DOI 10.4171/8ECM/18

Closed G_2-structures on compact quotients of Lie groups

Anna Fino and Alberto Raffero

Abstract. G_2-structures defined by a closed non-degenerate 3-form constitute the starting point in various known and potentially effective methods to obtain holonomy G_2-metrics on compact 7-manifolds. Albeit linear, the closed condition is quite restrictive, and no general results on the existence of closed G_2-structures on compact 7-manifolds are currently known. In this paper, we review some results regarding compact locally homogeneous spaces admitting invariant closed G_2-structures. In particular, we consider the case of compact quotients of simply connected Lie groups by discrete subgroups.

1. Introduction

A G_2-structure is a special type of G-structure that occurs on certain 7-dimensional smooth manifolds. More precisely, it is a reduction of the structure group of the frame bundle of a 7-manifold M from the general linear group $GL(7, \mathbb{R})$ to the compact exceptional Lie group G_2. The existence of a G_2-structure on M is equivalent to the orientability of M and the existence of a spin structure on it, namely to the vanishing of the the first and second Stiefel–Whitney classes of M.

Since every 7-manifold admitting G_2-structures is spin, it also admits almost contact structures. The interplay between the existence of special types of G_2-structures and of contact structures has been recently investigated in [2, 13, 26].

The existence of a G_2-structure on M can also be described in terms of differential forms. Indeed, it is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ with pointwise stabilizer isomorphic to G_2. This is also equivalent to requiring that φ is *non-degenerate*; namely that at each point p of M one has that

$$\iota_X\varphi \wedge \iota_X\varphi \wedge \varphi \neq 0,$$

for every non-zero tangent vector $X \in T_pM$, where ι_X denotes the contraction by X. Every such 3-form φ gives rise to a Riemannian metric g_φ and to an orientation on M. More precisely, g_φ and the corresponding Riemannian volume form dV_φ are

2020 Mathematics Subject Classification. Primary 53C10; Secondary 53C30.
Keywords. Closed G_2-structure, locally homogeneous space, Lie algebra, lattice, Laplacian soliton, exact G_2-structure.

related to φ as follows:

$$g_{\varphi}(X,Y)dV_{\varphi} = \frac{1}{6}\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi.$$

Moreover, at each point p of M, the 3-form φ can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

where $(e^1,\dots,e^7)$ is a g_{φ}-orthonormal basis of the cotangent space T_p^*M, and e^{ijk} denotes the wedge product $e^i \wedge e^j \wedge e^k$.

Let $*$ be the Hodge star operator determined by g_{φ} and the orientation, and let ∇ be the Levi-Civita connection of g_{φ}. By [19], the 3-form φ is parallel with respect to ∇ if and only if it is closed and co-closed; namely $d\varphi = 0$ and $d*\varphi = 0$. In this case, the G_2-structure is said to be *parallel* or *torsion-free*, its intrinsic torsion vanishes identically, the Riemannian metric g_{φ} is Ricci-flat (see also [4]), and $\mathrm{Hol}(g_{\varphi})$ is isomorphic to a subgroup of G_2. Notice that the conditions $\nabla\varphi = 0$ and $d*\varphi = 0$ are both non-linear in φ, as both ∇ and $*$ depend on g_{φ}, which is determined by φ.

The existence of Riemannian metrics with holonomy equal to G_2 was first proved by Bryant in [7], where some non-compact examples of Riemannian 7-manifolds with holonomy G_2 were given. The first complete (but still non-compact) examples were obtained by Bryant and Salamon in 1989 [9], and the first compact examples were constructed by Joyce in 1994 [35, 36]. Further compact examples admitting holonomy G_2 metrics were obtained in [12, 34, 38, 39].

A G_2-structure defined by a non-degenerate 3-form φ satisfying the linear condition $d\varphi = 0$ is said to be *closed* or *calibrated*, since φ defines a calibration on M, namely $\varphi|_{\xi} \leq \mathrm{vol}_{\xi}$, for every oriented tangent 3-plane ξ (cf. [30]). The codifferential of a closed G_2-structure φ is given by

$$d*\varphi = \tau \wedge \varphi,$$

for a unique 2-form τ belonging to the irreducible 14-dimensional space $\Lambda^2_{14} \cong \mathfrak{g}_2$. This 2-form is usually called the *torsion form* of the closed G_2-structure φ, and it satisfies the identities $\tau \wedge \varphi = -*\tau$ and $\tau \wedge *\varphi = 0$. Note that $\tau = d^*\varphi$, and therefore $d^*\tau = 0$. As a consequence, $d\tau = \Delta_{\varphi}\varphi$, where $\Delta_{\varphi} = dd^* + d^*d$ denotes the Hodge Laplacian of g_{φ}.

By [8], the scalar curvature of the metric g_{φ} induced by a closed G_2-structure is given by

$$\mathrm{Scal}(g_{\varphi}) = -\frac{1}{2}|\tau|^2,$$

and so it is non-positive. Notice that this is not a restrictive condition on compact manifolds.

By [56], a compact homogeneous 7-manifold cannot admit any invariant closed non-parallel G_2-structure. On the other hand, there exist many examples of compact locally homogeneous 7-manifolds admitting *invariant* G_2-structures of this type; see for instance [3, 8, 11, 14, 15, 23, 37, 50]. All these examples are compact quotients of simply connected Lie groups by co-compact discrete subgroups (lattices). Further examples of compact manifolds admitting closed non-parallel G_2-structures are given in [16, 51] and they are obtained resolving the singularities of 7-orbifolds.

In Section 2, we review known examples of compact locally homogeneous spaces admitting invariant closed G_2-structures and known classification results for Lie algebras admitting closed G_2-structures. A classification is currently available for 7-dimensional Lie algebras that are non-solvable [23] and for those having a non-trivial center [11, 26]. The classification of solvable Lie algebras with a trivial center admitting closed G_2-structures is still missing.

