DOI 10.4171/8ECM/26

On a class of nonlocal problems with fractional gradient constraint

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Abstract. We consider a Hilbertian and a charges approach to fractional gradient constraint problems of the type $|D^\sigma u| \leq g$, involving the distributional fractional Riesz gradient D^σ, $0 < \sigma < 1$, extending previous results on the existence of solutions and Lagrange multipliers of these nonlocal problems.

We also prove their convergence as $\sigma \nearrow 1$ towards their local counterparts with the gradient constraint $|Du| \leq g$.

1. Introduction

Recently, the distributional partial derivatives of the Riesz potentials of order $1-\sigma$, $0 < \sigma < 1$,

$$(D^\sigma u)_j = \frac{\partial}{\partial x_j}(I_{1-\sigma} u) = D_j(I_{1-\sigma} u), \quad j = 1, \dots, N,$$

where I_α, $0 < \alpha < 1$, is given by

$$I_\alpha u(x) = (I_\alpha * u)(x) = \gamma_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{d-\alpha}}\, dy, \quad \text{with } \gamma_{N,\alpha} = \frac{\Gamma\big(\frac{N-\alpha}{2}\big)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\big(\frac{\alpha}{2}\big)},$$

are shown to be a useful tool for a fractional vector calculus with the σ-gradient D^σ and σ-divergence $D^\sigma\cdot$ (see [5, 6, 12–14]). It leads to a new class of fractional partial differential equations and new problems in the calculus of variations [4]. As a consequence of the approximation of the identity by the Riesz kernel as $\alpha \to 0$ (see [7]), the σ-gradient converges to the classical gradient D as $\sigma \nearrow 1$, for instance, for smooth functions $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ (see also [4,6]). Among the nice properties of D^σ, in [12] it was shown, for $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, that

$$D^\sigma u \equiv D(I_{1-\sigma} * u) = I_{1-\sigma} * Du, \tag{1.1}$$

2020 Mathematics Subject Classification. Primary 35R11; Secondary 35J62, 49J40, 35J86, 26A33.
Keywords. Fractional gradient, nonlocal variational inequalities, gradient constraint, nonlocal Lagrange multiplier, elliptic quasilinear equations.

$$(-\Delta)^\sigma u = -D^\sigma \cdot (D^\sigma u), \tag{1.2}$$

where $(-\Delta)^\sigma$ is the classical fractional Laplacian in $\mathbb{R}^N$.

Here we are interested in complementing and extending some results of [10] on elliptic fractional equations of second σ-order, subjected to a σ-gradient constraint

$$|D^\sigma u| \leq g \quad \text{in } \mathbb{R}^N \tag{1.3}$$

and having the distributional form

$$-D^\sigma \cdot (AD^\sigma u + \Lambda^\sigma) = f_\# - D^\sigma \cdot \boldsymbol{f}. \tag{1.4}$$

We consider the homogeneous Dirichlet problem in a bounded open domain $\Omega \subset \mathbb{R}^N$, with Lipschitz boundary, so that the solution u is to be found in the fractional Sobolev space $H_0^\sigma(\Omega)$, $0 < \sigma < 1$, and may be extended by zero, belonging to $H^\sigma(\mathbb{R}^N)$. The Lipschitz boundary is sufficient for the $H_0^\sigma(\Omega)$-extension property, which is required in Section 4. Although in Sections 2 and 3 it is not strictly necessary, we prefer to keep this assumption in order to avoid delicate issues, in particular, with the definition of the classical space $H_0^\sigma(\Omega)$, which is the natural space to treat the Dirichlet boundary condition.

In (1.4), A is a coercive matrix with bounded variable coefficients (see (2.1), (2.2)) and $f_\#$ and $\boldsymbol{f}$ are given functions making the right-hand side an element f' of a suitable dual space.

The vector field Λ^σ is associated with the constraint (1.3) and may have two possible expressions. As we show in Section 2, with a Hilbertian approach, for $g \in L^2(\mathbb{R}^N)$, $g \geq 0$, and $f' \in H^{-\sigma}(\Omega) = (H_0^\sigma(\Omega))'$, $\Lambda^\sigma = D^\sigma \gamma$ for a unique $\gamma \in H_0^\sigma(\Omega)$ and it defines an element of the subdifferential of $\mathbb{K}_g^\sigma$, the convex subset of $H_0^\sigma(\Omega)$ of functions satisfying (1.3). The solution u is then the unique solution to the variational inequality (2.9) in $\mathbb{K}_g^\sigma$ for the operator $-D^\sigma \cdot (AD^\sigma \cdot) - f'$.

In the second case, with a strictly positive $g \in L^\infty(\mathbb{R}^N)$ and $f_\# \in L^1(\Omega)$, $\boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{R}^N) = L^1(\mathbb{R}^N)^N$, in Section 3, by approximating the unique solution u with a suitable quasilinear penalised Dirichlet problem, we show the existence of at least a generalised nonnegative Lagrange multiplier $\lambda^\sigma \in L^\infty(\mathbb{R}^N)'$, such that $\Lambda^\sigma = \lambda^\sigma D^\sigma u$ and $\lambda^\sigma(|D^\sigma u| - g) = 0$ in the sense of charges, i.e., as an element of $L^\infty(\mathbb{R}^N)'$.

We recall (see [15, Example 5, Section 9, Chapter IV]), for instance, that a charge or an element $\chi \in L^\infty(\mathcal{O})'$, in an open set $\mathcal{O} \subset \mathbb{R}^N$, can be represented by a finitely additive measure χ^*, with bounded total variation, which is also absolutely continuous with respect to the Lebesgue measure and may be given by a Radon integral

$$\langle \chi, \varphi \rangle = \int_{\mathcal{O}} \varphi \, d\chi^*, \quad \forall \varphi \in L^\infty(\mathcal{O}). \tag{1.5}$$

As a consequence, it is easy to show the Hölder inequality for nonnegative charges $\chi \in L^\infty(\mathcal{O})'$ and arbitrary functions $\varphi, \psi \in L^\infty(\mathcal{O})$:

$$\big|\langle \chi, \varphi\psi\rangle\big| \le \big\langle \chi, |\varphi|^p\big\rangle^{\frac{1}{p}} \big\langle \chi, |\psi|^{p'}\big\rangle^{\frac{1}{p'}}, \quad p > 1,\ p' = \frac{p}{p-1}. \tag{1.6}$$

It was proved in [12] that, similarly to the classical case $\sigma = 1$, the Sobolev, Trudinger, and Morrey inequalities also hold for the fractional D^σ; in particular, there exists a constant $C = C(N, p, \sigma) > 0$, such that, for $1 < p < \infty$, $\sigma \in (0, 1)$,

$$\|u\|_{L^q(\mathbb{R}^N)} \le C\|D^\sigma u\|_{L^p(\mathbb{R}^N)}, \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^N), \tag{1.7}$$

where $q = \frac{Np}{N-\sigma p}$ if $\sigma < \frac{N}{p}$, $q < \infty$ if $\sigma = \frac{N}{p}$, and $q = \infty$ if $\sigma > \frac{N}{p}$. In addition, when $\sigma > \frac{N}{p}$, we may take in the left-hand side of (1.7) the norm of the Hölder continuous functions $\mathcal{C}_c^\beta(\mathbb{R}^N)$, $0 < \beta = \sigma - \frac{N}{p} < 1$. As a consequence, we consider $H_0^\sigma(\Omega)$ with the equivalent Hilbertian norm $\|D^\sigma u\|_{L^2(\mathbb{R}^N)}$ (see [12]), which is also a consequence of the fractional Poincaré inequality (see [4]).

We observe that our results of Sections 2 and 3 also hold in the limit local case $\sigma = 1$, i.e., in $H_0^1(\Omega)$. We then show in Section 4, where we need to work with generalised sequences or nets, that the charges approach to the constrained problem yields the convergence, as $\sigma \nearrow 1$, of the solution u^σ and the generalised Lagrange multiplier λ^σ to the respective solution $(u, \lambda) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)'$ to the classical problem for D. We remark that, in this case, our results are new for data in L^1 and the general elliptic operator $-D \cdot (AD)$, extending [3], where the charges approach was introduced for $-\Delta$ with $f_\# \in L^2(\Omega)$ and $\boldsymbol{f} = 0$. For a recent survey on gradient type constrained problems, see [11].

