



# Positive harmonic functions on the Heisenberg group I

Yves Benoist

**Abstract.** We present the classification of positive harmonic functions on the Heisenberg group in the case of the southwest measure.

## 1. Introduction

In this self-contained paper, we present the classification of the positive harmonic functions on the Heisenberg group  $H_3(\mathbb{Z})$  in the special case of the *southwest measure*. This example is striking because the famous partition functions occur as positive harmonic functions. In this case, our main result tells us that roughly all positive harmonic functions are combinations of characters and partition functions (Theorem 1.1).

We will also explain with no proof how this result can be extended to finite positive measures on  $H_3(\mathbb{Z})$  (Theorem 3.8). The proof of this extension can be found in [2].

### 1.1. The partition function $p(x, y, z)$ as a potential

We first introduce the “partition function”  $p(x, y, z)$  for any integers  $x, y, z$  in  $\mathbb{Z}$ .

**1.1.1. The partition function.** This function counts the “number of Young diagrams of area  $z$ ,” also called “partitions of  $z$ ,” included in a rectangle with side lengths  $x$  and  $y$  (see Figure 1). More precisely, when  $x, y$ , and  $z$  are non-negative, one has that

$$p(x, y, z) = \left| \left\{ (n_1, \dots, n_y) \in \mathbb{Z}^y \mid x \geq n_1 \geq \dots \geq n_y \geq 0 \text{ and } n_1 + \dots + n_y = z \right\} \right|, \quad (1.1)$$

and  $p(x, y, z) = 0$  otherwise. The integers  $n_i$  are the lengths of the rows of the partition. By convention, for  $x \geq 0$ , one has that  $p(x, 0, z) = 0$  when  $z \neq 0$ , and

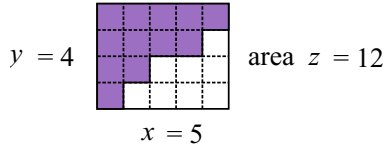


Figure 1. The partition  $12 = 5 + 4 + 2 + 1$  is included in a  $5 \times 4$  rectangle.

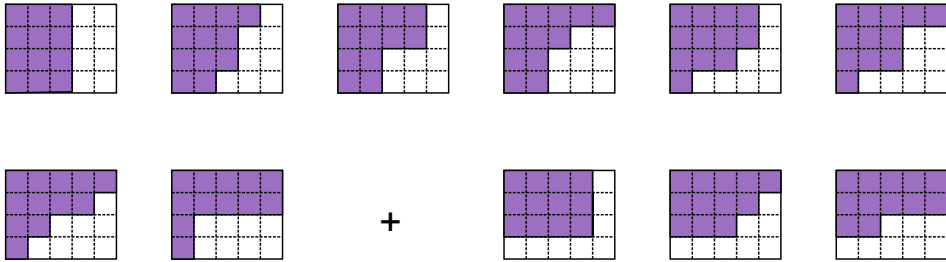


Figure 2. The 11 partitions in the equality  $p(5, 4, 12) = p(4, 4, 8) + p(5, 3, 12)$ .

that  $p(x, 0, 0) = 1$ . This partition function satisfies the functional equation, for all  $g = (x, y, z)$  in  $\mathbb{Z}^3$ ,  $g \neq (0, 0, 0)$ ,

$$p(x, y, z) = p(x - 1, y, z - y) + p(x, y - 1, z). \tag{1.2}$$

One checks it by splitting this set of partitions according to the color of the lower-left case of the rectangle as in Figure 2.

**1.1.2. The Heisenberg group.** Recall that the Heisenberg group  $G := H_3(\mathbb{Z})$  is the set  $\mathbb{Z}^3$  of triples seen as matrices

$$(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It is endowed with the product

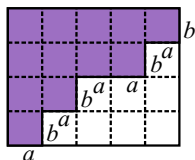
$$(x_0, y_0, z_0)(x, y, z) = (x_0 + x, y_0 + y, z_0 + z + x_0y). \tag{1.3}$$

Let  $\mu_0$  be the southwest measure on  $G$ . It is given by

$$\mu_0 = \delta_{a^{-1}} + \delta_{b^{-1}}, \quad \text{where } a := (1, 0, 0) \text{ and } b = (0, 1, 0). \tag{1.4}$$

Let  $e := (0, 0, 0)$  be the unity of  $G$  and let  $\mathbf{1}_{\{e\}}$  be the characteristic function of  $\{e\}$ . Equation (1.2) can be rewritten as, for all  $g = (x, y, z)$  in  $G$ ,

$$p(g) = p(a^{-1}g) + p(b^{-1}g) + \mathbf{1}_{\{e\}}(g). \tag{1.5}$$



**Figure 3.** The partition  $12 = 5 + 4 + 2 + 1$  associated to the word  $w = ababaabab$  gives the element  $g = g_w = ababaabab = (5, 4, 12) \in H_3(\mathbb{Z})$ .

In particular, the function  $f = p$  satisfies

$$f(g) \geq P_{\mu_0} f(g), \quad \text{where } P_{\mu_0} f(g) := f(a^{-1}g) + f(b^{-1}g). \quad (1.6)$$

This inequality (1.6) tells us that the function  $f$  is a  $\mu_0$ -superharmonic function on the Heisenberg group  $G$ .

**1.1.3. The potential.** More precisely, *the partition function  $p(g)$  is the potential of  $\mu_0$  at  $e$ .* This means that one has the equality

$$p = \sum_{n \geq 0} P_{\mu_0}^n \mathbf{1}_{\{e\}}.$$

Indeed, as can be seen in Figure 3, for  $g$  in  $G$ ,

$$p(g) \text{ is the number of ways to write } g \text{ as a word in } a \text{ and } b. \quad (1.7)$$

A function  $h$  on  $G$  is said to be  $\mu_0$ -harmonic if it satisfies

$$h(g) = P_{\mu_0} h(g), \quad \text{for all } g \text{ in } G, \text{ or equivalently} \quad (1.8)$$

$$h(x, y, z) = h(x - 1, y, z - y) + h(x, y - 1, z), \quad \text{for all } (x, y, z) \text{ in } \mathbb{Z}^3. \quad (1.9)$$

## 1.2. Construction of positive harmonic functions

We want to classify all the positive<sup>1</sup> solutions of (1.6), i.e., all the positive  $\mu_0$ -superharmonic functions  $h$  on  $G$ . We begin with five remarks.