A geometric flow evolving closed G_2-structures was introduced by Bryant in [8]. Self-similar solutions to this flow correspond to the so-called *Laplacian solitons*, namely to closed G_2-structures φ satisfying the condition $\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X\varphi$, for some real constant λ and some vector field X on M, where $\mathcal{L}_X\varphi$ denotes the Lie derivative of φ with respect to X. In Section 3, after reviewing general properties of the Laplacian flow and of Laplacian solitons, we present some recent results obtained in [26], where left-invariant Laplacian solitons on Lie groups with a non-trivial center were considered.

A Laplacian soliton φ is called *expanding* if $\lambda > 0$. In this case, the G_2-form φ has to be *exact*, i.e., $\varphi = d\alpha$, for some 2-form α on M. By [42, 44], a non-parallel Laplacian soliton on a compact 7-manifold must be expanding with $\mathcal{L}_X\varphi \neq 0$.

Currently, it is still not known whether exact G_2-structures may occur on compact 7-manifolds. In Section 4, we review the results of [18, 22, 28], where this problem was considered in the case when the compact 7-manifold M is the quotient of a 7-dimensional simply connected Lie group G by a co-compact discrete subgroup $\Gamma \subset$ G, and the exact G_2-structure on M is induced by a left-invariant one on G. In [18,28], it was shown that there are no examples of this type whenever the group G satisfies suitable extra assumptions. In the recent joint work with L. Martín Merchán [22], we extended the previous results, showing that every compact manifold $M = \Gamma\backslash G$ as above does not admit any exact G_2-structure which is induced by a left-invariant one on G.

2. Compact locally homogeneous examples and classification results for Lie algebras

Let M be a 7-manifold endowed with a G_2-structure φ and consider its automorphism group

$$\mathrm{Aut}(M,\varphi) := \left\{ f \in \mathrm{Diff}(M) \mid f^*\varphi = \varphi \right\}.$$

Note that $\mathrm{Aut}(M,\varphi)$ is a closed Lie subgroup of the full isometry group $\mathrm{Isom}(M, g_\varphi)$ of the Riemannian manifold (M, g_φ).

When M is compact, $\mathrm{Aut}(M,\varphi)$ is compact, too, and its Lie algebra is given by

$$\mathfrak{aut}(M,\varphi) = \left\{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\varphi = 0\right\}.$$

In particular, every $X \in \mathfrak{aut}(M,\varphi)$ is a Killing vector field for the metric g_φ; namely $\mathcal{L}_X g_\varphi = 0$.

When φ is parallel, g_φ is Ricci-flat, and it follows from the Bochner–Weitzenböck technique that every Killing vector field must be parallel with respect to the Levi-Civita connection of g_φ. Consequently, the Lie algebra $\mathfrak{aut}(M,\varphi)$ is abelian. Moreover, its possible dimensions are $0, 1, 3$ or 7, depending on $\mathrm{Hol}^0(g_\varphi)$ being equal to G_2, $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ or $\{1\}$, respectively.

If the G_2-structure φ is closed and non-parallel, namely $\tau = d^*\varphi \neq 0$, then for every $X \in \mathfrak{aut}(M,\varphi)$ the closed 2-form $\iota_X\varphi$ is Δ_φ-harmonic, since $*(\iota_X\varphi) = \frac{1}{2}\iota_X\varphi \wedge \varphi$ is also closed. There is then an injective map

$$X \in \mathfrak{aut}(M,\varphi) \mapsto \iota_X\varphi \in \mathcal{H}^2(M),$$

and thus $\dim \mathfrak{aut}(M,\varphi) \leq b_2(M)$, where $b_2(M) = \dim \mathcal{H}^2(M) = \dim H^2_{\mathrm{dR}}(M)$ is the second Betti number of M. Moreover, it is possible to prove the following.

Theorem 2.1 ([56]). *Let M be a compact 7-manifold with a closed non-parallel* G_2-*structure φ. Then, $\mathfrak{aut}(M,\varphi)$ is abelian and its dimension is at most* 6.

Therefore, the identity component of $\mathrm{Aut}(M,\varphi)$ is a compact abelian Lie group whose dimension is bounded above by $\min\{6, b_2(M)\}$. As a consequence, a compact 7-manifold M with a closed non-parallel G_2-structure φ cannot be homogeneous; namely neither $\mathrm{Aut}(M,\varphi)$ nor a subgroup thereof can act transitively on M. In contrast to this last result, it is possible to construct non-compact homogeneous examples; see for instance [55].

The first example of compact 7-manifold M admitting closed G_2-structures but not admitting any parallel G_2-structure was constructed by Fernández in [14]. In this example, $M = \Gamma\backslash N$ is a compact nilmanifold; i.e., the compact quotient of a 7-dimensional simply connected nilpotent Lie group N by a co-compact discrete subgroup (lattice) Γ. Moreover, the closed G_2-structure φ on $\Gamma\backslash N$ considered in [14] is induced by a left-invariant one on the Lie group N. In particular, the pair $(\Gamma\backslash N, \varphi)$ is a locally homogeneous space that is not globally homogeneous, as the transitive action of N on $\Gamma\backslash N$ does not preserve the 3-form φ. In other words, N is not a subgroup of $\mathrm{Aut}(\Gamma\backslash N, \varphi)$.

Remark. By Malcev's criterion [49], a nilpotent Lie group admits lattices if and only if its Lie algebra admits a basis with rational structure constants.

We now consider the following problem.

Problem 2.2. Study the existence of invariant closed G_2-structures on compact 7-manifolds of the form $\Gamma\backslash G$, where G is a 7-dimensional simply connected Lie group and $\Gamma \subset G$ is a co-compact discrete subgroup.