2. The Hilbertian approach with σ-gradient constraint in L^2

Let the not necessarily symmetric measurable matrix $A = A(x) : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ satisfy the coercive assumption, for some given $a_*, a^* > 0$,

$$A(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \ge a_*|\boldsymbol{\xi}|^2, \quad \text{a.e. } x \in \mathbb{R}^N,\ \forall \boldsymbol{\xi} \in \mathbb{R}^N, \tag{2.1}$$

and the boundedness conditions

$$A(x)\boldsymbol{\xi} \cdot \boldsymbol{\eta} \le a^*|\boldsymbol{\xi}|\,|\boldsymbol{\eta}|, \quad \text{a.e. } x \in \mathbb{R}^N,\ \forall \boldsymbol{\xi},\ \boldsymbol{\eta} \in \mathbb{R}^N. \tag{2.2}$$

Consider

$$f_\# \in L^{2^\#}(\Omega) \quad \text{and} \quad \boldsymbol{f} = (f_1, \dots, f_N) \in \boldsymbol{L}^2(\mathbb{R}^N), \tag{2.3}$$

where, by the Sobolev embeddings (1.7), $2^\# = \frac{2N}{N+2\sigma}$ if $0 < \sigma < \frac{N}{2}$, or $2^\# = q$ for any $q > 1$ when $\sigma = \frac{1}{2}$ and $2^\# = 1$ when $\frac{1}{2} < \sigma < 1$, so that

$$\langle f', v\rangle_\sigma = \int_\Omega f_\# v + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma v, \tag{2.4}$$

for arbitrary $v \in H_0^\sigma(\Omega)$, defines the linear form $f' \in H^{-\sigma}(\Omega) = H_0^\sigma(\Omega)'$, $0 < \sigma < 1$. We have

$$\exists! \phi \in H_0^\sigma(\Omega) : \int_{\mathbb{R}^N} D^\sigma \phi \cdot D^\sigma v = \langle f', v\rangle_\sigma, \quad \forall v \in H_0^\sigma(\Omega). \tag{2.5}$$

The validity of (2.5) is a consequence of the Fréchet–Riesz representation theorem and the choice of the left-hand side of this equality as the inner product in $H_0^\sigma(\Omega)$, as stated in Section 1. It follows that $\boldsymbol{F} = D^\sigma \phi \in \boldsymbol{L}^2(\Omega)$ belongs to the image of $H_0^\sigma(\Omega)$ by D^σ:

$$\Psi_\sigma = \left\{\boldsymbol{G} \in \boldsymbol{L}^2(\mathbb{R}^N) : \boldsymbol{G} = D^\sigma v,\ v \in H_0^\sigma(\Omega)\right\} = D^\sigma\left(H_0^\sigma(\Omega)\right), \tag{2.6}$$

which is a strict Hilbert subspace of $\boldsymbol{L}^2(\mathbb{R}^N)$, for the inner product

$$(\boldsymbol{F}, \boldsymbol{G})_{\Psi_\sigma} = \int_{\mathbb{R}^N} D^\sigma \phi \cdot D^\sigma v,$$

and Ψ_σ is isomorphic to $H^{-\sigma}(\Omega)$, by the Riesz theorem (2.5). Actually, this remark extends the well-known case $\sigma = 1$, when D^1 is the classical gradient D.

Consider the nonempty closed convex set

$$\mathbb{K}_g^\sigma = \left\{v \in H_0^\sigma(\Omega) : |D^\sigma v| \leq g \text{ a.e. in } \mathbb{R}^N\right\}, \tag{2.7}$$

where the σ-gradient threshold g is such that

$$g \in L^2(\mathbb{R}^N), \quad g(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^N. \tag{2.8}$$

Under the assumptions (2.1) and (2.2), A defines a continuous bounded coercive bilinear form over $H_0^\sigma(\Omega)$ and, as an immediate consequence of the Stampacchia theorem (see [9, p. 95], for instance), we have the existence, uniqueness, and continuous dependence of the solution u, with respect to the linear form (2.4), of the variational inequality

$$u \in \mathbb{K}_g^\sigma : \int_{\mathbb{R}^N} A D^\sigma u \cdot D^\sigma(v - u) \geq \int_\Omega f_\#(v-u) + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma(v-u), \quad \forall v \in \mathbb{K}_g^\sigma. \tag{2.9}$$

In particular, if C_* denotes the Sobolev constant, with $L^{2^*}(\Omega) = L^{2^\#}(\Omega)'$,

$$\|v\|_{L^{2^*}(\Omega)} \le C_* \|D^\sigma v\|_{\boldsymbol{L}^2(\mathbb{R}^N)}, \quad v \in H^\sigma_0(\Omega),\ 0 < \sigma \le 1,$$

and $\hat{u}$ is the solution corresponding to the data $\hat{f}_\#$, $\hat{\boldsymbol{f}}$, we have

$$\|u - \hat{u}\|_{H^\sigma_0(\Omega)} \le \frac{C_*}{a_*}\|f_\# - \hat{f}_\#\|_{L^{2^\#}(\Omega)} + \frac{1}{a_*}\|\boldsymbol{f} - \hat{\boldsymbol{f}}\|_{\boldsymbol{L}^2(\mathbb{R}^N)}. \tag{2.10}$$

It is well known (see [8, p. 203], for instance) that to solve (2.9) is equivalent to finding $u \in H^\sigma_0(\Omega)$, such that

$$\Gamma \equiv f' - \mathcal{L}^\sigma_A u \in \partial I_{\mathbb{K}^\sigma_g}(u) \quad \text{in } H^{-\sigma}(\Omega), \tag{2.11}$$

where $\mathcal{L}^\sigma_A : H^\sigma_0(\Omega) \to H^{-\sigma}(\Omega)$ is the linear continuous operator defined by

$$\langle \mathcal{L}^\sigma_A w, v\rangle_\sigma = \int_{\mathbb{R}^N} A D^\sigma w \cdot D^\sigma v, \quad \forall v, w \in H^\sigma_0(\Omega),$$

and $\Gamma = \Gamma(u) \in H^{-\sigma}(\Omega)$ is an element of the sub-gradient of the indicatrix function $I_{\mathbb{K}^\sigma_g}$ of the convex set $\mathbb{K}^\sigma_g$ at u:

$$I_{\mathbb{K}^\sigma_g}(v) = \begin{cases} 0 & \text{if } v \in \mathbb{K}^\sigma_g, \\ +\infty & \text{if } v \in H^\sigma_0(\Omega) \setminus \mathbb{K}^\sigma_g. \end{cases}$$

By the Riesz theorem, there exists a unique $\gamma = \gamma(u) \in H^\sigma_0(\Omega)$ corresponding to $\Gamma = \Gamma(u)$ given by (2.11) (recall (2.5)) and the couple $(u, \gamma) \in \mathbb{K}^\sigma_g \times H^\sigma_0(\Omega)$ solves the problem

$$\int_{\mathbb{R}^N} (A D^\sigma u + D^\sigma \gamma) \cdot D^\sigma v = \int_\Omega f_\# v + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma v, \quad \forall v \in H^\sigma_0(\Omega). \tag{2.12}$$

If we denote $\widehat{\gamma} = \gamma(\hat{u})$, with $\hat{u}$ solving (2.9) with $\hat{f}_\#$ and $\hat{\boldsymbol{f}}$ given in (2.3), using (2.10) and (2.2), we easily obtain, by the Riesz isometry $\|\Gamma\|_{H^{-\sigma}(\Omega)} = \|\gamma\|_{H^\sigma_0(\Omega)}$,

$$\|\gamma - \widehat{\gamma}\|_{H^\sigma_0(\Omega)} \le C_*\left(1 + \frac{a^*}{a_*}\right)\|f_\# - \hat{f}_\#\|_{L^{2^\#}(\Omega)} + \left(1 + \frac{a^*}{a_*}\right)\|\boldsymbol{f} - \hat{\boldsymbol{f}}\|_{\boldsymbol{L}^2(\mathbb{R}^N)}. \tag{2.13}$$

We have then proven the following result.

Theorem 2.1. *Under the previous assumptions, namely,* (2.1), (2.2), (2.3), *and* (2.8), *there exists a unique solution of* (2.9), *which also satisfies* (2.12) *with a unique* $\gamma = \gamma(u) \in H^\sigma_0(\Omega)$, *obtained through* (2.11) *and depending on the data through* (2.13).

Remark 2.2. This result extends to the Riesz fractional gradient the limit case $\sigma = 1$, where the classical gradients of u and γ are extended by zero in $\mathbb{R}^N \setminus \Omega$. A natural and important question is to find a more direct relation of the potential γ with the solution u through the existence of a Lagrange multiplier λ, such that

$$D^\sigma \gamma = \lambda D^\sigma u. \tag{2.14}$$

In the classical case $\sigma = 1$, with $A = Id$, $\Omega \subseteq \mathbb{R}^2$ simply connected, and f' and g given by positive constants, corresponding to the elasto-plastic torsion problem, Brézis has proven the existence and uniqueness of a bounded function

$$\lambda \geq 0 \quad \text{such that} \quad \lambda\big(|Du| - g\big) = 0 \text{ a.e. in } \Omega,$$

which is even continuous if Ω is convex (see [11] for references). Although (2.14) is an open question in the general case of Theorem 2.1, for strictly positive bounded threshold g, it has been shown to hold in the sense of finite additive measures in [10], following the case $\sigma = 1$ of [3].

Using a variant of a classical penalisation method proposed in [8, p. 376] with $\varepsilon \in (0, 1)$ and

$$k_\varepsilon(t) = 0,\ t \leq 0, \quad k_\varepsilon(t) = \frac{t}{\varepsilon},\ 0 \leq t \leq \frac{1}{\varepsilon}, \quad k_\varepsilon(t) = \frac{1}{\varepsilon^2},\ t \geq \frac{1}{\varepsilon}, \tag{2.15}$$

we may consider the approximating quasi-linear problem: find $u_\varepsilon \in H^\sigma_0(\Omega)$, such that

$$\int_{\mathbb{R}^N} \big(AD^\sigma u_\varepsilon + \widehat{k}_\varepsilon(u_\varepsilon)\, D^\sigma u_\varepsilon\big) \cdot D^\sigma v = \int_\Omega f_\# v + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma v,\ \forall v \in H^\sigma_0(\Omega), \tag{2.16}$$

where we set

$$\widehat{k}_\varepsilon = \widehat{k}_\varepsilon(u_\varepsilon) = k_\varepsilon\big(|D^\sigma u_\varepsilon|^2 - g^2\big) \quad \text{with } k_\varepsilon \text{ given by (2.15).}$$

In the proof of the approximation theorem, we shall require the following assumption: for each $R > 0$, there exists a g_R, such that

$$g(x) \geq g_R > 0, \quad \text{for a.e. } x \in B_R = \big\{x \in \mathbb{R}^N : |x| < R\big\}. \tag{2.17}$$

Theorem 2.3. *Under the assumptions of Theorem 2.1, let also (2.17) hold. Then, the unique solution $u_\varepsilon \in H^\sigma_0(\Omega)$ of (2.16), as $\varepsilon \to 0$, is such that*

$$u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \qquad \text{in } H^\sigma_0(\Omega)\text{-weak}, \tag{2.18}$$

$$\widehat{k}_\varepsilon D^\sigma u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} D^\sigma \gamma \quad \text{in } \Psi'_\sigma\text{-weak}, \tag{2.19}$$

where $(u,\gamma)\in\mathbb{K}_g^\sigma\times H_0^\sigma(\Omega)$ *is the unique couple given in Theorem 2.1 and satisfying* (2.12) *and* Ψ_σ *is the vector space defined in* (2.6).