**1.2.1. Choquet theorem.** By a theorem of Choquet in [5], every positive superharmonic function  $h$  is an average of extremal<sup>2</sup> positive superharmonic functions  $h_\alpha$ . Moreover, when  $h$  is harmonic, the  $h_\alpha$  are harmonic. By Riesz decomposition theorem [13, Thm. 2.1.4], every positive  $\mu_0$ -superharmonic function can be written in a unique way as the sum of a potential<sup>3</sup> and a positive  $\mu_0$ -harmonic function. Therefore, it is enough to describe the extremal positive  $\mu_0$ -harmonic functions on  $G$ .

<sup>1</sup>A function  $f$  on  $G$  is said to be positive if  $f(g) \geq 0$  for all  $g$  in  $G$  and  $f \neq 0$ .

<sup>2</sup>A positive (super)harmonic function is said to be extremal if it cannot be written as the sum of two non-proportional positive (super)harmonic functions.

<sup>3</sup>A potential is a function of the form  $f = \sum_{n \geq 0} P_{\mu_0}^n F$  for a positive function  $F$  on  $G$ .

			↑ $z$							
0	<b>0</b>	1	5	10	15	18	20	21	22	
0	<b>0</b>	1	4	8	11	13	14	15	15	
0	<b>0</b>	1	4	7	9	10	11	11	11	
0	<b>0</b>	1	3	5	6	7	7	7	7	
0	<b>0</b>	1	3	4	5	5	5	5	5	
0	<b>0</b>	1	2	3	3	3	3	3	3	
0	<b>0</b>	1	2	2	2	2	2	2	2	
0	<b>0</b>	1	1	1	1	1	1	1	1	
<b>0</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	→ $y$
0	<b>0</b>	0	0	0	0	0	0	0	0	0

**Figure 4.** The function  $p_y(z)$  satisfies  $p_y(z) = p_y(z - y) + p_{y-1}(z)$ .

**1.2.2. Choquet–Deny theorem.** If we look for a  $\mu_0$ -harmonic function  $h$  which does not depend on  $z$ , then (1.9) becomes

$$h(x, y) = h(x - 1, y) + h(x, y - 1), \quad \text{for all } (x, y) \text{ in } \mathbb{Z}^2. \tag{1.10}$$

This equation tells us that the function  $h$  is  $\mu_0$ -harmonic on the abelian quotient  $\mathbb{Z}^2$  of  $G$ . According to a theorem of Choquet and Deny in [6], since the support of the measure  $\mu_0$  spans the group  $\mathbb{Z}^2$ , every extremal positive  $\mu_0$ -harmonic function on this abelian group is proportional to a character<sup>4</sup>:

$$\chi(x, y, z) = r^x s^y \quad \text{with } r, s > 0 \text{ and } \frac{1}{r} + \frac{1}{s} = 1. \tag{1.11}$$

**1.2.3. The partition function as a harmonic function.** If we look for a  $\mu_0$ -harmonic function  $h$  which does not depend on  $x$ , then (1.9) becomes

$$h(y, z) = h(y, z - y) + h(y - 1, z), \quad \text{for all } (y, z) \text{ in } \mathbb{Z}^2. \tag{1.12}$$

A nice example is given in Figure 4 by the partition function  $(y, z) \mapsto p_y(z)$ , where

$$\begin{aligned} p_y(z) &= \sup_{x \in \mathbb{Z}} p(x, y, z) = \lim_{x \rightarrow \infty} p(x, y, z) = p(z, y, z) \\ &= \text{the number of partitions of } z \text{ with at most } y \text{ rows.} \end{aligned} \tag{1.13}$$

Hence the function  $h_0(x, y, z) := p_y(z)$  is a  $\mu_0$ -harmonic function on  $G$ .

---

<sup>4</sup>The proof is very short. One notices that equality (1.10) gives a decomposition of  $h$  as a sum of two positive harmonic functions and hence both of them are proportional to  $h$ .

**1.2.4. Margulis first theorem.** According to the first theorem of Margulis, a theorem he proved in [10] when he was not yet twenty, the Choquet–Deny theorem is still true on a finitely generated nilpotent group  $G$  as soon as the support of the measure spans  $G$  as a semigroup (see Fact 3.7). This is why it might look surprising at first glance that there exists a positive  $\mu_0$ -harmonic function  $h_0$  on  $H_3(\mathbb{Z})$  which is not invariant by the center. The reason it exists is that the support of  $\mu_0$  spans  $G$  as a group but not as a semigroup. What is more surprising is that this “new” positive harmonic function  $h_0$  is given by the famous partition function  $p_y(z)$ .

**1.2.5. Switching and translating harmonic functions.** We denote by  $\sigma$  the automorphism of  $G$  exchanging  $a$  and  $b$ . It is given by

$$\sigma(x, y, z) = (y, x, xy - z).$$

Since the function  $h_0$  is  $\mu_0$ -harmonic, the function

$$h_1 := h_0 \circ \sigma : (x, y, z) \mapsto p_x(xy - z)$$

is also  $\mu_0$ -harmonic. For  $g_0$  in  $G$ , we denote by  $\rho_{g_0} : g \mapsto gg_0$  the right translation by  $g_0$  on  $G$ . The translated functions  $h_0 \circ \rho_{g_0} : g \mapsto h_0(gg_0)$  and  $h_1 \circ \rho_{g_0} : g \mapsto h_1(gg_0)$  are also  $\mu_0$ -harmonic.

### 1.3. Classification of positive harmonic functions

We can now state our main result for the southwest measure  $\mu_0$  introduced in (1.4).

#### 1.3.1. Main result and strategy.

**Theorem 1.1.** *Let  $h$  be an extremal positive  $\mu_0$ -harmonic function on the Heisenberg group  $G := H_3(\mathbb{Z})$ . Then, up to a multiplicative scalar,*

- either  $h = \chi$  is a  $\mu_0$ -harmonic character  $\chi(x, y, z) = r^x s^y$  as in (1.11)
- or  $h = h_0 \circ \rho_{g_0}$  is a translate of the function  $h_0(x, y, z) := p_y(z)$
- or  $h = h_1 \circ \rho_{g_0}$  is a translate of the function  $h_1(x, y, z) := p_x(xy - z)$ .

