it would have some very interesting consequences. Concerning primes in short intervals, as a corollary in our work we obtain the following theorem.

Theorem 3.1 (Matomäki–Merikoski [17]). *Let $C \geq 2$. Let χ be a primitive quadratic character modulo $q \geq 2$ and assume that $L(s, \chi)$ has a real zero β_0 such that*

$$\beta_0 = 1 - \frac{1}{\eta \log q}.$$

for some $\eta \geq 10$.

Let $X \in [q^{10}, q^{\eta^{99/100}}]$ and let $2 \leq H \leq X^{1/3}$. Then

$$\int_X^{2X} \Bigg(\sum_{y<n\leq y+H} \Lambda(n) - H \Bigg)^2 dy = O_C\Bigg(H^2 X \bigg(\frac{\log X}{H} + \exp(-C\sqrt{\log \eta})\bigg)\Bigg).$$

This implies that as soon as

$$\eta \to \infty, \quad \frac{H}{\log X} \to \infty, \quad \text{and} \quad q^{10} \le X \le q^{\eta^{99/100}},$$

we get the asymptotic formula

$$\sum_{y<p\le y+H} 1 = \big(1+o(1)\big)\frac{H}{\log y}$$

for almost all $y \in [X, 2X]$.

Note that it is widely believed that such exceptional characters do not exist. But at least our result allows one to assume they do not exist when attacking primes in almost all short intervals.

3.3. Almost primes in short intervals

As discussed above, one expects that, for any $h \to \infty$ with $X \to \infty$, the interval $(x, x + h \log x]$ contains primes for almost all $x \in (X, 2X]$. This being far out of reach, one can ask similar questions about almost-primes, i.e., P_k numbers that have at most k prime factors or E_k numbers that have exactly k prime factors.

Here P_k numbers are easier to deal with since classical sieve methods can be applied. For instance Wu [26] has shown that, for all sufficiently large x, the interval $(x - x^{101/232}, x]$ contains P_2 numbers. This is significantly better than the corresponding result for the primes, where one could not cross the $1/2$ barrier even assuming the RH.

Due to the so-called parity barrier (see e.g. [2, Section 16.4]), classical sieves are unable to distinguish between numbers having an even and an odd number of prime factors. In particular, a sieve can be used to find P_2 numbers but, without additional input, it is impossible to tell whether it found primes or E_2-numbers.

However, E_k numbers are still easier to deal with than the primes, thanks to sums over them having a multilinear structure. Teräväinen has shown that for $k \ge 2$, there exists a constant C_k such that, for almost all $x \in (X, 2X]$, the interval $(x, x + (\log_{k-1} X)^{C_k} \log X]$ contains an E_k-number, where $\log_m X$ is m times iterated logarithm. Furthermore, in Teräväinen's result one can take $C_2 = 2.51$ and $C_3 = 6 + \varepsilon$.

Let us turn into discussing P_k numbers in almost all intervals. Following Friedlander [3,4], Friedlander and Iwaniec [2, Section 6.10] showed that as soon as $h \to \infty$ with $X \to \infty$, the interval $(x - h \log X, x]$ contains P_{19}-numbers for almost all $x \in (X/2, X]$. Furthermore, they say that, using more advanced techniques, one could obtain P_3 numbers. The author improved this in a recent preprint [16].

Theorem 3.2 (Matomäki [16]). *Let $h \to \infty$ with $X \to \infty$. Then the interval $(x - h \log X, x]$ contains P_2 numbers for almost all $x \le X$.*

4. The Möbius function

4.1. Introducing the Möbius function

Let $\mu(n)$ denote the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_i \text{ distinct;} \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $\Re s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)},$$

so $\mu(n)$ is closely related to $\Lambda(n)$ whose generating Dirichlet series was $-\zeta'/\zeta(s)$.

In particular, using similar contour integration arguments as in Section 2.3 one can show that

$$\text{PNT} \Leftrightarrow \zeta(s) \text{ has no zeros with } \Re s = 1 \Leftrightarrow \sum_{n \le x} \mu(n) = o(x)$$

$$\text{RH} \Leftrightarrow \sum_{n \le x} \mu(n) = O(x^{1/2+\varepsilon}) \quad \text{for all } \varepsilon > 0,$$

where the implied constant may depend on ε.

4.2. Möbius in short intervals

Until 2014 the story about the Möbius function in short intervals was exactly the same as for $\Lambda(n)$. In particular, Motohashi [20] and Ramachandra [22] independently adapted Huxley's proof of [11] to show that

$$\sum_{x < n \le x+H} \mu(n) = o(H) \quad \text{for } H \ge x^{7/12+\varepsilon}. \tag{4.1}$$

Analogously it was known by [22] that, for almost all $x \in (X, 2X]$,

$$\sum_{x < n \le x+H} \mu(n) = o(H) \quad \text{for } H \ge x^{1/6+\varepsilon}.$$

This almost-all interval result was significantly improved in the author's work with Radziwiłł [18] showing the following theorem.