Proof. Since the quasi-linear operator $\widehat{A}_\varepsilon : H_0^\sigma(\Omega)\to H^{-\sigma}(\Omega)$ defined by the left-hand side of (2.16) is bounded, strongly monotone, coercive, and hemicontinuous, the existence and uniqueness of u_ε solution to (2.16) is classical (see [8], for instance).

Taking $v=u_\varepsilon$ in (2.16) and recalling that $\widehat{\kappa}_\varepsilon(u_\varepsilon)\geq 0$, it is clear that we have, with $C_\sigma>0$ independent of ε, $0<\varepsilon<1$:

$$\|u_\varepsilon\|_{H_0^\sigma(\Omega)}\leq\frac{C_*}{a_*}\|f_\#\|_{L^{2^\#}(\Omega)}+\frac{1}{a_*}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^N)}\equiv C_\sigma, \tag{2.20}$$

so that we have (2.18) at least for a generalised subsequence and some $u\in H_0^\sigma(\Omega)$. Consequently, from (2.16), we also obtain

$$\|\widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon\|_{\Psi'_\sigma}=\sup_{\substack{v\in H_0^\sigma(\Omega)\\ \|v\|_{H_0^\sigma(\Omega)}=1}}\int_{\mathbb{R}^N}\widehat{\kappa}_\varepsilon(u_\varepsilon)D^\sigma u_\varepsilon\cdot Dv\leq(a_*+a^*)C_\sigma,$$

for all ε, $0<\varepsilon<1$, by using (2.20) and recalling (2.2). Here we use the definition (2.5) and we consider $\boldsymbol{L}^2(\mathbb{R}^N)$, identified to its dual, as a subspace of Ψ'_σ, the dual of $\Psi_\sigma\subseteq\boldsymbol{L}^2(\mathbb{R}^N)$. Hence, for a generalised subsequence $\varepsilon\to 0$, we also have

$$\widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon\underset{\varepsilon\to 0}{\rightharpoonup}\Lambda\quad\text{in }\Psi'_\sigma\text{-weak.} \tag{2.21}$$

In order to prove that $u\in\mathbb{K}_g^\sigma$, i.e., $|D^\sigma u|\leq g$ a.e. in $\mathbb{R}^N$, we consider, for $R>0$,

$$U_{\varepsilon,R}=\left\{x\in B_R : 0\leq\left|D^\sigma u_\varepsilon(x)\right|^2-g^2(x)\leq\sqrt{\varepsilon}\right\},$$
$$V_{\varepsilon,R}=\left\{x\in B_R : \left|D^\sigma u_\varepsilon(x)\right|^2-g^2(x)>\sqrt{\varepsilon}\right\}$$

and we observe that, using the assumptions (2.17), (2.20), and $\widehat{\kappa}_\varepsilon(|D^\sigma u^\varepsilon|^2-g^2)\geq 0$, from (2.16) it follows that

$$g_R^2\int_{B_R}\widehat{\kappa}_\varepsilon\leq\int_{\mathbb{R}^N}\widehat{\kappa}_\varepsilon g^2\leq\int_{\mathbb{R}^N}\widehat{\kappa}_\varepsilon|D^\sigma u_\varepsilon|^2\leq\frac{a_*}{2}C_\sigma^2,\qquad 0<\varepsilon<1. \tag{2.22}$$

Consequently, for all $R>0$, we conclude that $|D^\sigma u|\leq g$ in B_R from

$$\begin{aligned}\int_{B_R}\big(|D^\sigma u|-g\big)^+&\leq\varliminf_{\varepsilon\to 0}\int_{B_R}\big(|D^\sigma u_\varepsilon|-g\big)^+\\&=\varliminf_{\varepsilon\to 0}\left[\int_{U_{\varepsilon,R}}\big(|D^\sigma u_\varepsilon|-g\big)+\int_{V_{\varepsilon,R}}\big(|D^\sigma u_\varepsilon|-g\big)\right]\end{aligned}$$

since

$$\int_{U_{\varepsilon,R}} \big(|D^\sigma u_\varepsilon| - g\big) \le \frac{1}{g_R} \int_{U_{\varepsilon,R}} \big(|D^\sigma u_\varepsilon|^2 - g^2\big) \le \frac{|B_R|\sqrt{\varepsilon}}{g_R},$$

$$\int_{V_{\varepsilon,R}} \big(|D^\sigma u_\varepsilon| - g\big) \le |V_{\varepsilon,R}|^{\frac{1}{2}} \big(\|D^\sigma u_\varepsilon\|_{\boldsymbol{L}^2(B_R)} + \|g\|_{\boldsymbol{L}^2(B_R)}\big)$$
$$\le \big(C_\sigma + \|g\|_{\boldsymbol{L}^2(\mathbb{R}^N)}\big)|V_{\varepsilon,R}|^{\frac{1}{2}}$$

with

$$|V_{\varepsilon,R}| = \int_{V_{\varepsilon,R}} 1 \le \int_{V_{\varepsilon,R}} \frac{\widehat{\kappa}_\varepsilon}{k_\varepsilon(\sqrt{\varepsilon})} \le \sqrt{\varepsilon} \int_{B_R} \widehat{\kappa}_\varepsilon \le \frac{a_* C_\sigma^2}{2 g_R^2} \sqrt{\varepsilon}.$$

Now, observing that for arbitrary $v \in \mathbb{K}_g^\sigma$ we have

$$\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon \cdot D^\sigma(v - u_\varepsilon) \le \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon| \big(|D^\sigma v| - |D^\sigma u_\varepsilon|\big) \le 0$$

(since $\widehat{\kappa}_\varepsilon > 0$ if $|D^\sigma u_\varepsilon| > g \ge |D^\sigma v|$), from (2.16) we obtain

$$\int_{\mathbb{R}^N} A D^\sigma u_\varepsilon \cdot D^\sigma(v - u_\varepsilon) \ge \int_\Omega f_\#(v - u_\varepsilon) + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma(v - u_\varepsilon), \quad \forall v \in \mathbb{K}_g^\sigma,$$

and, passing to the limit as $\varepsilon \to 0$, we conclude that u solves (2.9), by using (2.18) and the lower semi-continuity

$$\varliminf_{\varepsilon \to 0} \int_{\mathbb{R}^N} A D^\sigma u_\varepsilon \cdot D^\sigma u_\varepsilon \ge \int_{\mathbb{R}^N} A D^\sigma u \cdot D^\sigma u. \tag{2.23}$$

Finally, taking an arbitrary $\boldsymbol{G} = D^\sigma v \in \Psi_\sigma$ and taking $\varepsilon \to 0$ in (2.16), by recalling (2.21), (2.12), and (2.5) we find

$$\langle \Lambda, \boldsymbol{G} \rangle_{\Psi_\sigma} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon \cdot D^\sigma v = \int_{\mathbb{R}^N} (D^\sigma \phi - A D^\sigma u) \cdot D^\sigma v$$
$$= \int_{\mathbb{R}^N} D^\sigma \gamma \cdot D^\sigma v,$$

yielding the conclusion (2.19), by the uniqueness of u and γ. ■

3. The charges approach with a σ-gradient constraint in L^∞

In the framework of the previous section, we consider now the convex set $\mathbb{K}_g^\sigma$ defined by (2.7) with the assumption

$$g \in L^\infty(\mathbb{R}^N), \quad 0 < g_* \le g(x) \le g^* \text{ a.e. } x \text{ in } \mathbb{R}^N, \tag{3.1}$$

for some constants g_* and g^*. It is clear that $\mathbb{K}_g^\sigma$ is still closed for the topology of $H_0^\sigma(\Omega)$ in the space

$$\Upsilon_\infty^\sigma(\Omega) = \{v \in H_0^\sigma(\Omega) : D^\sigma v \in \boldsymbol{L}^\infty(\mathbb{R}^N)\}, \quad 0 < \sigma \le 1, \tag{3.2}$$

and therefore, by the fractional Morrey–Sobolev inequality (1.7) for $\sigma > \frac{N}{p}$, we have, for all $0 < \beta < \sigma$,

$$\mathbb{K}_g^\sigma \subset \Upsilon_\infty^\sigma(\Omega) \subset \mathcal{C}^{0,\beta}(\bar{\Omega}) \subset L^\infty(\Omega). \tag{3.3}$$

Here $\mathcal{C}^{0,\beta}(\bar{\Omega})$ is the space of the Hölder continuous functions with exponent β. As observed in [10], (3.3) is a consequence of Theorem 7.63 of [1] (see also [12, Theorem 2.2]), which yields

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\le C_p \|D^\sigma u\|_{\boldsymbol{L}^p(\mathbb{R}^N)} \\ &\le C_p \|D^\sigma u\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)}^{1-\frac{2}{p}} \|D^\sigma u\|_{\boldsymbol{L}^2(\mathbb{R}^N)}^{\frac{2}{p}}, \quad \forall u \in \Upsilon_\infty^\sigma(\Omega), \end{aligned} \tag{3.4}$$

where $C_p > 0$ is the Sobolev constant corresponding to any $p > \frac{N}{\sigma} \vee 2$.