This classification has been announced on May 28th 2019 in a short informal videotaped speech at the Cetraro conference “Dynamics of group actions.” This video can be found on the author’s web page.

As we will see, the partition function  $p(x, y, z)$  will play a crucial role in the proof of Theorem 1.1. Indeed, in Chapter 2, we will prove a ratio limit theorem for the partition function  $p(x, y, z)$ . In Chapter 3, we will deduce from this ratio limit theorem the proof of Theorem 1.1.

Notice that the positive  $\mu_0$ -harmonic function  $h_0$  vanishes. In particular, it does not satisfy the Harnack inequality. This contrasts with the case studied in [10], where the support of  $\mu$  spans  $G$  as a semigroup.

In the last section (Section 3.4), we will present the classification of the positive  $\mu$ -harmonic functions, for all finitely supported measures  $\mu$  on  $G$ .

**1.3.2. Dealing with a probability measure.** At first glance it might look a little bit weird to deal with a  $\mu_0$ -harmonic function for a measure  $\mu_0$  which is not a probability measure. We could have worked instead with the probability measure

$$\tilde{\mu}_0 = \frac{1}{2}(\delta_{a^{-1}} + \delta_{b^{-1}}), \quad \text{where } a := (1, 0, 0) \text{ and } b = (0, 1, 0)$$

which is the law for the *southwest random walk on  $\mathbb{H}_3(\mathbb{Z})$* . The  $\tilde{\mu}_0$ -harmonic functions  $\tilde{h}$  on  $G$  are the functions satisfying

$$\tilde{h} = P_{\tilde{\mu}_0}h, \quad \text{where } P_{\tilde{\mu}_0}h(x, y, z) = \frac{1}{2}(\tilde{h}(x - 1, y, z - y) + \tilde{h}(x, y - 1, z))$$

is the expected value of the function  $h$  after one step of the random walk.

It is easy to see that

$$h(x, y, z) \text{ is } \mu_0\text{-harmonic if and only if } 2^{-x-y}h(x, y, z) \text{ is } \tilde{\mu}_0\text{-harmonic.}$$

Therefore, classifying positive  $\mu_0$ -harmonic functions is equivalent to classifying positive  $\tilde{\mu}_0$ -harmonic functions. The main reason we are using  $\mu_0$  instead of  $\tilde{\mu}_0$  is to get rid of all these factors  $2^{-x-y}$ .

**1.3.3. Extremal superharmonic functions.** We have seen in (1.5) that the partition function  $p$  is  $\mu_0$ -superharmonic and more precisely that it is the potential of  $\mu_0$  at  $e$ . For every  $g_0$  in  $G$ , the function  $p \circ \rho_{g_0}$  is also a potential of  $\mu_0$  at  $g_0^{-1}$ . By Riesz decomposition theorem, those potentials are exactly the extremal positive  $\mu_0$ -superharmonic functions on  $G$  which are not harmonic. Therefore,

*every extremal positive  $\mu_0$ -superharmonic function  $f$  on  $G$  which is not harmonic is a translate  $f = p \circ \rho_{g_0}$  of the function  $p(x, y, z)$ .*

We would like to end this introduction by pointing out other limit theorems for random walks on the Heisenberg group and other nilpotent groups as [3, 4, 7, 8] even though we will not use them here.

## 2. The partition function

The aim of this chapter is to prove the ratio limit theorem (Proposition 2.2) for the partition function  $p(x, y, z)$ .

### 2.1. The unimodality of the partition functions

We recall that, for  $x, y, z \geq 0$ , the partition function  $p(x, y, z)$  counts the number of partitions of  $z$  included in a rectangle with side lengths  $x$  and  $y$ . See definition (1.1) and Figure 1.

This function is non-zero for  $0 \leq z \leq xy$  and satisfies the equalities

$$p(x, y, z) = p(y, x, z) = p(x, y, xy - z). \tag{2.1}$$

This function is well studied. For instance, one has the following fact.

**Fact 2.1** (Cayley, Sylvester 1850). *The sequence  $z \mapsto p(x, y, z)$  is unimodal; i.e., it is increasing for  $z \leq xy/2$ .*

The proof below relies on the theory of finite dimensional representations of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ . This proof is due to Hughes in [9]. See [12] for an elementary proof and [14, p. 522] for a survey of various generalizations.

*Sketch of proof of Fact 2.1.* Let  $n := x + y$  and let  $(Y, H, X)$  be the principal  $\mathfrak{sl}_2$ -triple in the Lie algebra  $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R})$  so that  $H = \text{diag}(n - 1, n - 3, \dots, -n + 1)$ . This Lie algebra  $\mathfrak{g}$  has a natural representation in the space  $V := \Lambda^x(\mathbb{R}^n)$ . One checks that  $p(x, y, z) = \dim V_{xy-2z}$ , where  $V_\lambda$  denotes the eigenspace of  $H$  in  $V$  for the eigenvalue  $\lambda$ . The theory of representations of  $\mathfrak{sl}(2, \mathbb{R})$  tells us that for  $\lambda > 0$ , one always has that  $\dim V_\lambda \leq \dim V_{\lambda-2}$ . ■

### 2.2. The ratio limit theorem

Here is the *ratio limit theorem* for  $p(x, y, z)$ .

**Proposition 2.2.** *One has that*

$$\lim_{\substack{z \rightarrow \infty \\ xy-z \rightarrow \infty}} \frac{p(x, y, z - 1)}{p(x, y, z)} = 1.$$

This limit is taken along sequences of positive triples  $(x, y, z)$  such that  $z \rightarrow \infty$  and  $xy - z \rightarrow \infty$ .

With this generality this theorem seems to be new, even though there already exist very precise estimates of  $p(x, y, z)$  in certain ranges. For instance, when  $x, y \geq z$ , the partition function  $p(x, y, z) = p(z, z, z)$  depends only on  $z$ . It is the classical partition function  $p(z)$  which admits a famous asymptotic expansion due to Hardy and Ramanujan in 1920 (see [1, Chap. 5]). These estimates have been extended to larger ranges of  $(x, y, z)$  as in [11, 15]. We will not use them.