Theorem 4.1 (Matomäki–Radziwiłł [18]). *Let $H \to \infty$ with $x \to \infty$. Then, for almost all $x \in (X, 2X]$, one has*

$$\sum_{x < n \le x+H} \mu(n) = o(H).$$

Our result is more general and has led to numerous advancements, e.g., concerning Chowla's conjecture (see e.g. [24]). In the proof we crucially used the fact that a typical n has prime factors from certain convenient intervals—something that is certainly not true for $n \in \mathbb{P}$.

A natural question is whether one can improve also on (4.1) along similar lines. Recently, the author and J. Teräväinen [19] obtained such a result.

Theorem 4.2 (Matomäki–Teräväinen [19]). *One has*

$$\sum_{x<n\le x+H} \mu(n) = o(H) \quad \text{for } H \ge x^{0.55+\varepsilon}.$$

Note that $7/12 = 0.5833\ldots$, and that even under RH one cannot get beyond $1/2$, so we get significantly closer to this natural barrier.

5. Proof strategy

5.1. The general strategy

We have already discussed how contour integration can be used to relate questions about primes and the Möbius function to questions about the Riemann zeta function. However, there is another more flexible way to go which we will describe in this section.

In this strategy for proving results on primes or the Möbius function there are two steps: a combinatorial step and an analytic step. In the combinatorial step a combinatorial identity or a sieve method is used to reduce the problem to that of estimating so-called type I and type II sums. In the analytic step these type I and type II sums are estimated.

This overall strategy works for various problems concerning primes, including problems for which no other strategy is known. On the other hand, it can also be used e.g. to reprove Huxley's PNT (3.1) without appealing to zero density results; see e.g. [7, Section 7.3].

5.2. The combinatorial step

When one is looking for an asymptotic formula for the number of primes in some interesting set, the combinatorial step is often done using Vaughan's identity or Heath-Brown's identity (see [14, Sections 13.3–13.4]). A special case of Vaughan's identity (see e.g. [14, Proposition 13.4]) implies that, for any (α_n),

$$\sum_{X<n\le 2X} \alpha_n \Lambda(n) = \sum_{\substack{X<bc\le 2X \\ b\le X^{1/3}}} \alpha_{bc}\mu(b)\log c$$

$$- \sum_{\substack{X<abc\leq 2X \\ b,c\leq X^{1/3}}} \alpha_{abc}\mu(b)\Lambda(c) + \sum_{\substack{X<abc\leq 2X \\ b,c>X^{1/3}}} \alpha_{abc}\mu(b)\Lambda(c).$$

From this one can see that instead of $\sum_{X<n\leq 2X} \alpha_n \Lambda(n)$ it suffices to study type I sums

$$\sum_{\substack{X<mn\leq 2X \\ m\leq X^{1/3}}} \alpha_{mn} a_m \quad \text{and} \quad \sum_{\substack{X<mn\leq 2X \\ m\leq X^{1/3}}} \alpha_{mn} a_m \log n \tag{5.1}$$

with arbitrary bounded coefficients a_m and type II sums

$$\sum_{\substack{X<mn\leq 2X \\ X^{1/3}\leq m\leq X^{2/3}}} \alpha_{mn} a_m b_n$$

with arbitrary bounded coefficients a_m and b_n.

Heath-Brown's identity is a more flexible variant of Vaughan's identity in terms of the different sums it produces, and it is of benefit to be able to deal e.g. with type I_2 sums

$$\sum_{\substack{X<\ell mn\leq 2X \\ m\sim M \\ n\sim N}} \alpha_{\ell mn} a_\ell.$$

5.3. The analytic step

In the analytic step one estimates the resulting type I and type II sums. In type I sums (5.1) there is a smooth variable n and one often wants to bring the sum over n inside. For instance if $\alpha_n = 1_{n\in(X,X+X^{3/4}]}$, then

$$\sum_{\substack{X<mn\leq 2X \\ m\leq X^{1/3}}} \alpha_{mn} a_m = \sum_{m\leq X^{1/3}} a_m \sum_{X/m<n\leq (X+X^{3/4})/m} 1 = X^{3/4} \sum_{m\leq X^{1/3}} \frac{a_m}{m} + O(X^{1/3}),$$

so that we get an asymptotic formula for such a type I sums.

In type II sums we have genuine bilinear structure and quite often one applies Cauchy–Schwarz at some point, either to separate the variables or to dispose of some of the coefficients.

For instance when working on problems concerning short intervals, one can use Dirichlet polynomials through contour integration (2.7). One gets that

$$\frac{1}{H} \sum_{x<mn\leq x+H} a_m b_n \approx \frac{1}{X} \sum_{X<mn\leq 2X} a_m b_n$$

essentially if

$$\int_{(\log X)^{100}}^{X/H} \left| \sum_{mn\sim X} \frac{a_m b_n}{(mn)^{1/2+it}} \right| dt = O\left(\frac{X^{1/2}}{(\log X)^{100}} \right).$$

Such mean values can be estimated through mean and large value results for Dirichlet polynomials; see e.g. [7, Chapter 7].