Therefore, in this case, we can extend the result of the solvability of the variational inequality (2.9) with data in L^1:

$$f_\# \in L^1(\Omega) \quad \text{and} \quad \boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{R}^N). \tag{3.5}$$

Theorem 3.1. *Under the assumptions* (2.1)*,* (2.2)*,* (2.3)*, and* (3.1)*, the unique solution u to* (2.9) *also satisfies the continuous dependence estimates* (2.10)*. Moreover, if in addition $(\boldsymbol{f}, f_\#)$ and $(\hat{\boldsymbol{f}}, \hat{f}_\#)$ also satisfy* (3.5)*, the following estimate holds:*

$$\|u - \hat{u}\|_{H_0^\sigma(\Omega)} \le a_p \|f_\# - \hat{f}_\#\|_{L^1(\Omega)}^{\frac{1}{2-\frac{2}{p}}} + b_1 \|\boldsymbol{f} - \hat{\boldsymbol{f}}\|_{\boldsymbol{L}^1(\mathbb{R}^N)}^{\frac{1}{2}}, \tag{3.6}$$

where $p > \frac{N}{\sigma} \vee 2$ as in (3.4) *and $a_p, b_1 > 0$ are constants.*

Consequently, the variational inequality (2.9) *is also uniquely solvable with the assumption* (2.3) *replaced by* (3.5) *and the estimate* (3.6) *still holds in this case.*

Proof. While the first part of this theorem is also a direct consequence of the Stampacchia theorem, the estimate (3.6) follows easily from (2.9). Indeed, if we set $\bar{u} = u - \hat{u}$, $\bar{f}_\# = f_\# - \hat{f}_\#$, and $\bar{\boldsymbol{f}} = \boldsymbol{f} - \hat{\boldsymbol{f}}$, we have

$$\begin{aligned} a_* \|\bar{u}\|_{H_0^\sigma(\Omega)}^2 &= a_* \int_{\mathbb{R}^N} \|D^\sigma \bar{u}\|^2 \\ &\le \|\bar{u}\|_{L^\infty(\Omega)} \|\bar{f}_\#\|_{L^1(\Omega)} + \|D^\sigma \bar{u}\|_{\boldsymbol{L}^\infty(\Omega)} \|\bar{\boldsymbol{f}}\|_{\boldsymbol{L}^1(\Omega)} \\ &\le C_p (2g^*)^{1-\frac{2}{p}} \|D^\sigma \bar{u}\|_{\boldsymbol{L}^2(\Omega)}^{\frac{2}{p}} \|\bar{f}_\#\|_{L^1(\Omega)} + 2g^* \|\bar{\boldsymbol{f}}\|_{\boldsymbol{L}^1(\Omega)}, \end{aligned} \tag{3.7}$$

by (3.4) and the assumption (3.1). Hence, (3.6) follows easily by applying Young's inequality and $\sqrt{\phi + \psi} \leq \sqrt{\phi} + \sqrt{\psi}$ to the right-hand side of (3.7), where we obtain the constants a_p and b_1 depending on C_p, a_*, g^*, and $p > \frac{N}{\sigma} \vee 2$. The solvability of (2.9) under the assumption (3.5) can be easily obtained using (3.6), approximating the solution by a Cauchy sequence in $H_0^\sigma(\Omega)$ of solutions $u_\nu \xrightarrow[\nu\to 0]{} u$, where u_ν solves (2.9) with approximating sequences

$$f_{\#\nu} \xrightarrow[\nu\to 0]{} f_\# \text{ in } L^1(\Omega) \quad \text{and} \quad \boldsymbol{f}_\nu \xrightarrow[\nu\to 0]{} \boldsymbol{f} \text{ in } L^1(\mathbb{R}^N) \tag{3.8}$$

with $f_{\#\nu} \in L^2(\Omega)$ and $\boldsymbol{f}_\nu \in \boldsymbol{L}^2(\mathbb{R}^N)$, for instance, with $f_\nu = (f \wedge \frac{1}{\nu}) \vee (-\frac{1}{\nu})$ by truncation. ■

Remark 3.2. This result with L^1-data extends Theorem 2.1 of [10] which considered only the case $\boldsymbol{f} \equiv 0$. If the data $f_\# \in L^{2^\#}(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^N) \cap \boldsymbol{L}^1(\mathbb{R}^N)$ hold, our approximation Theorem 2.3 also holds for the solution (u, γ) to (2.11)-(2.12) under the assumption (3.1), which implies $g \in L^2(B_R)$ for all $R > 0$, since the proof is the same.

It is also possible to obtain with L^1-data the $\frac{1}{2}$-Hölder continuity of the map $L^\infty(\mathbb{R}^N) \ni g \mapsto u \in H_0^\sigma(\Omega)$ with g satisfying (3.1) and u solution to (2.9), extending Theorem 2.2 of [10].

Theorem 3.3. *Under the assumptions (2.1), (2.2), and (3.5), let u and $\hat{u}$ be the solutions to (2.9) corresponding to g and $\hat{g}$ satisfying (3.1). Then, there exists a constant $C_* > 0$, depending on g_* and the data, but independent of the solutions, such that*

$$\|u - \hat{u}\|_{H_0^\sigma(\Omega)} \leq C_* \|g - \hat{g}\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{2}}. \tag{3.9}$$

Proof. Denote $\delta = \|g - \hat{g}\|_{L^\infty(\mathbb{R}^N)}$, and take as test functions in (2.9), respectively,

$$w = \frac{g_*}{g_* + \delta}\hat{u} \in \mathbb{K}_g^\sigma \quad \text{and} \quad \hat{w} = \frac{g_*}{g_* + \delta}u \in \mathbb{K}_{\hat{g}}^\sigma$$

for the variational inequality for u and for $\hat{u}$.

Observing that

$$|u - \hat{w}| \leq \frac{\delta}{g_*}|u| \quad \text{and} \quad \left|D^\sigma(u - \hat{w})\right| \leq \frac{\delta}{g_*}|D^\sigma u|$$

and similarly for $\hat{u} - w$, we obtain (3.9) from

$$\begin{aligned} a_*\|u - \hat{u}\|_{H_0^\sigma(\Omega)}^2 &\leq \int_{\mathbb{R}^N} AD^\sigma(u - \hat{u}) \cdot D^\sigma(u - \hat{u}) \\ &= \int_{\mathbb{R}^N} AD^\sigma u \cdot D^\sigma(u - w) + \int_{\mathbb{R}^N} AD^\sigma u \cdot D^\sigma(w - \hat{u}) \\ &\quad + \int_{\mathbb{R}^N} AD^\sigma \hat{u} \cdot D^\sigma(\hat{u} - \hat{w}) + \int_{\mathbb{R}^N} AD^\sigma \hat{u} \cdot D^\sigma(\hat{w} - u) \end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega f_\#\big((u-w)+(\hat u-\hat w)\big)+\int_{\mathbb{R}^N} \boldsymbol{f}\cdot D^\sigma\big((u-w)+(\hat u-\hat w)\big)\\
&\quad+\frac{2\delta}{g_*}\int_{\mathbb{R}^N}|AD^\sigma u\cdot D^\sigma \hat u|\\
&=\int_\Omega f_\#\big((u-\hat w)+(\hat u-w)\big)+\int_{\mathbb{R}^N} \boldsymbol{f}\cdot D^\sigma\big((u-\hat w)+(\hat u-w)\big)\\
&\quad+\frac{2\delta}{g_*}\int_{\mathbb{R}^N}|AD^\sigma u\cdot D^\sigma \hat u|\\
&\le \frac{2\delta}{g_*}\big(C_p {g^*}^{1-\frac{2}{p}}\eta_p^{\frac{2}{p}}\|f_\#\|_{L^1(\Omega)}+g^*\|\boldsymbol{f}\|_{\boldsymbol{L}^1(\mathbb{R}^N)}+a^*\eta_p^2\big),
\end{aligned}
$$

by using (3.4) and $\eta_p = a_p\|f_\#\|_{L^1(\Omega)}^{\frac{1}{2-\frac{2}{p}}} + b_1\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^N)}^{\frac{1}{2}}$, which is a general upper bound for $\|D^\sigma u\|_{L^2(\mathbb{R}^N)}$ and $\|D^\sigma \hat u\|_{L^2(\mathbb{R}^N)}$, just by taking $v\equiv 0$ in (2.9) and calculating as in (3.6). ■

Remark 3.4. This theorem allows to obtain solutions to quasi-variational inequalities of the type (2.9), with the solution dependent on the convex sets $\mathbb{K}^\sigma_{G[u]}$ as in (2.7) with $g = G[u]$, where $G : L^{2^*}(\Omega)\to L^\infty_{g_*}(\mathbb{R}^N)$, being $L^\infty_{g_*}(\mathbb{R}^N) = \{h\in L^\infty(\mathbb{R}^N) : h(x)\ge g_* > 0 \text{ a.e. } x\in\mathbb{R}^N\}$, or $G : \mathcal{C}(\overline{\Omega})\to L^\infty_{g_*}(\mathbb{R}^N)$ are continuous and bounded operators, as in [10, Section 4], where only the case $f_\#\in L^2(\Omega)$ and $\boldsymbol{f}\equiv 0$ was considered.