The proof of Proposition 2.2 is tricky but elementary. The rough idea is to introduce a relation between the set of partitions  $w$  of  $z$  and the set of partitions  $w'$  of  $z - 1$  such that “most of the time” when  $w$  and  $w'$  are related, they are related to approximately the same number of partitions (see Lemma 2.5).

Because of (2.1), we can assume that  $y \leq x$  and  $z \leq xy/2$ .

### 2.3. When the height of the rectangles is bounded

In this section, we deal with the easy case when the height  $y$  remains bounded.

**Lemma 2.3.** *For all  $y \geq 1$ , one has that*

$$\lim_{\substack{x, z \rightarrow \infty \\ z \leq xy/2}} \frac{p(x, y, z - 1)}{p(x, y, z)} = 1.$$

Note that in this limit  $y$  is fixed, and  $x, z$  go to  $\infty$  with  $z \leq xy/2$ .

*Proof of Lemma 2.3.* This follows from Lemma 2.4 and the inequalities

$$0 \leq p(x, y, z) - p(x, y, z - 1) \leq p(x, y - 1, z).$$

The first inequality is the unimodality of the partition function.

For the second inequality, just notice that one can inject the set of partitions of  $z$  of height exactly  $y$  inside the set of partitions of  $z - 1$  of height at most  $y$  by removing the last square in the bottom row of each partition. ■

We have used the following lemma.

**Lemma 2.4.** (a) *For all  $x, y, z \geq 1$ , one has that  $p(x, y, z) \leq z^{y-1}$ .*

(b) *For all  $y \geq 1$ , there exists  $\alpha_y > 0$  such that, for all  $x, z \geq 1$  with  $z \leq xy/2$ , one has that  $p(x, y, z) \geq \alpha_y z^{y-1}$ .*

*Proof of Lemma 2.4.* (a) The lengths of the last  $y - 1$  rows of the partition are bounded by  $z - 1$  and the first row is deduced from the others.

(b) Choose  $y - 1$  integers  $m_1, \dots, m_{y-1}$  in the interval  $[0, \frac{z}{y^2}]$  and keep only those for which the system

$$n_1 - n_2 = m_1, \dots, n_{y-1} - n_y = m_{y-1} \quad \text{and} \quad n_1 + \dots + n_y = z$$

has a solution  $(n_1, \dots, n_y)$  in  $\mathbb{Z}^y$ . But then one has that

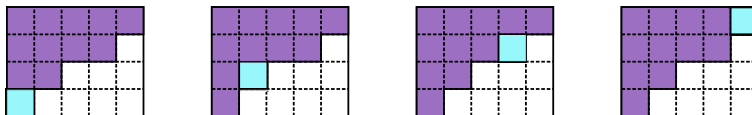
$$\begin{aligned} n_y &= \frac{1}{y}(z - m_1 - 2m_2 - \dots - (y - 1)m_{y-1}) \geq 0, \\ n_1 &= n_y + m_1 + \dots + m_{y-1} \leq \frac{z}{y} + \frac{z}{y} \leq x. \end{aligned}$$

This gives about  $\frac{1}{y}(\frac{z}{y^2})^{y-1}$  partitions of  $z$  with  $x \geq n_1 \geq \dots \geq n_y \geq 0$ . ■

### 2.4. Inner and outer corner of a partition

We now introduce notations that will strengthen the connection between partitions and words in the Heisenberg group.





**Figure 5.** The fiber  $\pi^{-1}(w)$  of the word  $w = ababaabab$  has size  $f_w = 4$ .

We recall that  $a = (1, 0, 0)$  and  $b = (0, 1, 0)$  are the generators of the Heisenberg group  $G = H_3(\mathbb{Z})$ . Let

$$G^+ := \{g = (x, y, z) \in G \mid x, y \geq 0 \text{ and } 0 \leq z \leq xy\}$$

be the semigroup generated by  $a$  and  $b$  and let

$$c = aba^{-1}b^{-1} = (0, 0, 1) \tag{2.2}$$

be the generator of the center  $Z$  of  $G$ .

Let  $B_n := \{a, b\}^n$  be the set of finite words  $w$  in  $a, b$  of length  $\ell_w = n$  and let  $B := \bigcup_{n \geq 0} B_n$ . Using the product law in  $G$ , to each word  $w \in B$ , we can associate an element  $g_w$  in  $G^+$ . The partition function gives the size of the fibers of this map:

$$p(g) = |B_g|, \quad \text{where } B_g := \{w \in B \mid g_w = g\}. \tag{2.3}$$

Indeed, as explained in Figure 3, when  $g = (x, y, z)$ , each word  $w$  in  $B_g$  corresponds uniquely to a partition of  $z$  included in a rectangle with side lengths  $x$  and  $y$ . We introduce now the following relation  $\mathcal{R}$  on  $B$ :

$$\mathcal{R} := \{(w, w') \in B \times B \mid w = w_0abw_1 \text{ and } w' = w_0baw_1 \text{ for some } w_0, w_1 \in B\}.$$

Let  $\pi : \mathcal{R} \rightarrow B$  and  $\pi' : \mathcal{R} \rightarrow B$  be the two projections

$$\pi(w, w') = w \quad \text{and} \quad \pi'(w, w') = w'.$$

For  $w, w'$  in  $B$ , the cardinality of the fiber  $f_w := |\pi^{-1}(w)|$  is the number of pairs  $ab$  occurring in the word  $w$ . The size  $f_w$  is also the number of *inner corners* of the partition associated to  $w$  (see Figure 5). Similarly the cardinality of the fiber  $f'_{w'} := |\pi'^{-1}(w')|$  is the number of pairs  $ba$  occurring in the word  $w'$ . It is equal to the number of *outer corners* of the partition associated to  $w'$ .

The following lemma compares the size of these fibers.

**Lemma 2.5.** (a) For all  $(w, w') \in \mathcal{R}$ , one has that  $g_w = g_{w'}c$ .

(b) For all  $(w, w') \in \mathcal{R}$ , one has that  $|f_w - f'_{w'}| \leq 2$ .

In particular, one also has that  $f_w \leq 3f'_{w'}$ .

*Proof of Lemma 2.5.* (a) This follows from the equality  $c = aba^{-1}b^{-1}$ .