We recall that a G_2-structure on $\Gamma\backslash G$ is said to be *invariant* if it is induced by a left-invariant one on the Lie group G. Therefore, an invariant closed G_2-structure on $\Gamma\backslash G$ is completely determined by a G_2-structure φ on the Lie algebra $\mathfrak{g}$ of G which is closed with respect to the Chevalley–Eilenberg differential d of $\mathfrak{g}$.

A 3-form φ on a 7-dimensional Lie algebra $\mathfrak{g}$ defines a G_2-structure if and only if the symmetric bilinear map

$$b_\varphi : \mathfrak{g} \times \mathfrak{g} \to \Lambda^7\mathfrak{g}^*, \quad b_\varphi(v, w) = \frac{1}{6}\iota_v\varphi \wedge \iota_w\varphi \wedge \varphi$$

satisfies the condition $\det(b_\varphi)^{1/9} \neq 0 \in \Lambda^7\mathfrak{g}^*$ and the symmetric bilinear form

$$g_\varphi := \det(b_\varphi)^{-1/9} b_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

is positive definite; see e.g. [32]. In particular, for any choice of orientation on $\mathfrak{g}$, the map

$$b_\varphi : \mathfrak{g} \times \mathfrak{g} \to \Lambda^7\mathfrak{g}^* \cong \mathbb{R}$$

has to be positive or negative definite.

By [52], a simply connected Lie group G admits lattices only if its Lie algebra $\mathfrak{g}$ is unimodular; i.e., $\mathrm{tr}(\mathrm{ad}_X) = 0$, for every $X \in \mathfrak{g}$.

In the sequel, the structure equations of an n-dimensional Lie algebra with respect to a basis of covectors $(e^1, \dots, e^n)$ of $\mathfrak{g}^*$ will be specified by the n-tuple $(de^1, \dots, de^n)$. Moreover, we will use the shortening $e^{ijk\cdots}$ to denote the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge \cdots$.

In [23], we classified all unimodular non-solvable Lie algebras admitting closed G_2-structures, up to isomorphism, obtaining the following result.

Theorem 2.3 ([23]). *A unimodular non-solvable Lie group* G *admits left-invariant closed* G_2*-structures if and only if its Lie algebra* $\mathfrak{g}$ *is isomorphic to one of the following:*

$$\mathfrak{q}_1 = \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \frac{1}{2}e^{46} - e^{47}, \frac{1}{2}e^{47}\right),$$

$$\mathfrak{q}_2 = \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, -\mu e^{46}, (1+\mu)e^{47}\right), \quad -1 < \mu \leq -\frac{1}{2},$$

$$\mathfrak{q}_3 = \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -\mu e^{45}, \frac{\mu}{2}e^{46} - e^{47}, e^{46} + \frac{\mu}{2}e^{47}\right), \quad \mu > 0,$$

$$\mathfrak{q}_4 = (-e^{23}, -2e^{12}, 2e^{13}, -e^{14} - e^{25} - e^{47}, e^{15} - e^{34} - e^{57}, 2e^{67}, 0).$$

The first three Lie algebras in the previous list decompose as a product of the form $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, where the radical $\mathfrak{r}$ is unimodular and centerless. The Lie algebra $\mathfrak{q}_4$ is indecomposable and its Levi decomposition is given by $\mathfrak{q}_4 \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{r}$, where $\mathfrak{r} \cong \mathbb{R} \ltimes \mathbb{R}^3$.

As a consequence of the previous result, a unimodular Lie algebra with a non-trivial center admitting closed G_2-structures must be solvable.

It is well known that every nilpotent Lie algebra is unimodular and has a non-trivial center. Nilpotent Lie algebras admitting closed G_2-structures were considered in [11], where the following classification result was obtained.

Theorem 2.4 ([11]). *A 7-dimensional nilpotent Lie algebra admits closed* G_2*-structures if and only if it is isomorphic to one of the following:*

$$\begin{aligned}
\mathfrak{n}_1 &= (0,0,0,0,0,0,0),\\
\mathfrak{n}_2 &= (0,0,0,0,e^{12},e^{13},0),\\
\mathfrak{n}_3 &= (0,0,0,e^{12},e^{13},e^{23},0),\\
\mathfrak{n}_4 &= (0,0,e^{12},0,0,e^{13}+e^{24},e^{15}),\\
\mathfrak{n}_5 &= (0,0,e^{12},0,0,e^{13},e^{14}+e^{25}),\\
\mathfrak{n}_6 &= (0,0,0,e^{12},e^{13},e^{14},e^{15}),\\
\mathfrak{n}_7 &= (0,0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}),\\
\mathfrak{n}_8 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}),\\
\mathfrak{n}_9 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25}),\\
\mathfrak{n}_{10} &= (0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}),\\
\mathfrak{n}_{11} &= (0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26}),\\
\mathfrak{n}_{12} &= (0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-2e^{16}+2e^{25}).
\end{aligned}$$

In [26], we dealt with the more general case of unimodular solvable non-nilpotent Lie algebras with a non-trivial center admitting closed G_2-structures. There, we obtained a characterization that is based on the following observation. Let W be a 7-dimensional vector space endowed with a G_2-structure φ. Choosing a non-zero vector $z \in W$ and a complementary vector subspace $V \subset W$ so that $W \cong V \oplus \mathbb{R}z$, one can write

$$\varphi = \widetilde{\omega} \wedge \theta + \rho,$$

where $\theta \in W^*$ is the dual of z, $\widetilde{\omega} \in \Lambda^2 V^*$, and $\rho \in \Lambda^3 V^*$. The 3-form φ defines a G_2-structure on W if and only if it is definite; namely for each non-zero vector $w \in W$ the contraction $\iota_w \varphi$ has rank six. Moreover, the 3-form φ on W is definite if and only if the 3-form ρ on V is definite; i.e., for each non-zero vector $v \in V$ the contraction

$\iota_v \rho$ has rank four, and $\widetilde{\omega}$ is a taming form for the complex structure J induced by ρ and one of the two orientations of V; namely $\widetilde{\omega}(v, Jv) > 0$ for every non-zero vector $v \in V$.