As we observed in Remark 3.2, the solution u to the variational inequality with bounded σ-gradient constraint and data satisfying (2.3) also solves (2.12), but the extra terms involving γ can be interpreted with a Lagrange multiplier λ in a generalised sense extending Theorem 3.1 of [10] to L^1-data. Here we use the duality in $L^\infty(\mathbb{R}^N)$ and in $\boldsymbol{L}^\infty(\mathbb{R}^N)$ with the notation

$$
\langle\!\langle \lambda\boldsymbol{\alpha},\boldsymbol{\beta}\rangle\!\rangle = \langle \lambda,\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\rangle,\quad \forall\lambda\in L^\infty(\mathbb{R}^N)'\ \forall \boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{L}^\infty(\mathbb{R}^N). \tag{3.10}
$$

Theorem 3.5. *Under the assumptions* (2.1), (2.2), (3.1), *and* (2.3) *or* (3.5), *there exists* $(u,\lambda)\in\Upsilon^\sigma_\infty(\Omega)\times L^\infty(\mathbb{R}^N)'$, *such that*

$$
\begin{aligned}
&\int_{\mathbb{R}^N} AD^\sigma u\cdot D^\sigma w + \langle\!\langle \lambda D^\sigma u, D^\sigma w\rangle\!\rangle\\
&\qquad = \int_\Omega f_\# w + \int_{\mathbb{R}^N}\boldsymbol{f}\cdot D^\sigma w,\quad \forall w\in\Upsilon^\sigma_\infty(\Omega),
\end{aligned}
\tag{3.11}
$$

$$
|D^\sigma u|\le g \text{ a.e. in } \mathbb{R}^N,\quad \lambda\ge 0\quad\text{and}\quad \lambda\big(|D^\sigma u|-g\big)=0 \text{ in } L^\infty(\mathbb{R}^N)'. \tag{3.12}
$$

Moreover, u is the unique solution to the variational inequality (2.9).

Proof. (i) First we suppose (2.3), i.e., $f_\# \in L^2(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^N)$, and, from the approximation problem (2.16), in addition to (2.20), we obtain the *a priori* estimates independent of $0 < \varepsilon < 1$:

$$\|\widehat{\kappa}_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq \frac{a_*}{2g_*^2} C_\sigma^2 \equiv \frac{C_1}{g_*^2}, \tag{3.13}$$

$$\|\widehat{\kappa}_\varepsilon\|_{L^\infty(\mathbb{R}^N)'} \leq \frac{C_1}{g_*^2}, \tag{3.14}$$

$$\|\widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)'} \leq \frac{C_1}{g_*}. \tag{3.15}$$

Indeed, (3.13) follows from (2.22) with the assumption (3.1), which implies (3.14), by definition of the dual norm, as well as (3.15), by using (3.13) and again (2.22):

$$\begin{aligned}\|\widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)'} &= \sup_{\substack{\boldsymbol{\beta} \in \boldsymbol{L}^\infty(\mathbb{R}^N) \\ \|\boldsymbol{\beta}\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)}=1}} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon \cdot \boldsymbol{\beta} \\ &\leq \left(\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon \right)^{\frac{1}{2}} \leq \frac{C_1}{g_*}.\end{aligned}$$

By the estimates (3.14), (3.15), and the Banach–Alaoglu–Bourbaki theorem, at least for some generalised subsequence $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ in $H_0^\sigma(\Omega)$ also

$$\widehat{\kappa}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \lambda \text{ weakly in } L^\infty(\mathbb{R}^N)' \quad \text{and} \quad \widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \Lambda \text{ weakly in } \boldsymbol{L}^\infty(\mathbb{R}^N)'.$$

Since $\widehat{\kappa}_\varepsilon \geq 0$ a.e., $\lambda \geq 0$ in $L^\infty(\mathbb{R}^N)'$, and letting $\varepsilon \to 0$ in (2.16) with $w \in \Upsilon_\infty^\sigma(\Omega)$, u and Λ satisfy

$$\begin{aligned}&\int_{\mathbb{R}^N} AD^\sigma u \cdot D^\sigma w + \langle\!\langle \Lambda, D^\sigma w \rangle\!\rangle \\ &\qquad = \int_\Omega f_\# w + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma w, \quad \forall w \in \Upsilon_\infty^\sigma(\Omega).\end{aligned} \tag{3.16}$$

Letting $\varepsilon \to 0$ in (2.16) with $v = u_\varepsilon$ and using (2.23), we easily find that

$$\overline{\lim_{\varepsilon\to 0}} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon|^2 \leq \langle\!\langle \Lambda, D^\sigma u \rangle\!\rangle.$$

Recalling that $(|D^\sigma u_\varepsilon|^2 - g^2)\widehat{\kappa}_\varepsilon \geq 0$ and $|D^\sigma u| \leq g$ a.e. $x \in \mathbb{R}^N$, we obtain

$$\big\langle \lambda, |D^\sigma u|^2 \big\rangle \leq \langle \lambda, g^2 \rangle = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon g^2 \leq \overline{\lim_{\varepsilon\to 0}} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon|^2 \leq \langle\!\langle \Lambda, D^\sigma u \rangle\!\rangle.$$

Since we get the opposite inequality from

$$
\begin{aligned}
0 &\le \overline{\lim_{\varepsilon\to 0}} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon \big|D^\sigma(u_\varepsilon - u)\big|^2 \\
&= \overline{\lim_{\varepsilon\to 0}} \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon|^2 - 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma u_\varepsilon \cdot D^\sigma u + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u|^2 \\
&\le \langle\!\langle \Lambda, D^\sigma u\rangle\!\rangle - 2\langle\!\langle \Lambda, D^\sigma u\rangle\!\rangle + \big\langle \lambda, |D^\sigma u|^2\big\rangle = -\langle\!\langle \Lambda, D^\sigma u\rangle\!\rangle + \big\langle \lambda, |D^\sigma u|^2\big\rangle,
\end{aligned}
$$

we conclude $\langle\!\langle \Lambda, D^\sigma u\rangle\!\rangle = \langle \lambda, |D^\sigma u|^2\rangle$ and

$$
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon \big|D^\sigma(u_\varepsilon - u)\big|^2 = 0. \tag{3.17}
$$

Hence, for any $\boldsymbol{\beta} \in \boldsymbol{L}^\infty(\mathbb{R}^N)$, we have

$$
\begin{aligned}
\big|\langle\!\langle \Lambda - \lambda D^\sigma u, \boldsymbol{\beta}\rangle\!\rangle\big| &= \lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma(u_\varepsilon - u)\cdot\boldsymbol{\beta}\right| \\
&\le \lim_{\varepsilon\to 0}\left[\left(\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon \big|D^\sigma(u_\varepsilon - u)\big|^2\right)^{\frac{1}{2}} \|\widehat{\kappa}_\varepsilon\|_{L^1(\mathbb{R}^N)} \|\boldsymbol{\beta}\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)}\right] = 0,
\end{aligned}
$$

showing that

$$
\Lambda = \lambda D^\sigma u \quad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^N)'
$$

and that, in fact, (3.16) is equivalent to (3.11).

It remains to show the last equation of (3.12) which follows easily from (recall (3.1))

$$
\begin{aligned}
0 &= \big\langle \lambda, (g^2 - |D^\sigma u|^2)\varphi\big\rangle = \big\langle \lambda, \big(g - |D^\sigma u|\big)\big(g + |D^\sigma u|\big)\varphi\big\rangle \\
&\ge g_* \big\langle \lambda, \big(g - |D^\sigma u|\big)\varphi\big\rangle = g_* \big\langle \lambda\big(g - |D^\sigma u|\big), \varphi\big\rangle \ge 0
\end{aligned}
$$

for arbitrarily $\varphi \in L^\infty(\Omega)$, $\varphi \ge 0$, which holds provided that we show

$$
\big\langle \lambda, \big(g^2 - |D^\sigma u|^2\big)\varphi\big\rangle = 0. \tag{3.18}
$$

As above, using (3.17), we have first

$$
\begin{aligned}
\langle \lambda, g^2\varphi\rangle &\le \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u_\varepsilon|^2\varphi \\
&= \lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon \big|D^\sigma(u_\varepsilon - u)\big|^2\varphi\right. \\
&\qquad \left. + 2\int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon D^\sigma(u_\varepsilon - u)\cdot D^\sigma u\varphi + \int_{\mathbb{R}^N} \widehat{\kappa}_\varepsilon |D^\sigma u|^2\varphi\right) \\
&= \big\langle \lambda, |D^\sigma u|^2\varphi\big\rangle
\end{aligned}
$$

and, since $u \in \mathbb{K}^\sigma_g$ and $\varphi, \lambda \geq 0$, it also holds that

$$\big\langle \lambda, \big(g^2 - |D^\sigma u|^2\big)\varphi \big\rangle \geq 0.$$

To show that u is the unique solution to (2.9), it suffices to take $w = u - v$, with an arbitrary $v \in \mathbb{K}^\sigma_g$, and observe that, by (3.18),

$$\begin{aligned} \langle\!\langle \lambda D^\sigma u, D^\sigma(v-u) \rangle\!\rangle &\leq \big\langle \lambda, |D^\sigma u|\big(|D^\sigma v| - |D^\sigma u|\big)\big\rangle \\ &\leq \big\langle \lambda, |D^\sigma u|\big(g - |D^\sigma u|\big)\big\rangle \\ &= \left\langle \lambda\big(g^2 - |D^\sigma u|^2\big), \frac{|D^\sigma u|}{g + |D^\sigma u|} \right\rangle = 0. \end{aligned}$$

(ii) In the second case, if (3.5) holds, we can use approximation by solutions (u_ν, λ_ν) of (3.11)-(3.12) corresponding to data $f_{\#\nu} \in L^{2^\#}(\Omega)$ and $\boldsymbol{f}_\nu \in \boldsymbol{L}^2(\mathbb{R}^N)$ satisfying (3.8), as in Theorem 3.1.