(b) Comparing the number of pairs  $ab$  and pairs  $ba$  occurring in  $w$  and in  $w'$ , one gets  $|f_w - f_{w'}| \leq 1$  and  $|f_{w'} - f'_w| \leq 1$ . ■

**2.5. Partitions with bounded number of corners**

We will need to control the number  $p_{\leq i}(x, y, z)$  of partitions of  $z$  included in a rectangle with side length  $x, y$  that have at most  $i$  inner corner.

The following Lemma 2.6 tells us that  $p_{\leq i}(x, y, z)$  is negligible compared to the total number of partitions  $p(x, y, z)$ .

**Lemma 2.6.** *For all  $i \geq 0$ , one has that*

$$\lim_{\substack{x, y, z \rightarrow \infty \\ z \leq xy/2}} \frac{p_{\leq i}(x, y, z)}{p(x, y, z)} = 0.$$

The limit is taken along sequences where all coordinates  $x, y, z$  go to  $\infty$  and  $z \leq xy/2$ .

*Proof of Lemma 2.6.* Use the following slight upgrade of Lemma 2.4. ■

**Lemma 2.7.** (a) *For all  $x, y, z, i \geq 1$ , one has that  $p_{\leq i}(x, y, z) \leq (2z)^{2i}$ .*

(b) *For all  $j > 1$ , there exists  $z_0 = z_0(j) \geq 1$  such that, for all  $x, y, z \geq 1$  with  $4j \leq y \leq x$  and  $z_0 \leq z \leq xy/2$ , one has that  $p(x, y, z) \geq z^j$ .*

*Proof of Lemma 2.7.* It is similar to Lemma 2.4.

(a) We can assume that  $x = y = z$ . We want to choose integers  $a_1, \dots, a_i \geq 1$  and  $m_1, \dots, m_i \geq 0$ , bounded by  $z$  such that  $a_1 m_1 + \dots + a_i m_i = z$ . There are at most  $(2z)^{2i}$  possibilities.

(b) We give a rough count. Choose  $L_y \leq y$  as large as possible such that, setting  $\ell_y = [L_y/2]$  and  $\ell_x = [z/L_y]$ , one has that  $\ell_y \leq \ell_x \leq x/2$ . There exists a partition  $w_0$  of  $z$  with  $L_y$  rows and all of whose rows have length  $\ell_x$  or  $\ell_x + 1$ . For every sequence  $\ell_x > m_1 \geq \dots \geq m_{\ell_y} \geq 0$ , we can modify this partition  $w_0$  by adding  $m_j$  spots to the  $j$ th highest row of  $w_0$  and removing  $m_j$  spots to the  $j$ th lowest row of  $w_0$ , for all  $j \leq \ell_y$ . This gives  $N$  different partitions of  $z$ , where  $N := \binom{\ell_x + \ell_y - 1}{\ell_y} \geq \max(2, \ell_x/\ell_y)^{\ell_y}$ . Hence, one has that  $p(x, y, z) \geq N$ .

*First case:* when  $z \leq y^2/2$ . In this case, we have that  $L_y = [\sqrt{2z}]$ .

One gets  $N \geq 2^{\ell_y} \geq 2^{\sqrt{z}/2} \geq z^j$ .

*Second case:* when  $z \geq y^2/2$ . In this case, we have that  $L_y = y$ .

If  $z \leq y^4$ , one gets  $N \geq 2^{\ell_y} \geq 2^{\sqrt[4]{z}/4} \geq z^j$ .

If  $z \geq y^4$ , one gets  $N \geq \left(\frac{\ell_x}{\ell_y}\right)^{\ell_y} \geq \left(\frac{z}{y^2}\right)^{\ell_y} \geq \sqrt{z}^{\ell_y} \geq z^{y/4} \geq z^j$ . ■

### 2.6. When the height of the rectangles is unbounded

We can now explain the proof of the ratio limit theorem.

*Proof of Proposition 2.2.* By (2.1) and Lemma 2.3, we can assume that the three positive integers  $x, y, z$  are going to  $\infty$  with  $y \leq x$  and  $z \leq xy/2$ . For  $g = (x, y, z)$  in  $G^+$ , one sets  $\mathcal{R}_g := \{(w, w') \in \mathcal{R} \mid g_w = g\}$ , and one computes

$$p(g) = |B_g| = \varepsilon_g + \sum_{(w,w') \in \mathcal{R}_g} \frac{1}{f_w}, \tag{2.4}$$

where  $\varepsilon_g = 1$  if  $\mathcal{R}_g = \emptyset$  and  $\varepsilon_g = 0$  otherwise. Similarly, by Lemma 2.5 (a), one has that

$$p(gc^{-1}) = |B_{gc^{-1}}| = \varepsilon'_g + \sum_{(w,w') \in \mathcal{R}_g} \frac{1}{f_{w'}}, \tag{2.5}$$

where  $\varepsilon'_g = 0$  or  $1$ . Combining (2.4), (2.5), and Lemma 2.5 (b), one gets

$$|p(g) - p(gc^{-1})| \leq 2 + \sum_{(w,w') \in \mathcal{R}_g} \frac{2}{f_w f_{w'}} \leq 2 + \sum_{(w,w') \in \mathcal{R}_g} \frac{6}{f_w^2} \leq 2 + \sum_{\substack{w \in B_g \\ f_w \neq 0}} \frac{6}{f_w}.$$

We recall that  $p_{\leq i}(g)$  is the number of  $w$  with  $f_w \leq i$ . Therefore, one has that

$$|p(g) - p(gc^{-1})| \leq 2 + 6p_{\leq i}(g) + \frac{6}{i}p(g) \quad \text{for all } i \geq 1.$$

We let  $x, y, z$  go to infinity with  $z \leq xy/2$ . According to Lemma 2.6, for all  $i \geq 1$ , the ratios  $p_{\leq i}(g)/p(g)$  converge to 0. Therefore,

$$\limsup \left| \frac{p(gc^{-1})}{p(g)} - 1 \right| \leq \frac{6}{i},$$

and therefore the sequence  $\frac{p(gc^{-1})}{p(g)}$  converges to 1 as required. ■

### 3. Positive harmonic functions

We now start the classification of extremal positive  $\mu_0$ -harmonic functions  $h$ . In Section 3.1, we deal with the case where  $h$  has a non-zero limit along an orbit of  $a^{-1}$  or  $b^{-1}$ . In Sections 3.2 and 3.3, we deal with the case where  $h$  goes to zero along all orbits of  $a^{-1}$  and  $b^{-1}$ . In Section 3.4, we present the generalization of this classification to any finitely supported measure  $\mu$  on  $G$ .