Using this property, in [26] we proved that a Lie algebra $\mathfrak{g}$ with a non-trivial center endowed with a closed G_2-structure φ must be the central extension of a 6-dimensional Lie algebra $\mathfrak{h}$ by means of a closed 2-form $\omega_0 \in \Lambda^2\mathfrak{h}^*$; namely $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ and its Lie bracket is given by

$$[z, \mathfrak{h}] = 0, \quad [x, y] = -\omega_0(x, y)z + [x, y]_{\mathfrak{h}}, \quad \forall x, y \in \mathfrak{h}.$$

Moreover, $\varphi = \widetilde{\omega} \wedge \theta + \rho$, where θ is a 1-form on $\mathfrak{g}$ satisfying the condition $d\theta = \omega_0$, ρ is a definite 3-form on $\mathfrak{h}$ such that $d\rho = -\omega_0 \wedge \widetilde{\omega}$, and $\widetilde{\omega}$ is a symplectic form on $\mathfrak{h}$ that tames the almost complex structure induced by ρ and a suitable orientation. If the 2-form $\widetilde{\omega}$ is symplectic, the 1-form θ is a contact form on $\mathfrak{g}$ and $(\mathfrak{g}, \theta)$ is the *contactization* of $(\mathfrak{h}, \widetilde{\omega})$; see [1]. In this last case, the Lie algebra $\mathfrak{g}$ admits both a closed G_2-structure and a contact structure. This is reminiscent of the Boothby–Wang construction in [5].

As a first consequence of this characterization, we determined all isomorphism classes of nilpotent Lie algebras admitting closed G_2-structures that arise as the contactization of a 6-dimensional symplectic nilpotent Lie algebra $(\mathfrak{h}, \omega_0)$, showing that any such Lie algebra must be isomorphic to one of the following Lie algebras: $\mathfrak{n}_9$, $\mathfrak{n}_{10}$, $\mathfrak{n}_{11}$, $\mathfrak{n}_{12}$.

Then, we proved that there exist eleven unimodular solvable non-nilpotent Lie algebras with a non-trivial center admitting closed G_2-structures, up to isomorphism, achieving in this way the classification of all isomorphism classes of unimodular Lie algebras with a non-trivial center admitting closed G_2-structures.

Theorem 2.5 ([26]). *Let $\mathfrak{g}$ be a 7-dimensional unimodular solvable non-nilpotent Lie algebra with a non-trivial center. Then, $\mathfrak{g}$ admits closed G_2-structures if and only if it is isomorphic to one of the following:*

$$\mathfrak{s}_1 = (e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0, 0),$$
$$\mathfrak{s}_2 = (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, 0),$$
$$\mathfrak{s}_3 = (-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0, 0),$$
$$\mathfrak{s}_4 = (0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}, 0),$$
$$\mathfrak{s}_5 = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0, 0),$$
$$\mathfrak{s}_6 = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0, 0), \quad \alpha > 0,$$
$$\mathfrak{s}_7 = (e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0, 0),$$
$$\mathfrak{s}_8 = (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, e^{34}),$$

$$\mathfrak{s}_9 = (-e^{26}+e^{35}, e^{16}+e^{45}, -e^{46}, e^{36}, 0, 0, e^{34}),$$
$$\mathfrak{s}_{10} = (e^{23}, -e^{36}, e^{26}, e^{26}-e^{56}, e^{36}+e^{46}, 0, 2e^{16}+e^{25}-e^{34}+\sqrt{3}e^{24}+\sqrt{3}e^{35}),$$
$$\mathfrak{s}_{11} = (e^{23}, -e^{36}, e^{26}, e^{26}-e^{56}, e^{36}+e^{46}, 0, 2e^{16}+e^{25}-e^{34}-\sqrt{3}e^{24}-\sqrt{3}e^{35}).$$

In particular, $\mathfrak{g}$ is the contactization of a symplectic Lie algebra if and only if it is isomorphic either to $\mathfrak{s}_{10}$ or to $\mathfrak{s}_{11}$.

By the characterization above, we know that $\mathfrak{g}$ has to be the central extension of a unimodular symplectic Lie algebra $\mathfrak{h}$ endowed with a closed (possibly non-degenerate) 2-form ω_0 and a suitable pair of forms $(\tilde{\omega}, \rho)$. Such an extension is determined by any representative in the cohomology class $[\omega_0] \in H^2(\mathfrak{h})$, and the proof of the theorem follows after an inspection of all 6-dimensional unimodular symplectic Lie algebras that exist up to isomorphism (cf. [20, 47]).

As far as we know, the following problem remains open.

Problem 2.6. Classify all 7-dimensional solvable Lie algebras with a trivial center admitting closed G_2-structures, up to isomorphism.

3. Laplacian solitons

A special class of closed G_2-structures that has attracted a lot of attention in recent years is given by *Laplacian solitons*. These G_2-structures are closely related to the self-similar solutions to the *Laplacian flow* for closed G_2-structures, a geometric flow that was introduced by Bryant in [8] as a tool to potentially deform a closed G_2-structure towards a parallel one.

Definition 3.1 ([8]). Let φ_0 be a closed G_2-structure on a 7-manifold M. The *Laplacian flow* starting at φ_0 is the initial value problem

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\Delta_{\varphi(t)}$ is the Hodge Laplacian of $g_{\varphi(t)}$.