Using the estimate (3.6), it is clear that

$$u_\nu \underset{\nu\to 0}{\longrightarrow} u \quad \text{in } H^\sigma_0(\Omega) \tag{3.19}$$

and u solves (2.9).

For $\varphi \in L^\infty(\mathbb{R}^N)$, setting $b = \frac{\|\varphi\|_{L^\infty(\mathbb{R}^N)}}{g_*^2}$, recalling (3.1), and using (3.11) and (3.12) for λ_ν, which also implies that $\langle \lambda_\nu, g^2 - |D^\sigma u_\nu|^2 \rangle = 0$, we have

$$\begin{aligned} \big|\langle \lambda_\nu, \varphi \rangle\big| &\leq \langle \lambda_\nu, bg^2 \rangle \\ &= b\big\langle \lambda_\nu, |D^\sigma u_\nu|^2 \big\rangle = b\langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma u_\nu \rangle\!\rangle \\ &\leq b\left(\int_\Omega f_\# u_\nu + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma u_\nu \right) \leq C\,\frac{\|\varphi\|_{L^\infty(\mathbb{R}^N)}}{g_*^2}, \end{aligned} \tag{3.20}$$

where the constant $C > 0$ depends only on the L^1-norms of $f_\#$ and $\boldsymbol{f}$ and on the constants a_p and b_1 of (3.6), being consequently independent of ν. Then, λ_ν is uniformly bounded in $L^\infty(\mathbb{R}^N)'$ and we may assume, for some generalised subsequence,

$$\lambda_\nu \underset{\nu\to 0}{\rightharpoonup} \lambda \text{ in } L^\infty(\mathbb{R}^N)'\text{-weakly}^*, \quad \text{with } \lambda \geq 0, \tag{3.21}$$

and, since $\Lambda_\nu = \lambda_\nu D^\sigma u_\nu$ is also bounded in $\boldsymbol{L}^\infty(\mathbb{R}^N)'$ (recall $\|D^\sigma u_\nu\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)} \leq g^*$), also

$$\Lambda_\nu \underset{\nu\to 0}{\rightharpoonup} \Lambda \quad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^N)'\text{-weakly}^*. \tag{3.22}$$

Therefore, taking the limit $\nu \to 0$ in (3.11), we find that (u, λ) solves

$$\begin{aligned} &\int_{\mathbb{R}^N} A D^\sigma u \cdot D^\sigma w + \langle\!\langle \Lambda, D^\sigma w \rangle\!\rangle \\ &\qquad = \int_\Omega f_\# w + \int_{\mathbb{R}^N} \boldsymbol{f} \cdot D^\sigma w, \quad \forall w \in \Upsilon^\sigma_\infty(\Omega). \end{aligned} \tag{3.23}$$

Recalling (3.18) with $\varphi = 1$, we have

$$\langle \lambda_\nu, |D^\sigma u|^2 \rangle \le \langle \lambda_\nu, g^2 \rangle = \langle \lambda_\nu, |D^\sigma u_\nu|^2 \rangle. \tag{3.24}$$

Using the equalities (3.24) and (3.19), we have

$$\begin{aligned}
0 &\le \frac{1}{2}\langle \lambda_\nu, \left|D^\sigma(u_\nu - u)\right|^2 \rangle \\
&= \frac{1}{2}\left(\langle \lambda_\nu, |D^\sigma u_\nu|^2 \rangle - 2\langle \lambda_\nu, D^\sigma u_\nu \cdot D^\sigma u \rangle + \langle \lambda_\nu, |D^\sigma u|^2 \rangle\right) \\
&\le \langle \lambda_\nu, |D^\sigma u_\nu|^2 \rangle - \langle \lambda_\nu, D^\sigma u_\nu \cdot D^\sigma u \rangle = \langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma (u_\nu - u) \rangle\!\rangle \\
&= \int_\Omega f_{\#\nu}(u_\nu - u) + \int_{\mathbb{R}^N} \boldsymbol{f}_\nu \cdot D^\sigma(u_\nu - u) \\
&\quad - \int_{\mathbb{R}^N} A D^\sigma u_\nu \cdot D^\sigma(u_\nu - u) \xrightarrow[\nu \to 0]{} 0,
\end{aligned} \tag{3.25}$$

being the last equality satisfied because (u_ν, λ_ν) solves problem (3.11)-(3.12) with data $f_{\#\nu}$ and $\boldsymbol{f}_\nu$.

Then, from (3.23) we can conclude that u in fact solves (3.11) from the equality

$$\begin{aligned}
\langle\!\langle \Lambda, D^\sigma w \rangle\!\rangle &= \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma w \rangle\!\rangle \\
&= \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma u, D^\sigma w \rangle\!\rangle + \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma (u_\nu - u), D^\sigma w \rangle\!\rangle \\
&= \lim_{\nu \to 0} \langle \lambda_\nu, D^\sigma u \cdot D^\sigma w \rangle = \langle \lambda, D^\sigma u \cdot D^\sigma w \rangle = \langle\!\langle \lambda D^\sigma u, D^\sigma w \rangle\!\rangle,
\end{aligned} \tag{3.26}$$

which is valid for all $w \in \Upsilon^\sigma_\infty(\Omega)$ since (3.25) implies that

$$\begin{aligned}
\left|\langle\!\langle \lambda_\nu D^\sigma (u_\nu - u), D^\sigma w \rangle\!\rangle\right| &= \left|\langle \lambda_\nu, D^\sigma (u_\nu - u) \cdot D^\sigma w \rangle\right| \\
&\le \langle \lambda_\nu, \left|D^\sigma (u_\nu - u)\right| |D^\sigma w| \rangle \\
&\le \left(\langle \lambda_\nu, \left|D^\sigma (u_\nu - u)\right|^2 \rangle\right)^{\frac{1}{2}} \left(\langle \lambda_\nu, |D^\sigma w|^2 \rangle\right)^{\frac{1}{2}} \xrightarrow[\nu \to 0]{} 0,
\end{aligned}$$

where we have used the Hölder inequality for charges in the last inequality.

From (3.26), we find $\langle\!\langle \Lambda, D^\sigma u \rangle\!\rangle = \langle \lambda, |D^\sigma u|^2 \rangle$ and

$$\begin{aligned}
\langle \lambda, g^2 \rangle &= \lim_{\nu \to 0} \langle \lambda_\nu, g^2 \rangle = \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma u_\nu \rangle\!\rangle \\
&= \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma u \rangle\!\rangle + \lim_{\nu \to 0} \langle\!\langle \lambda_\nu D^\sigma u_\nu, D^\sigma (u_\nu - u) \rangle\!\rangle \\
&= \lim_{\nu \to 0} \langle\!\langle \Lambda_\nu, D^\sigma u \rangle\!\rangle = \langle\!\langle \Lambda, D^\sigma u \rangle\!\rangle = \langle \lambda, |D^\sigma u|^2 \rangle.
\end{aligned}$$

Finally, we can now complete the proof of the theorem by using this equality in the form $\langle \lambda(g^2 - |D^\sigma u|^2), 1 \rangle = 0$ and again the Hölder inequality to conclude the third condition in (3.12) with an arbitrarily $\varphi \in L^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\big|\langle\lambda(g-|D^\sigma u|),\varphi\rangle\big| &\le \langle\lambda(g-|D^\sigma u|),|\varphi|\rangle\\ &= \left\langle\lambda(g^2-|D^\sigma u|^2),\frac{|\varphi|}{g+|D^\sigma u|}\right\rangle\\ &\le \langle\lambda(g^2-|D^\sigma u|^2),1\rangle^{\frac12}\left\langle\lambda(g^2-|D^\sigma u|^2),\frac{|\varphi|^2}{(g+|D^\sigma u|)^2}\right\rangle^{\frac12}\\ &= 0.\end{aligned}$$

∎

The second part of this proof actually shows a generalised continuous dependence of the solution and of the Lagrange multiplier with respect to the L^1-data.

Corollary. *Under the assumptions (2.1), (2.2), (3.1), and (3.5), if $(u_\nu,\lambda_\nu)\in\Upsilon^\sigma_\infty(\Omega)\times L^\infty(\mathbb{R}^N)'$ are the solutions to (3.11) and (3.12) corresponding to L^1-data satisfying (3.8), as $\nu\to 0$, we have the convergence, for some generalised subsequence or net,*

$$u_\nu \underset{\nu\to0}{\longrightarrow} u \text{ in } H^\sigma_0(\Omega) \quad\text{and}\quad \lambda_\nu \underset{\nu\to0}{\rightharpoonup} \lambda \text{ in } L^\infty(\mathbb{R}^N)'\text{-weakly}^*,$$

where $(u,\lambda)\in\Upsilon^\sigma_\infty(\Omega)\times L^\infty(\mathbb{R}^N)'$ also solves (3.11)-(3.12).

4. Convergence to the local problem as $\sigma \nearrow 1$

It is easy to check that all the theorems of the preceding two sections hold in the limit case $\sigma=1$, when $D^\sigma = D$ is the classical gradient and the data $f_\#$ and $\boldsymbol{f}$ satisfy (2.3) (with $f_\#\in L^{\frac{2N}{N+2}}(\Omega)$, if $N>2$, $f_\#\in L^q(\Omega)$, $\forall q<\infty$ if $N=2$ and $q=\infty$ if $N=1$) or (3.5), and g satisfies (2.8), (2.17) or (3.1), respectively.