### 3.1. The partition function as a harmonic function

In this section, we characterize the functions  $h_0 \circ \rho_{g_0}$  and  $h_1 \circ \rho_{g_0}$  among extremal positive  $\mu_0$ -harmonic functions by their behavior along the orbits  $a^{-\mathbb{N}}g_0$  and  $b^{-\mathbb{N}}g_0$  of  $G$ .

We recall that  $a = (1, 0, 0)$  and  $b = (0, 1, 0)$  are the generators of the Heisenberg group  $G = H_3(\mathbb{Z})$ , that  $\mu_0 = \delta_{a^{-1}} + \delta_{b^{-1}}$ , and that  $h_0$  and  $h_1$  are the  $\mu_0$ -harmonic functions  $h_0(x, y, z) = p_y(z)$  and  $h_1(x, y, z) = p_x(xy - z)$ .

We first begin by an alternative construction of the function  $h_0$ . Let  $H_0$  be the abelian subgroup of  $G$  generated by  $a$  and let  $\psi_0 := \mathbf{1}_{H_0}$  be the characteristic function of  $H_0$ . One has that

$$\psi_0(x, y, z) = \begin{cases} 1 & \text{when } y = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 3.1.** *One has the equality  $h_0 = \lim_{n \rightarrow \infty} P_{\mu_0}^n \psi_0$ .*

**Remark.** Since the function  $\psi_0$  is  $\mu_0$ -subharmonic, i.e.,  $\psi_0 \leq P_{\mu_0} \psi_0$ , the sequence  $n \mapsto P_{\mu_0}^n \psi_0$  is increasing.

*Proof of Lemma 3.1.* One can compute explicitly this function  $P_{\mu_0}^n \psi_0$ . It does not depend on  $x$ . Indeed,  $P_{\mu_0}^n \psi_0(x, y, z)$  is the number of ways of writing the element  $(n - y, y, z)$  as a word  $w$  of length  $n$  in  $a$  and  $b$ . This proves the equality, involving the partition function,

$$P_{\mu_0}^n \psi_0(x, y, z) = p(n - y, y, z).$$

Letting  $n$  go to  $\infty$ , we conclude using (1.13). ■

**Lemma 3.2.** *Let  $g_0 \in G$  and let  $h$  be an extremal positive  $\mu_0$ -harmonic function on  $G$  such that  $\limsup_{n \rightarrow \infty} h(a^{-n}g_0) > 0$ . Then one has that  $h = \lambda h_0 \circ \rho_{g_0}$  with  $\lambda > 0$ .*

*In particular, the positive  $\mu_0$ -harmonic function  $h_0 \circ \rho_{g_0}$  is extremal.*

*Proof of Lemma 3.2.* We can assume that  $g_0 = e$ . Since the function  $h$  is positive and  $\mu_0$ -harmonic, the sequence  $n \mapsto h(a^{-n})$  is decreasing. Hence it has a limit  $\lambda$ . By assumption, this limit  $\lambda$  is positive. By construction, one has the equality  $h \geq \lambda \psi_0$ . Since  $h$  is  $\mu_0$ -harmonic, one also has the inequality  $h \geq \lambda P_{\mu_0}^n \psi_0$  for all  $n \geq 0$ . Therefore, by Lemma 3.1, one gets  $h \geq \lambda h_0$ . Since  $h$  is extremal, it has to be proportional to  $h_0$  and therefore one has that  $h = \lambda h_0$ .

It remains to check that  $h_0$  is extremal. If one can write  $h_0 = h'_0 + h''_0$  with both  $h'_0$  and  $h''_0$  positive  $\mu_0$ -harmonic, for at least one of them, say  $h'_0$ , the sequence  $h'_0(a^{-n})$  does not converge to 0 for  $n \rightarrow \infty$ . Hence, by the previous discussion,  $h'_0$  is proportional to  $h_0$ . This proves that  $h_0$  is extremal. ■

Exchanging the role of  $a$  and  $b$  we get the following corollary.

**Corollary 3.3.** *Let  $h$  be an extremal positive  $\mu_0$ -harmonic function on  $G$  such that  $\limsup_{n \rightarrow \infty} h(b^{-n}g_0) > 0$ . Then one has that  $h = \lambda h_1 \circ \rho_{g_0}$  for some  $\lambda > 0$ . In particular, the positive  $\mu_0$ -harmonic function  $h_1 \circ \rho_{g_0}$  is extremal.*

**3.2. Harmonic functions that decay on cosets**

We now discuss positive harmonic functions on  $G$  that decay to 0 along the orbits  $a^{-\mathbb{N}}g_0$  and  $b^{-\mathbb{N}}g_0$ .

Let  $G_n$  be the subset of  $G$  consisting of elements of “degree”  $n$ ,

$$G_n = \{g = (x, y, z) \in G \mid x + y = n\}.$$

By definition and by (1.7), a positive  $\mu_0$ -harmonic function  $h$  on  $G$  satisfies the equality, for all  $n \geq 1$ ,

$$h(g_0) = \sum_{w \in B_n} h(g_w^{-1}g_0) = \sum_{g \in G_n} p(g)h(g^{-1}g_0). \tag{3.1}$$

For an integer  $A > 0$ , we set

$$\begin{aligned} G_{n,A} &= \{g = (x, y, z) \in G_n \mid z \leq A\}, \\ G_{n,A}^\sigma &= \{g = (x, y, z) \in G_n \mid xy - z \leq A\}. \end{aligned} \tag{3.2}$$

The following lemma tells us when the contributions of  $G_{n,A}$  and  $G_{n,A}^\sigma$  in formula (3.1) are negligible.