The stationary points of the Laplacian flow are parallel G_2-structures, even on non-compact manifolds (see [43] for the explicit computation in the non-compact case). If $\varphi(t)$ is a family of closed G_2-structures solving the Laplacian flow, then $\varphi(t) \in [\varphi_0] \in H^3_{\mathrm{dR}}(M)$; namely the de Rham cohomology class $[\varphi(t)]$ is constant in t. Moreover, the evolution equation of the metric $g_{\varphi(t)}$ induced by $\varphi(t)$ coincides with the Ricci flow of $g_{\varphi(t)}$ up to lower order terms; namely

$$\partial_t g_{\varphi(t)} = -2\,\mathrm{Ric}(g_{\varphi(t)}) + \mathrm{l.o.t.}$$

Remark. On a compact manifold M, the Laplacian flow is the gradient flow of Hitchin's volume functional

$$\mathcal{V} : \varphi \in [\varphi_0] \mapsto \int_M \varphi \wedge *\varphi.$$

This functional is monotonically increasing along the flow, its critical points are parallel G_2-structures, and they are strict local maxima. See [6,43] and the arXiv version of [31] for more details.

The short-time existence and uniqueness of the solution to the Laplacian flow on a compact manifold were proved by Bryant and Xu in [6].

Theorem 3.2 ([6]). *Let M be a compact 7-manifold with a closed G_2-structure φ_0. Then, the Laplacian flow starting at φ_0 has a unique solution defined for short time $t \in [0, \varepsilon)$, with ε depending on φ_0.*

The geometric and analytic properties of the Laplacian flow have been deeply investigated by Lotay and Wei in [44–46], and further results are available in [10, 21, 57]. Moreover, various lower-dimensional reductions of the flow were studied in [21, 24, 27, 40]. Explicit examples of solutions to the flow are also known; see for instance [17, 24, 41] for examples on simply connected Lie groups with left-invariant closed G_2-structures, and [33] for a cohomogeneity one example on the 7-torus.

A closed G_2-structure φ on a 7-manifold M is said to be a *Laplacian soliton* if it satisfies the equation

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi,$$

for some real constant λ and some vector field X on M. These G_2-structures give rise to self-similar solutions to the Laplacian flow, namely to solutions of the form $\varphi(t) = \sigma(t) f_t^* \varphi$, where $\sigma(t)$ is a real-valued function of t, and $f_t \in \mathrm{Diff}(M)$. Laplacian solitons are expected to model finite time singularities of the Laplacian flow; see [43] for more details.

Depending on the sign of λ, one can introduce the following definitions.

Definition 3.3. A Laplacian soliton φ is called *shrinking* if $\lambda < 0$, *steady* if $\lambda = 0$ and *expanding* if $\lambda > 0$.

Some restrictions to the existence of a Laplacian soliton on a compact manifold are known.

Theorem 3.4 ([42, 44]). *On a compact 7-manifold, a non-parallel Laplacian soliton φ must satisfy the equation $\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi$, with $\lambda > 0$ and $\mathcal{L}_X \varphi \neq 0$. Moreover, the only steady Laplacian solitons are given by parallel G_2-structures.*

Thus, a non-parallel Laplacian soliton on a compact manifold must be expanding. The following problem is still open.

Problem 3.5. Do there exist expanding Laplacian solitons on compact manifolds?

The non-compact setting is less restrictive, and various homogeneous examples of steady, shrinking, and expanding solitons are known [3, 24, 25, 41, 53, 54]. More recently, complete inhomogeneous examples of steady and shrinking solitons were obtained in [3, 27]. These examples are of gradient type; i.e., X is a gradient vector field.

By [41], any left-invariant Laplacian soliton φ on a Lie group G is *semi-algebraic*; i.e., the vector field X is defined by a 1-parameter group of automorphisms induced by a derivation D of the Lie algebra $\mathfrak{g}$. Some results on semi-algebraic solitons on unimodular Lie algebras with a non-trivial center have been recently obtained in [26]. For instance, under a natural assumption on the derivation D, it is possible to relate the constant λ to a certain eigenvalue of D and to the norm of the torsion form τ of the semi-algebraic soliton φ. Moreover, the following result can be proved.

Theorem 3.6 ([26]). *Let $\mathfrak{g}$ be a unimodular Lie algebra with a non-trivial center $\mathfrak{z}(\mathfrak{g})$ admitting a semi-algebraic soliton φ. Then the following conditions hold:*

(1) *if $\mathfrak{g}$ is the contactization of a symplectic Lie algebra, then $\lambda = |\tau|^2$ and thus φ must be expanding;*

(2) *if $\dim \mathfrak{z}(\mathfrak{g}) = 2$, then $\mathfrak{g}$ has to be isomorphic to one of the following Lie algebras: $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$, $\mathfrak{n}_4$, $\mathfrak{n}_5$, $\mathfrak{n}_6$, $\mathfrak{n}_7$, $\mathfrak{s}_5$, $\mathfrak{s}_6$, $\mathfrak{s}_7$.*

If $\dim \mathfrak{z}(\mathfrak{g}) = 1$, some non-existence results for semi-algebraic solitons on certain Lie algebras are also known [26], but a general result is still missing.

Remark. All known examples of Lie algebras admitting shrinking or steady Laplacian solitons have a trivial center. It would be interesting to establish whether the existence of these types of solitons forces the Lie algebra to be centerless.

4. Exact G_2-structures

An expanding Laplacian soliton φ is an *exact* G_2-structure. Indeed, since φ is closed and $\Delta_\varphi \varphi = d\tau$, the condition $\Delta_\varphi \varphi = \lambda\varphi + \mathscr{L}_X \varphi$ can be rewritten as follows:

$$\varphi = d\left(\frac{1}{\lambda}(\tau - \iota_X \varphi)\right).$$

In the literature, all known examples of compact 7-manifolds M admitting closed G_2-structures, but not admitting parallel G_2-structures, have $b_1(M) > 0$ and $b_3(M) > 0$; see [11, 14–16, 50, 51]. A longstanding open question concerns the existence of closed G_2-structures on compact 7-manifolds with $b_3(M) = 0$, such as the 7-sphere. Notice that, in this case, any closed G_2-structure would be defined by an exact 3-form. A natural question is then the following.

Problem 4.1. Does there exist a compact 7-manifold admitting exact G_2-structures?