In this section, we show a continuous dependence of the solution u^σ and of the Lagrange multiplier λ^σ when $\sigma\nearrow 1$. For the sake of simplicity, we take $f_\#=0$ and $\boldsymbol{f}\in \boldsymbol{L}^1(\mathbb{R}^N)$, so that the limit variational inequality reads

$$u\in\mathbb{K}_g=\{v\in H^1_0(\Omega): |Dv|\le g \text{ a.e. in } \Omega\}, \tag{4.1}$$

$$\int_\Omega ADu\cdot D(v-u)\ge\int_\Omega \boldsymbol{f}\cdot D(v-u),\quad \forall v\in\mathbb{K}_g. \tag{4.2}$$

Likewise, observing that setting $\sigma=1$ in (3.2) we have $\Upsilon_\infty(\Omega)=W^{1,\infty}_0(\Omega)$, we can write the limit Lagrange multiplier problem in the following form: find $(u,\lambda)\in W^{1,\infty}_0(\Omega)\times L^\infty(\Omega)'$

$$\int_\Omega ADu\cdot Dw+\langle\!\langle\lambda Du,Dw\rangle\!\rangle=\int_\Omega \boldsymbol{f}\cdot Dw,\quad\forall w\in W^{1,\infty}_0(\Omega), \tag{4.3}$$

$$|Du|\le g \text{ a.e. in }\Omega,\quad \lambda\ge0\quad\text{and}\quad \lambda(|Du|-g)=0 \text{ in } L^\infty(\Omega)'. \tag{4.4}$$

In (4.3), we denote the duality in $\boldsymbol{L}^\infty(\Omega)$ similarly to (3.10), as we can always consider the solution and the test functions extended by zero in $\mathbb{R}^N \setminus \Omega$, since $\partial\Omega$ is $\mathcal{C}^{0,1}$.

We first recall an important consequence of the fact that the Riesz kernel is an approximation of the identity, as remarked by Kurokawa in [7].

Proposition 4.1. *If $h \in L^p(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$, for some $p \geq 1$, is bounded and uniformly continuous in $\mathbb{R}^N$, then*

$$\lim_{\alpha \to 0} \|I_\alpha * h - h\|_{L^\infty(\mathbb{R}^N)} = 0.$$

As a consequence, we have

$$D^\sigma w \xrightarrow[\sigma \nearrow 1]{} Dw \quad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^N), \text{ for all } w \in \mathcal{C}^1_c(\mathbb{R}^N). \tag{4.5}$$

Proof. In [7, Proposition 2.10], it is proved that

$$I_\alpha * h(x) \xrightarrow[\alpha \to 0]{} h(x)$$

at each point of continuity of any function $h \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and it is not difficult to check that this convergence is uniform in $x \in \mathbb{R}^N$ for bounded and uniformly continuous functions (see [2]). Then, (4.5) is an immediate consequence of Theorem 1.2 of [12], which established that $D^s w = I_{1-s} * Dw$ for all $w \in \mathcal{C}^\infty_c(\mathbb{R}^N)$, being the proof equally valid for functions only in $\mathcal{C}^1_c(\mathbb{R}^N)$. ∎

Remark 4.2. The convergence (4.5), as well as in $\boldsymbol{L}^p(\mathbb{R}^N)$ for $p \geq 1$, has been shown in [6, Proposition 4.4] for functions of $\mathcal{C}^2_c(\mathbb{R}^N)$. By density of $\mathcal{C}^\infty_c(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ for $p \geq 1$, in [4] it was shown that the convergence $D^\sigma h \xrightarrow[\sigma \nearrow 1]{} Dh$ holds in $L^p(\mathbb{R}^N)$, for $1 < p < \infty$, if $h \in W^{1,p}(\mathbb{R}^N)$.

For $\chi \in L^\infty(\mathbb{R}^N)'$, we denote its restriction to $\Omega \subset \mathbb{R}^N$ by $\chi_\Omega \in L^\infty(\Omega)'$, defined by

$$\langle \chi_\Omega, \varphi \rangle = \langle \chi, \widetilde{\varphi} \rangle, \quad \forall \varphi \in L^\infty(\Omega),$$

where $\widetilde{\varphi}$ is the extension of φ by zero to $\mathbb{R}^N \setminus \Omega$,

Theorem 4.3. *Let $\boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{R}^N)$ ($f_\# = 0$) and let g be given as in (3.1). Then, if $(u^\sigma, \lambda^\sigma) \in \Upsilon^\sigma_\infty(\Omega) \times L^\infty(\mathbb{R}^N)'$ are the solutions to (3.11)-(3.12), we have, for a generalised subsequence, the convergences, for any s, $0 < s < \sigma < 1$:*

$$u^\sigma \xrightarrow[\sigma \nearrow 1]{} u \text{ in } H^s_0(\Omega) \quad \text{and} \quad \lambda^\sigma_\Omega \xrightarrow[\sigma \nearrow 1]{} \lambda \text{ in } L^\infty(\Omega)'\text{-weakly}^*, \tag{4.6}$$

where $(u, \lambda) \in W^{1,\infty}_0(\Omega) \times L^\infty(\Omega)'$ is a solution to (4.3)-(4.4) and u is the unique solution to (4.1)-(4.2).

Proof. Setting $v = 0$ in (2.9), or $w = u^\sigma$ in (3.11), we immediately obtain

$$\|u^\sigma\|_{H_0^\sigma(\Omega)} = \|D^\sigma u^\sigma\|_{\boldsymbol{L}^2(\mathbb{R}^N)} \leq \left(\frac{g^*}{a_*}\|\boldsymbol{f}\|_{L^1(\mathbb{R}^N)}\right)^{\frac{1}{2}} \equiv C_1, \tag{4.7}$$

where C_1 is independent of σ, $0 < \sigma < 1$. Hence, arguing as in (3.20), using (3.11)-(3.12), it also follows easily that

$$\|\lambda^\sigma\|_{L^\infty(\mathbb{R}^N)'} = \sup_{\substack{\varphi \in L^\infty(\mathbb{R}^N) \\ \|\varphi\|_{L^\infty(\mathbb{R}^N)}=1}} \langle \lambda^\sigma, \varphi\rangle \leq \frac{\|\boldsymbol{f}\|_{\boldsymbol{L}^1(\mathbb{R}^N)}}{g_*^2}. \tag{4.8}$$

Then, using $\Lambda^\sigma = \lambda^\sigma D^\sigma u^\sigma$ and recalling $\|D^\sigma u^\sigma\|_{\boldsymbol{L}^\infty(\mathbb{R}^N)} \leq g^*$, from the estimates (4.7) and (4.8), we may take a generalised subsequence $\sigma \nearrow 1$ such that, by the compactness of $H_0^\sigma(\Omega) \hookrightarrow H_0^s(\Omega)$, $0 < s < \sigma \leq 1$,

$$\begin{cases} u^\sigma \xrightarrow[\sigma\nearrow 1]{} u & \text{in } H_0^s(\Omega), \\ D^\sigma u^\sigma \underset{\sigma\nearrow 1}{\rightharpoonup} \boldsymbol{\chi} & \text{in } \boldsymbol{L}^2(\mathbb{R}^N)'\text{-weak and } \boldsymbol{L}^\infty(\mathbb{R}^N)'\text{-weak}^*, \end{cases} \tag{4.9}$$

$$\lambda^\sigma \underset{\sigma\nearrow 1}{\rightharpoonup} \widetilde{\lambda} \text{ in } L^\infty(\mathbb{R}^N)'\text{-weak}^*, \quad \Lambda^\sigma \underset{\sigma\nearrow 1}{\rightharpoonup} \widetilde{\Lambda} \text{ in } \boldsymbol{L}^\infty(\mathbb{R}^N)'\text{-weak}. \tag{4.10}$$

Denoting by $\tilde{u}^\sigma$ the extension of u^σ by zero to $\mathbb{R}^N \setminus \Omega$, from (4.9) we conclude that $\boldsymbol{\chi} = D\tilde{u}$ and in fact $u \in H_0^1(\Omega)$, and then $D\tilde{u} = \widetilde{Du}$. Indeed, recalling the convergence (4.5), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \boldsymbol{\chi}\cdot\boldsymbol{\varphi} &= \lim_{\sigma\nearrow 1}\int_{\mathbb{R}^N} D^\sigma u^\sigma \cdot \boldsymbol{\varphi} = -\lim_{\sigma\nearrow 1}\int_{\mathbb{R}^N} \tilde{u}^\sigma (D^\sigma\cdot\boldsymbol{\varphi}) \\ &= -\int_{\mathbb{R}^N} \tilde{u}(D\cdot\boldsymbol{\varphi}) = \int_{\mathbb{R}^N} D\tilde{u}\cdot\boldsymbol{\varphi}, \end{aligned}$$

with an arbitrary $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^N)$.

On the other hand, given any measurable set $\omega \subset \Omega$, we have now

$$\int_\omega |Du|^2 \leq \varliminf_{\sigma\nearrow 1}\int_\omega |D^\sigma u^\sigma|^2 \leq \int_\omega g^2$$

and therefore $|Du| \leq g$ a.e. in Ω, which yields $u \in \mathbb{K}_g \subset W_0^{1,\infty}(\Omega)$.

Passing to the limit $\sigma \nearrow 1$ in (3.11), first with $w \in \mathcal{C}_c^\infty(\Omega)$

$$\int_{\mathbb{R}^N} AD^\sigma u^\sigma \cdot D^\sigma w + \langle\!\langle \Lambda^\sigma, D^\sigma w\rangle\!\rangle = \int_{\mathbb{R}^N} \boldsymbol{f}\cdot D^\sigma w$$

and using (4.5), (4.9), and (4.10), since $\boldsymbol{\chi} = \widetilde{Du}$ and $D\tilde{w} = \widetilde{Dw}$, we obtain

$$\int_\Omega ADu \cdot Dw + \langle\!\langle \Lambda, Dw \rangle\!\rangle = \int_\Omega \boldsymbol{f} \cdot Dw, \tag{4.11}$$

by setting $\Lambda = \widetilde{\Lambda}_\Omega$ and $\langle\!\langle \Lambda, Dw \rangle\!\rangle = \langle\!\langle \widetilde{\Lambda}, D\tilde{w} \rangle\!\rangle$.