**Lemma 3.4.** *Let  $h$  be a positive  $\mu_0$ -harmonic function on  $G$  such that*

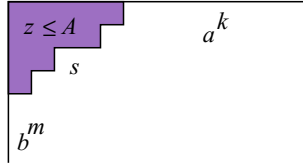
$$\lim_{n \rightarrow \infty} h(a^{-n}g_0) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} h(b^{-n}g_0) = 0 \quad \text{for all } g_0 \text{ in } G. \tag{3.3}$$

*Then, for all  $A > 0$  and  $g_0$  in  $G$ , one has that*

$$\lim_{n \rightarrow \infty} \sum_{g \in G_{n,A} \cup G_{n,A}^\sigma} p(g)h(g^{-1}g_0) = 0. \tag{3.4}$$

*Proof of Lemma 3.4.* It is enough to prove (3.4) with  $g_0 = e$ . Moreover, since  $G_{n,A}^\sigma$  is the image of  $G_{n,A}$  by the involution  $\sigma$  which exchanges  $a$  and  $b$ , it is enough to prove (3.4) with  $g \in G_{n,A}$ . Equivalently, it is enough to prove that

$$\lim_{n \rightarrow \infty} \sum_{w \in B_{n,A}} h(g_w^{-1}) = 0, \quad \text{where } B_{n,A} := \{w \in B_n \mid g_w^{-1} \in G_{n,A}\}. \tag{3.5}$$



**Figure 6.** The decomposition  $w = b^m s a^k$  for a word  $w \in B_{n,A}$ .

Note that, when  $n > A$ , every word  $w \in B_{n,A}$  can be written as

$$w = b^m s a^k$$

with  $s \in B_{A+1}$  a word of length  $A + 1$  (see Figure 6). One splits the set  $B_{n,A}$  according to  $m \geq A$  or  $m < A$ . Therefore, for  $n \geq 2A$ , one has the inclusion

$$B_{n,A} \subset b^A B_{n-A} \cup B_{2A} a^{n-2A}.$$

Therefore, using (3.1), one gets the inequalities

$$\begin{aligned} \sum_{w \in B_{n,A}} h(g_w^{-1}) &\leq \sum_{w \in B_{n-A}} h(g_w^{-1} b^{-A}) + \sum_{w \in B_{2A}} h(a^{-(n-2A)} g_w^{-1}) \\ &= h(b^{-A}) + \sum_{w \in B_{2A}} h(a^{-(n-2A)} g_w^{-1}). \end{aligned}$$

For all  $\varepsilon > 0$ , we choose  $A$  large enough so that, by the second assumption (3.3), one has that  $h(b^{-A}) \leq \varepsilon$ . Then the last sum is a sum over the fixed finite set  $B_{2A}$ , and, by the first assumption (3.3), this last sum converges to 0 when  $n$  goes to infinity. This proves (3.5) as required. ■

### 3.3. Using the ratio limit theorem

Combining Lemma 3.4 with the ratio limit theorem, we can finish the last case of the proof of Theorem 1.1.

**Lemma 3.5.** *Let  $h$  be a positive  $\mu_0$ -harmonic function on  $G$  such that, for all  $g_0$  in  $G$ ,  $\lim_{n \rightarrow \infty} h(a^{-n} g_0) = \lim_{n \rightarrow \infty} h(b^{-n} g_0) = 0$ . Then  $h$  is invariant by the center  $Z = c^{\mathbb{Z}}$  of  $G$ .*

*Proof of Lemma 3.5.* Using (3.1) with  $g_0$  and  $g_0 c$ , we compute

$$h(g_0) - h(g_0 c) = \sum_{g \in G_n} (p(g) - p(gc)) h(g^{-1} g_0). \tag{3.6}$$

We fix  $\varepsilon > 0$ . According to the ratio limit theorem (Proposition 2.2), there exists an integer  $A > 0$  such that, for all  $g = (x, y, z)$  in  $G^+$  with  $z \geq A$  and  $xy - z \geq A$ , one has that

$$|p(g) - p(gc)| \leq \varepsilon p(g). \tag{3.7}$$

Therefore, using (3.6), (3.7), and definition (3.2), one gets

$$|h(g_0) - h(g_0c)| \leq \sum_{g \in G_n} \varepsilon p(g) h(g^{-1}g_0) + \sum_{g \in G_{n,A} \cup G_{n,A}^\sigma} p(g) (h(g^{-1}g_0) + h(g^{-1}g_0c)).$$

By (3.1), the first term is equal to  $\varepsilon h(g_0)$ . Therefore, using twice Lemma 3.4 and letting  $n$  go to infinity, one gets  $|h(g_0) - h(g_0c)| \leq \varepsilon h(g_0)$ . Since  $\varepsilon$  is arbitrary small, this proves that  $h(g_0) = h(g_0c)$  as required. ■

**Corollary 3.6.** *Let  $h$  be an extremal positive  $\mu_0$ -harmonic function on  $G$  such that, for all  $g_0$  in  $G$ ,  $\lim_{n \rightarrow \infty} h(a^{-n}g_0) = \lim_{n \rightarrow \infty} h(b^{-n}g_0) = 0$ . Then  $h$  is a character of  $G$ .*

*In particular, every  $\mu_0$ -harmonic character of  $G$  is an extremal positive  $\mu_0$ -harmonic function.*

*Proof of Corollary 3.6.* By Lemma 3.5, the function  $h$  is  $\mu_0$ -harmonic on the abelian group  $G/Z$ . By Choquet–Deny theorem, it is a character.

It remains to check that a  $\mu_0$ -harmonic character  $\chi$  is extremal. Assume that  $\chi = h' + h''$  with both  $h'$  and  $h''$  positive  $\mu_0$ -harmonic. For all  $g_0$  in  $G$ , the sequences  $h'(a^{-n}g_0)$  and  $h'(b^{-n}g_0)$  converge to 0 for  $n \rightarrow \infty$ . Hence, by the previous discussion and by Choquet’s theorem, the function  $h'$  is an integral  $h' = \int_C \chi' d\sigma(\chi')$ , where  $\sigma$  is a finite positive measure on the set  $C$  of (harmonic) character  $\chi'$  of  $G$ . Since  $h' \leq \chi$ , the measure  $\sigma$  must be supported by  $\chi$ . This proves that  $\chi$  is extremal. ■

This ends the proof of Theorem 1.1.

### 3.4. Extension to finitely supported measures

In this section, we give the classification of the positive  $\mu$ -harmonic functions on the Heisenberg group for all finitely supported measure  $\mu$ .

Let  $G = H_3(\mathbb{Z})$  be the Heisenberg group and let  $S$  be a finite subset of  $G$ . We denote by  $G_S$  the subgroup of  $G$  generated by  $S$ . Let  $\mu = \sum_{s \in S} \mu_s \delta_s$  be a positive measure on  $G$  with support  $S$ .