In this section, we consider this problem in the case when the manifold is the compact quotient of a simply connected unimodular Lie group G by a lattice.

The following example constructed in [18] shows that exact G_2-structures occur on unimodular Lie algebras.

Example 4.2. Let $\mathfrak{s}$ be the 7-dimensional unimodular solvable Lie algebra with structure equations

$$\begin{aligned} de^1 &= -2e^{17}, \quad de^2 = -4e^{27}, \quad de^3 = \frac{9}{2}e^{37}, \\ de^4 &= \frac{5}{2}e^{47} - e^{13}, \quad de^5 = \frac{1}{2}e^{57} - 6e^{37} - e^{14} - e^{23}, \\ de^6 &= -\frac{3}{2}e^{67} - 6e^{47} + 3e^{13} + e^{15} + e^{24}, \quad de^7 = 0. \end{aligned}$$

This Lie algebra is a semidirect product of the form $\mathfrak{s} = \mathbb{R} \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is a codimension one 4-step nilpotent ideal, and it satisfies the conditions $b_2(\mathfrak{s}) = 0 = b_3(\mathfrak{s})$. Moreover, $\mathfrak{s}$ admits the exact G_2-structure

$$\begin{aligned} \varphi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \\ &= d\left(\frac{1}{6}e^{12} + \frac{23}{7}e^{34} + 2e^{36} - 2e^{45} + e^{56}\right). \end{aligned}$$

Consequently, the simply connected solvable Lie group S with Lie algebra $\mathfrak{s}$ is endowed with a left-invariant exact G_2-structure obtained from φ via left multiplication.

As we already recalled, a Lie group G admitting lattices must be unimodular. In the case of solvable Lie groups, a stronger necessary condition for the existence of lattices is known; namely the group must be strongly unimodular (cf. [29, Prop. 3.3]). We recall the definition here.

Definition 4.3 ([29]). A solvable Lie group G with Lie algebra $\mathfrak{g}$ and nilradical $\mathfrak{n}$ is said to be *strongly unimodular* if $\mathrm{tr}(\mathrm{ad}_X)|_{\mathfrak{n}^i/\mathfrak{n}^{i+1}} = 0$, for every $X \in \mathfrak{g}$, where $\mathfrak{n}^0 = \mathfrak{n}$, and $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$, $i \geq 1$, is the ith term in the descending central series of $\mathfrak{n}$.

For instance, the simply connected solvable Lie group S in Example 4.2 is unimodular but not strongly unimodular, so it does not admit any compact quotient by a lattice.

In [18], we showed that a strongly unimodular (2, 3)-trivial Lie algebra $\mathfrak{g}$, namely with $b_2(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$, does not admit any exact G_2-structure. Therefore, there are no compact examples of the form $\Gamma\backslash G$ admitting invariant exact G_2-structures whenever the Lie algebra of G is (2, 3)-trivial. To prove this result, we used the property that a $(2,3)$-trivial Lie algebra $\mathfrak{g}$ is solvable and $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ a codimension one nilpotent ideal (see [48]), and we classified all 7-dimensional strongly unimodular $(2,3)$-trivial Lie algebras.

One can then investigate what happens if either $b_3(\mathfrak{g}) = 0$ and $b_2(\mathfrak{g}) \neq 0$ or if no conditions on the Betti numbers of $\mathfrak{g}$ are imposed. A first partial answer to this problem was given in [28].

Theorem 4.4 ([28]). *If the Lie algebra $\mathfrak{g}$ of G has a codimension one nilpotent ideal, then any compact quotient $\Gamma\backslash\mathrm{G}$ does not admit any invariant exact G_2-structure. If in addition G is completely solvable, namely ad_X has only real eigenvalues for every $X \in \mathfrak{g}$, then $\Gamma\backslash\mathrm{G}$ does not have any exact G_2-structure at all.*

In [22], we investigated the existence of invariant exact G_2-structures on compact quotients of Lie groups without introducing any extra assumption on the Lie algebra $\mathfrak{g}$, and we proved the following result.

Theorem 4.5 ([22]). *A potential compact 7-manifold M with an exact G_2-structure φ cannot be of the form $M = \Gamma\backslash\mathrm{G}$, where G is a 7-dimensional simply connected Lie group, $\Gamma \subset \mathrm{G}$ is a lattice, and the exact G_2-structure φ on M is invariant.*

To prove this result, we focused on 7-dimensional unimodular Lie algebras $\mathfrak{g}$ and we studied the non-solvable case and the solvable case separately. By Theorem 2.3, there are four non-solvable unimodular Lie algebras admitting closed G_2-structures, up to isomorphism. The first three Lie algebras are decomposable, and by a direct computation we showed that b_φ is never definite for every exact 3-form φ on each one of them. The remaining Lie algebra $\mathfrak{q}_4$ is indecomposable, and for this we proved that the corresponding simply connected Lie group does not admit any lattice. In the solvable case, $\mathfrak{g}$ has a codimension one unimodular ideal $\mathfrak{s}$, and the existence of a G_2-structure φ on $\mathfrak{g}$ allows one to consider the g_φ-orthogonal decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the orthogonal complement of $\mathfrak{s}$. As a Lie algebra, $\mathfrak{g}$ is then a semidirect product of the form $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$, for some derivation D of $\mathfrak{s}$. Moreover, the G_2-structure φ on $\mathfrak{g}$ can be written as follows:

$$\varphi = \omega \wedge \eta + \rho,$$

where $\eta := z^\flat$ is the metric dual of a unit vector $z \in \mathbb{R}$, and the pair (ω, ρ) defines an SU(3)-structure on $\mathfrak{s}$. By imposing that φ is an exact non-degenerate 3-form and using that $\mathfrak{g}$ has to be strongly unimodular, one sees that no examples can be found also in the solvable case.