5.4. Sieve methods

If one does not require an asymptotic formula, one can use a sieve method. The most popular prime-detecting sieve is Harman's sieve (for a comprehensive account, see [7]) that has been used e.g. in proofs of (3.4) and (3.5).

For $\mathcal{A} \subset \mathbb{N}$ and $z \geq 2$, write $P(z) = \prod_{p<z} p$ and

$$S(\mathcal{A}, z) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} 1.$$

If now $\mathcal{A} \subseteq (X, 2X] \cap \mathbb{N}$, then

$$\mathcal{A} \cap \mathbb{P} = S(\mathcal{A}, 2X^{1/2}).$$

Writing also $\mathcal{A}_d = \{n \in \mathcal{A} : d \mid n\}$, one has the Buchstab identity

$$S(\mathcal{A}, z) = S(\mathcal{A}, w) - \sum_{w \leq p < z} S(\mathcal{A}_p, p).$$

Harman's sieve method consists of consecutive applications of Buchstab's identity to reach type I and type II sums. Some sums with a positive sign can be dropped if one looks for a lower bound.

When one is looking for P_k numbers, one can use more classical sieve methods that require only type I information. For instance in a lower bound sieve one replaces the identity

$$S(\mathcal{A}, z) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} 1 = \sum_{n \in \mathcal{A}} \sum_{d \mid (n, P(z))} \mu(d)$$

by an inequality

$$S(\mathcal{A}, z) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} 1 \geq \sum_{n \in \mathcal{A}} \sum_{d \mid (n, P(z))} \mu^-(d)$$

for an appropriate chosen sequence $\mu^-(d)$ which is supported only on $d \leq D$. Now one encounters a type I sum

$$\sum_{\substack{dn \in \mathcal{A} \\ d \mid P(z) \\ d \leq D}} \mu^-(d).$$

Unfortunately, such a sieve can produce a non-trivial lower bound only when $D > z^2$.

5.5. Implementation of the strategy

In this subsection we briefly discuss the combinatorial and analytic steps in the proofs of Theorems 3.1, 3.2, 4.1, and 4.2.

In the proof of Theorem 3.2 on P_2 numbers in almost all short intervals, the combinatorial tool used is Richert's weighted sieve with β-sieve (for a comprehensive account of these sieves, see [2, Chapters 25 and 11]). These sieves reduce the problem to understanding type I sums. As mentioned in Section 5.4, sieves using only type I information as input are incapable of catching primes, but here our goal is P_2 numbers. Then in the analytic step we reduce estimating the type I sums in almost all intervals into estimating averages of Kloosterman sums which can be done by known results.

Let us now turn to the proof of Theorem 4.1 on the Möbius function for almost all intervals. The combinatorial step uses Ramaré's identity in the form saying that, for $(P, Q] \subseteq (1, H]$, one has

$$\sum_{x<n\leq x+H} \mu(n) = -\sum_{\substack{x<pm\leq x+H\\ P<p\leq Q}} \frac{\mu(pm)}{\#\{P<q\leq Q : q \mid m\} + 1_{p\nmid m}} + O\left(H\frac{\log P}{\log Q}\right),$$

where the error term comes from those n that do not have a prime factor in the interval $(P, Q]$. This combinatorial step leads to type II sums with one of the variables (i.e., p) being very small. In the analytic step we reduce estimating such sums to mean square estimates for Dirichlet polynomials. In order to reach very short intervals, we need to use an iterative argument, with several applications of Ramaré's identity.

In the proof of Theorem 4.2 on the Möbius function in all short intervals, in the combinatorial step we use both Ramaré's identity and Heath-Brown's identity. Ramaré's identity allows us to extract a very small prime factor from the sum over $\mu(n)$ before using the Heath-Brown identity to split into type I, type II, and type I/II sums. In the analytic step we again use estimates on Dirichlet polynomials. This method actually works in greater generality. For instance we obtain also the following theorem.

Theorem 5.1 (Matomäki–Teräväinen [19]).

$$\sum_{\substack{x<p_1p_2\leq x+H\\ p_j\in\mathbb{P}}} 1 = H\frac{\log\log x}{\log x} + O\left(H\frac{\log\log\log x}{\log x}\right), \qquad H \geq x^{0.55+\varepsilon}.$$

The proof of Theorem 3.1 works somewhat differently though there are similar steps. Thanks to the assumption on the existence of exceptional characters, the relevant type II sums become quite easy to bound and then one just needs to obtain enough type I information to find primes. In the analytic step for the type I sums one again reduces the problem to that of bounding Kloosterman sums.

Acknowledgments. The author would like to thank Juho Salmensuu and the referee for helpful comments on the manuscript.

Funding. Research supported by the Academy of Finland grant no. 285894.

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