Note that for each $w \in W_0^{1,\infty}(\Omega)$ we may choose $w_\nu \in \mathcal{C}_c^\infty(\Omega)$ such that $w_\nu \xrightarrow[\nu\to\infty]{} w$ in $H_0^1(\Omega)$ and $Dw_\nu \underset{\nu\to\infty}{\rightharpoonup} Dw$ in $\boldsymbol{L}^\infty(\Omega)$-weak* in (4.11) and we may pass to the generalised limit $\nu \to \infty$, concluding that (4.11) also holds for all $w \in W_0^{1,\infty}(\Omega)$. So, in order to see that u and $\lambda = \tilde{\lambda}_{|\Omega}$, i.e., the restriction to Ω of the limit charge $\tilde{\lambda}$ in (4.10), solve (4.3), we need to show that

$$\langle\!\langle \Lambda, Dw \rangle\!\rangle = \langle\!\langle \lambda Du, Dw \rangle\!\rangle = \langle \lambda, Du \cdot Dw \rangle, \quad \forall w \in W_0^{1,\infty}(\Omega). \tag{4.12}$$

We show first (4.12) for $w = u$, i.e., $\langle\!\langle \Lambda, Du \rangle\!\rangle = \langle \lambda, |Du|^2 \rangle$, in two steps.

Observing that $\tilde{\lambda} \geq 0$ and $|Du| \leq g$, we have $\langle \lambda, |Du|^2 \rangle \leq \langle\!\langle \Lambda, Du \rangle\!\rangle$ from

$$\begin{aligned} \left\langle \lambda, |Du|^2 \right\rangle &\leq \langle \tilde{\lambda}, g^2 \rangle = \lim_{\sigma \nearrow 1} \langle \lambda^\sigma, g^2 \rangle = \lim_{\sigma \nearrow 1} \left\langle \lambda^\sigma, |D^\sigma u^\sigma|^2 \right\rangle \\ &= \lim_{\sigma \nearrow 1} \langle\!\langle \lambda^\sigma D^\sigma u^\sigma, D^\sigma u^\sigma \rangle\!\rangle \\ &= \overline{\lim_{\sigma \nearrow 1}} \int_{\mathbb{R}^N} (\boldsymbol{f} - AD^\sigma u^\sigma) \cdot D^\sigma u^\sigma \\ &\leq \int_{\mathbb{R}^N} (\boldsymbol{f} - AD\tilde{u}) \cdot D\tilde{u} = \langle\!\langle \widetilde{\Lambda}, D\tilde{u} \rangle\!\rangle = \langle\!\langle \Lambda, Du \rangle\!\rangle. \end{aligned} \tag{4.13}$$

Note that $D^\sigma u^\sigma \underset{\sigma\nearrow 1}{\rightharpoonup} D\tilde{u}$ in $\boldsymbol{L}^2(\mathbb{R}^N)$-weak and hence

$$\underline{\lim_{\sigma \nearrow 1}} \int_{\mathbb{R}^N} AD^\sigma u^\sigma \cdot D^\sigma u^\sigma \geq \int_{\mathbb{R}^N} AD\tilde{u} \cdot D\tilde{u} = \int_\Omega ADu \cdot Du.$$

On the other hand, we find $\langle\!\langle \Lambda, Du \rangle\!\rangle \leq \langle \lambda, |Du|^2 \rangle$ by noting that $\Lambda^\sigma = \lambda^\sigma D^\sigma u^\sigma$ and, similarly,

$$0 \leq \left\langle \lambda^\sigma, |D^\sigma u^\sigma - D\tilde{u}|^2 \right\rangle = \langle\!\langle \lambda^\sigma D^\sigma u^\sigma, D^\sigma u^\sigma \rangle\!\rangle - 2\langle\!\langle \Lambda^\sigma, D\tilde{u} \rangle\!\rangle + \left\langle \lambda^\sigma, |D\tilde{u}|^2 \right\rangle \tag{4.14}$$

yields

$$\begin{aligned} 2\langle\!\langle \widetilde{\Lambda}, D\tilde{u} \rangle\!\rangle = 2 \lim_{\sigma \nearrow 1} \langle\!\langle \Lambda^\sigma, D\tilde{u} \rangle\!\rangle &\leq \overline{\lim_{\sigma \nearrow 1}} \int_{\mathbb{R}^N} (\boldsymbol{f} - AD^\sigma u^\sigma) \cdot D^\sigma u^\sigma + \lim_{\sigma \nearrow 1} \left\langle \lambda^\sigma, |D\tilde{u}|^2 \right\rangle \\ &\leq \int_{\mathbb{R}^N} (\boldsymbol{f} - AD\tilde{u}) \cdot D\tilde{u} + \left\langle \lambda, |Du|^2 \right\rangle = \langle\!\langle \widetilde{\Lambda}, D\tilde{u} \rangle\!\rangle + \left\langle \lambda, |Du|^2 \right\rangle. \end{aligned}$$

As a consequence of $\langle\!\langle \Lambda, Du \rangle\!\rangle = \langle \lambda, |Du|^2 \rangle$, from (4.14) we deduce

$$\lim_{\sigma \nearrow 1} \langle \lambda^\sigma, |D^\sigma u^\sigma - D\tilde{u}|^2 \rangle = 0, \tag{4.15}$$

which by the Hölder inequality yields, for any $\boldsymbol{\beta} \in \boldsymbol{L}^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \left| \langle\!\langle \widetilde{\Lambda} - \widetilde{\lambda} D\tilde{u}, \boldsymbol{\beta} \rangle\!\rangle \right| &= \lim_{\sigma \nearrow 1} \left| \langle\!\langle \Lambda^\sigma - \lambda^\sigma D\tilde{u}, \boldsymbol{\beta} \rangle\!\rangle \right| = \lim_{\sigma \nearrow 1} \left| \langle\!\langle \lambda^\sigma (D^\sigma u^\sigma - D\tilde{u}), \boldsymbol{\beta} \rangle\!\rangle \right| \\ &\le \lim_{\sigma \nearrow 1} \langle \lambda^\sigma, |D^\sigma u^\sigma - D\tilde{u}| \, |\boldsymbol{\beta}| \rangle \\ &\le \lim_{\sigma \nearrow 1} \langle \lambda^\sigma, |D^\sigma u^\sigma - D\tilde{u}|^2 \rangle^{\frac{1}{2}} \langle \lambda^\sigma, |\boldsymbol{\beta}|^2 \rangle^{\frac{1}{2}} = 0, \end{aligned}$$

and, consequently, (4.12) follows from

$$\Lambda = \lambda Du \quad \text{in } \boldsymbol{L}^\infty(\Omega)'.$$

This equality in (4.12) with $g > 0$ implies that

$$\langle \lambda, |Du|^2 \rangle = \langle \widetilde{\lambda}, g^2 \rangle \ge \langle \lambda, g^2_{|\Omega} \rangle \ge \langle \lambda, |Du|^2 \rangle,$$

and $\langle \lambda, |Du|^2 - g^2 \rangle = 0$ (here $g = g_{|\Omega}$). Then, exactly the same argument as at the end of the proof of Theorem 3.4 shows that λ and u satisfy the third condition of (4.4).

Finally, since we also have

$$\langle\!\langle \lambda Du, D(v - u) \rangle\!\rangle \le 0, \quad \forall v \in \mathbb{K}_g,$$

(4.3) implies (4.2) and this concludes the proof of the theorem. ■

Remark 4.4. In the Hilbertian case of $g \in L^2(\Omega)$, $g \ge 0$, and $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^N)$, it is easy to show the convergence of the solutions $(u^\sigma, \gamma^\sigma) \in \Upsilon^\sigma_\infty(\Omega) \times H^\sigma_0(\Omega)$ given by Theorem 2.1, also in the case $f_\# = 0$ to simplify, as $\sigma \nearrow 1$ to the local problem for $(u, \gamma) \in W^{1,\infty}_0(\Omega) \times H^1_0(\Omega)$, satisfying (2.11) with $\sigma = 1$ and

$$\int_\Omega (ADu + D\sigma) \cdot Dv = \int_\Omega \boldsymbol{f} \cdot Dv, \quad \forall v \in H^1_0(\Omega). \tag{4.16}$$

Indeed, as in (2.10) and (2.13), the a priori estimates

$$\|u^\sigma\|_{H^\sigma_0(\Omega)} \le \frac{1}{a_*} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^N)} \quad \text{and} \quad \|\gamma^\sigma\|_{H^\sigma_0(\Omega)} \le \left(1 + \frac{a^*}{a_*}\right) \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^N)}$$

allow us to take sequences

$$u^\sigma \xrightarrow[\sigma \nearrow 1]{} u \text{ and } \gamma^\sigma \xrightarrow[\sigma \nearrow 1]{} \gamma \text{ in } H^s_0(\Omega), \quad 0 < s < 1,$$

in (2.12) with $v \in H_0^1(\Omega) \subset H_0^\sigma(\Omega)$, in order to obtain (4.16) and, using (2.18), the $\Gamma = \Gamma(u) \in H^{-\sigma}(\Omega)$ corresponding to γ satisfies (2.11) with $\sigma = 1$.

Funding. The research of José-Francisco Rodrigues was partially done under the framework of the Project PTDC/MATPUR/28686/2017 at CMAFcIO/ULisboa. The research of Assis Azevedo and Lisa Santos was partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Projects UIDB/00013/2020 and UIDP/00013/2020.

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