Acknowledgments. The first author would like to thank Mahir Bilen Can, Sergey Grigorian, and Sema Salur for the invitation to give a seminar in the Minisymposium “Geometries Defined by Differential Forms” at 8ECM.

Funding. The authors were partly supported by GNSAGA of INdAM and by the project PRIN 2017 "Real and Complex Manifolds: Topology, Geometry and Holomorphic Dynamics."

References

[1] D. V. Alekseevskiĭ, Contact homogeneous spaces. *Funktsional. Anal. i Prilozhen.* **24** (1990), no. 4, 74–75 Zbl 0721.53042 MR 1092805

[2] M. F. Arikan, H. Cho, and S. Salur, Existence of compatible contact structures on G_2-manifolds. *Asian J. Math.* **17** (2013), no. 2, 321–333 Zbl 1337.53064 MR 3078933

[3] G. Ball, Quadratic closed G_2-structures. 2020, arXiv:2006.14155

[4] E. Bonan, Sur des variétés riemanniennes à groupe d'holonomie G_2 ou spin (7). *C. R. Acad. Sci. Paris Sér. A-B* **262** (1966), A127–A129 Zbl 0134.39402 MR 196668

[5] W. M. Boothby and H. C. Wang, On contact manifolds. *Ann. of Math. (2)* **68** (1958), 721–734 Zbl 0084.39204 MR 112160

[6] R. Bryant and F. Xu, Laplacian flow for closed G_2-structures: short time behavior. 2011, arXiv:1101.2004

[7] R. L. Bryant, Metrics with exceptional holonomy. *Ann. of Math. (2)* **126** (1987), no. 3, 525–576 Zbl 0637.53042 MR 916718

[8] R. L. Bryant, Some remarks on G_2-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pp. 75–109, Gökova Geometry/Topology Conference (GGT), Gökova, 2006 Zbl 1115.53018 MR 2282011

[9] R. L. Bryant and S. M. Salamon, On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.* **58** (1989), no. 3, 829–850 Zbl 0681.53021 MR 1016448

[10] G. Chen, Shi-type estimates and finite-time singularities of flows of G_2 structures. *Q. J. Math.* **69** (2018), no. 3, 779–797 Zbl 1425.53033 MR 3859207

[11] D. Conti and M. Fernández, Nilmanifolds with a calibrated G_2-structure. *Differential Geom. Appl.* **29** (2011), no. 4, 493–506 Zbl 1222.53059 MR 2811660

[12] A. Corti, M. Haskins, J. Nordström, and T. Pacini, G_2-manifolds and associative submanifolds via semi-Fano 3-folds. *Duke Math. J.* **164** (2015), no. 10, 1971–2092 Zbl 1343.53044 MR 3369307

[13] X. de la Ossa, M. Larfors, and M. Magill, Almost contact structures on manifolds with a G_2 structure. 2021, arXiv:2101.12605

[14] M. Fernández, An example of a compact calibrated manifold associated with the exceptional Lie group G_2. *J. Differential Geom.* **26** (1987), no. 2, 367–370 Zbl 0604.53013 MR 906398

[15] M. Fernández, A family of compact solvable G_2-calibrated manifolds. *Tohoku Math. J. (2)* **39** (1987), no. 2, 287–289 Zbl 0609.53011 MR 887944

[16] M. Fernández, A. Fino, A. Kovalev, and V. Muñoz, A compact G_2-calibrated manifold with first Betti number $b_1 = 1$. *Adv. Math.* **381** (2021), Paper No. 107623 Zbl 1472.53061 MR 4206793

[17] M. Fernández, A. Fino, and V. Manero, Laplacian flow of closed G_2-structures inducing nilsolitons. *J. Geom. Anal.* **26** (2016), no. 3, 1808–1837 Zbl 1344.53041 MR 3511459

[18] M. Fernández, A. Fino, and A. Raffero, Exact G_2-structures on unimodular Lie algebras. *Monatsh. Math.* **193** (2020), no. 1, 47–60 Zbl 1455.53049 MR 4127433

[19] M. Fernández and A. Gray, Riemannian manifolds with structure group G_2. *Ann. Mat. Pura Appl. (4)* **132** (1982), 19–45 (1983) Zbl 0524.53023 MR 696037

[20] M. Fernández, V. Manero, A. Otal, and L. Ugarte, Symplectic half-flat solvmanifolds. *Ann. Global Anal. Geom.* **43** (2013), no. 4, 367–383 Zbl 1266.53035 MR 3038540

[21] J. Fine and C. Yao, Hypersymplectic 4-manifolds, the G_2-Laplacian flow, and extension assuming bounded scalar curvature. *Duke Math. J.* **167** (2018), no. 18, 3533–3589 Zbl 07009771 MR 3881202

[22] A. Fino, L. Martín-Merchán, and A. Raffero, Exact G_2-structures on compact quotients of Lie groups. 2021, arXiv:2108.11664

[23] A. Fino and A. Raffero, Closed G_2-structures on non-solvable Lie groups. *Rev. Mat. Complut.* **32** (2019), no. 3, 837–851 Zbl 1428.53032 MR 3995432

[24] A. Fino and A. Raffero, Closed warped G_2-structures evolving under the Laplacian flow. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20** (2020), no. 1, 315–348 Zbl 1451.53127 MR 4088743

[25] A. Fino and A. Raffero, Remarks on homogeneous solitons of the G_2-Laplacian flow. *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 4, 401–406 Zbl 1475.53107 MR 4134249

[26] A. Fino, A. Raffero, and F. Salvatore, Closed G_2-structures on unimodular Lie algebras with non-trivial center. *Transform. Groups* (2022), DOI 10.1007/s00031-021-09683-8

[27] U. Fowdar, S^1-invariant Laplacian flow. *J. Geom. Anal.* **32** (2022), no. 1, Paper No. 17 Zbl 07446091 MR 4349461

[28] M. Freibert and S. Salamon, Closed G_2-eigenforms and exact G_2-structures. *Rev. Mat. Iberoam.* **38** (2022), no. 6, 1827–1866 Zbl 07628545 MR 4516173

[29] H. Garland, On the cohomology of lattices in solvable Lie groups. *Ann. of Math. (2)* **84** (1966), 175–196 Zbl 0142.26702 MR 207916

[30] R. Harvey and H. B. Lawson, Jr., Calibrated geometries. *Acta Math.* **148** (1982), 47–157 Zbl 0584.53021 MR 666108

[31] N. Hitchin, The geometry of three-forms in six dimensions. *J. Differential Geom.* **55** (2000), no. 3, 547–576 Zbl 1036.53042 MR 1863733

[32] N. Hitchin, Stable forms and special metrics. In *Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Bilbao, 2000)*, pp. 70–89, Contemp. Math. 288, Amer. Math. Soc., Providence, RI, 2001 Zbl 1004.53034 MR 1871001

[33] H. Huang, Y. Wang, and C. Yao, Cohomogeneity-one G_2-Laplacian flow on the 7-torus. *J. Lond. Math. Soc. (2)* **98** (2018), no. 2, 349–368 Zbl 1403.53056 MR 3873112

[34] D. Joyce and S. Karigiannis, A new construction of compact torsion-free G_2-manifolds by gluing families of Eguchi-Hanson spaces. *J. Differential Geom.* **117** (2021), no. 2, 255–343 Zbl 1464.53067 MR 4214342

[35] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I. *J. Differential Geom.* **43** (1996), no. 2, 291–328 Zbl 0861.53022 MR 1424428

[36] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. II. *J. Differential Geom.* **43** (1996), no. 2, 329–375 Zbl 0861.53023 MR 1424428

[37] I. Kath and J. Lauret, A new example of a compact ERP G_2-structure. *Bull. Lond. Math. Soc.* **53** (2021), no. 6, 1692–1710 MR 4375925

[38] A. Kovalev, Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.* **565** (2003), 125–160 Zbl 1043.53041 MR 2024648

[39] A. Kovalev and N.-H. Lee, $K3$ surfaces with non-symplectic involution and compact irreducible G_2-manifolds. *Math. Proc. Cambridge Philos. Soc.* **151** (2011), no. 2, 193–218 Zbl 1228.53064 MR 2823130

[40] B. Lambert and J. D. Lotay, Spacelike mean curvature flow. *J. Geom. Anal.* **31** (2021), no. 2, 1291–1359 Zbl 1465.53081 MR 4215263

[41] J. Lauret, Laplacian flow of homogeneous G_2-structures and its solitons. *Proc. Lond. Math. Soc. (3)* **114** (2017), no. 3, 527–560 Zbl 1380.53074 MR 3653239

[42] C. Lin, Laplacian solitons and symmetry in G_2-geometry. *J. Geom. Phys.* **64** (2013), 111–119 Zbl 1259.53066 MR 3004019

[43] J. D. Lotay, Geometric flows of G_2 structures. In *Lectures and Surveys on* G_2*-Manifolds and Related Topics*, pp. 113–140, Fields Inst. Commun. 84, Springer, New York, 2020 Zbl 1447.53086 MR 4295856

[44] J. D. Lotay and Y. Wei, Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. *Geom. Funct. Anal.* **27** (2017), no. 1, 165–233 Zbl 1378.53078 MR 3613456

[45] J. D. Lotay and Y. Wei, Laplacian flow for closed G_2 structures: real analyticity. *Comm. Anal. Geom.* **27** (2019), no. 1, 73–109 Zbl 1421.53032 MR 3951021

[46] J. D. Lotay and Y. Wei, Stability of torsion-free G_2 structures along the Laplacian flow. *J. Differential Geom.* **111** (2019), no. 3, 495–526 Zbl 07036514 MR 3934598

[47] M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras. *Differential Geom. Appl.* **31** (2013), no. 1, 112–129 Zbl 1263.53045 MR 3010082

[48] T. B. Madsen and A. Swann, Multi-moment maps. *Adv. Math.* **229** (2012), no. 4, 2287–2309 Zbl 1238.53017 MR 2880222

[49] A. I. Malcev, On a class of homogeneous spaces. *Amer. Math. Soc. Translation* **1951** (1951), no. 39, 33 Zbl 0034.01701 MR 0039734

[50] V. Manero, Compact solvmanifolds with calibrated and cocalibrated G_2-structures. *Manuscripta Math.* **162** (2020), no. 3-4, 315–339 Zbl 1441.53017 MR 4109489

[51] L. Martín-Merchán, A compact non-formal closed G_2 manifold with $b_1 = 1$. 2020, *Math. Nachr.* (to appear); arXiv:2005.04924

[52] J. Milnor, Curvatures of left invariant metrics on Lie groups. *Advances in Math.* **21** (1976), no. 3, 293–329 Zbl 0341.53030 MR 425012

[53] M. Nicolini, Laplacian solitons on nilpotent Lie groups. *Bull. Belg. Math. Soc. Simon Stevin* **25** (2018), no. 2, 183–196 Zbl 1393.37088 MR 3819121

[54] M. Nicolini, New examples of shrinking Laplacian solitons. *Q. J. Math.* **73** (2022), no. 1, 239–259 MR 4395079

[55] F. Podestà and A. Raffero, Closed G_2-structures with a transitive reductive group of automorphisms. 2019, *Asian J. Math.* (to appear); arXiv:1911.13052

[56] F. Podestà and A. Raffero, On the automorphism group of a closed G_2-structure. *Q. J. Math.* **70** (2019), no. 1, 195–200 Zbl 1414.53019 MR 3927848

[57] F. Xu and R. Ye, Existence, convergence and limit map of the Laplacian flow. 2009, arXiv:0912.0074

Anna Fino
Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy; and Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA; annamaria.fino@unito.it, afino@fiu.edu

Alberto Raffero
Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy; alberto.raffero@unito.it