We recall that a function  $h$  on  $G$  is said to be  $\mu$ -harmonic if

$$h = P_\mu h, \quad \text{where } P_\mu h(g) := \sum_{s \in S} \mu_s h(sg). \tag{3.8}$$

We want to describe the cone  $\mathcal{H}^+$  of positive  $\mu$ -harmonic functions  $h$  on  $G$ . By Choquet’s theorem, it is enough to describe the extremal rays of this cone  $\mathcal{H}^+$ .

There are two constructions of extremal positive  $\mu$ -harmonic functions.

**3.4.1. The harmonic characters  $\chi$ .** By definition, the  $\mu$ -harmonic characters are the characters  $\chi : G \rightarrow \mathbb{R}_{>0}$  of  $G$  such that  $\sum_{s \in S} \mu_s \chi(s) = 1$ . Such a function  $h = \chi$  is an extremal positive  $\mu$ -harmonic function on  $G$  which is invariant by the center  $Z$  of  $G$ .

We now recall Margulis’s theorem which tells us that this first construction is the only possible when  $G_\mu^+ = G$ .

**Fact 3.7 (Margulis).** *Let  $\mu$  be a finite positive measure on a finitely generated nilpotent group  $G$ . If the semigroup  $G_\mu^+$  generated by the support of  $\mu$  is equal to  $G$ , then every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  is a character.*

*Sketch of proof of Fact 3.7 for  $G = H_3(\mathbb{Z})$ .* Because of the assumption  $G_\mu^+ = G$ , we can assume that  $\mu_c > 0$  and  $\mu_a > 0$ . The first part of the argument is as in the abelian case: since  $h(x, y, z) \geq \mu_c h(x, y, z + 1)$ , these two  $\mu$ -harmonic functions are proportional and we get that, for some  $t > 0$ , one has that  $h(x, y, z) = h(x, y, 0)t^z$ . We now want to prove that  $t = 1$ .

Let  $K_t$  be the set of positive harmonic functions  $h_0(x, y, z) = \psi_0(x, y)t^z$  with  $h_0(e) = 1$ . Since  $G_\mu^+ = G$ , the convex set  $K_t$  is compact for the pointwise convergence. The element  $a \in G$  acts continuously by “right-translation and renormalization” on  $K_t$ . By Schauder’s fixed point theorem, this action has a fixed point  $h_0$  in  $K_t$ . It can be written as  $h_0(x, y, z) = r^x \varphi_0(y)t^z$  with  $r > 0$ . But then one writes  $h_0(g) \geq \mu_a h_0(ag)$  for all  $g$  in  $G$ , or equivalently  $\varphi_0(y) \geq \mu_a r \varphi_0(y)t^y$  for all  $y \in \mathbb{Z}$ . This proves that  $t = 1$ . ■

When  $G_\mu^+ \neq G$ , a second construction is possible.

**3.4.2. The functions  $h_{S_0, \chi_0}$  induced from a harmonic character.** Let  $S_0 \subset S$  be an abelian subset. Denote by  $\mu_{S_0} := \sum_{s \in S_0} \mu_s \delta_s$  the measure restriction of  $\mu$  to  $S_0$ . Let  $\chi_0$  be a  $\mu_{S_0}$ -harmonic character of  $G_{S_0}$ . We extend  $\chi_0$  as a function

$$\psi_0 := \chi_0 \mathbf{1}_{G_{S_0}}$$

on  $G$  which is 0 outside  $G_{S_0}$ . This function  $\psi_0$  is  $\mu$ -subharmonic, so that the sequence  $P_\mu^n \psi_0$  is increasing. We set

$$h_{S_0, \chi_0} = \lim_{n \rightarrow \infty} P_\mu^n \psi_0.$$

We can tell exactly for which pairs  $(S_0, \chi_0)$  the function  $h_{S_0, \chi_0}$  is finite (see [2]). In this case, the function  $h_{S_0, \chi_0}$  is an extremal positive  $\mu$ -harmonic function on  $G$ .



We can now state the extension of Theorem 1.1 to a more general finitely supported measure  $\mu$  on  $G$ .

**Theorem 3.8.** *Let  $G = H_3(\mathbb{Z})$  and  $\mu$  a positive measure on  $G$  whose finite support  $S$  generates the group  $G$ . Then every extremal positive  $\mu$ -harmonic function  $h$  on  $G$  is proportional either to a character  $\chi$  of  $G$  or to a translate  $h_{S_0, \chi_0} \circ \rho_{g_0}$  of a function induced from a harmonic character.*

**Corollary 3.9.** *Let  $G = H_3(\mathbb{Z})$ ,  $Z$  its center, and  $\mu$  a probability measure on  $G$  whose finite support  $S$  generates the group  $G$ . The following are equivalent.*

- (i) *Every positive  $\mu$ -harmonic function on  $G$  is  $Z$ -invariant.*
- (ii)  *$G_\mu^+$  contains two non-central elements whose product is in  $Z \setminus \{0\}$ .*

Theorem 3.8 and Corollary 3.9 are proven in the sequel paper [2].

We will also see in [2] that on the nilpotent group of rank 4 with cyclic center, there exist extremal positive harmonic functions which are neither a harmonic character nor a function induced from a harmonic character.

## References

- [1] G. E. Andrews, *The Theory of Partitions*. Encyclopedia Math. Appl. 2, Addison-Wesley, Reading, Mass., 1976 Zbl [0371.10001](#) MR [0557013](#)
- [2] Y. Benoist, Positive harmonic functions on the Heisenberg group II. *J. Éc. polytech. Math.* **8** (2021), 973–1003 Zbl [1472.31012](#) MR [4257161](#)
- [3] E. Breuillard, Local limit theorems and equidistribution of random walks on the Heisenberg group. *Geom. Funct. Anal.* **15** (2005), no. 1, 35–82 Zbl [1083.60008](#) MR [2140628](#)
- [4] E. Breuillard, Equidistribution of dense subgroups on nilpotent Lie groups. *Ergodic Theory Dynam. Systems* **30** (2010), no. 1, 131–150 Zbl [1194.22012](#) MR [2586348](#)
- [5] G. Choquet, Représentations intégrales dans les cônes convexes sans base compacte. *C. R. Acad. Sci. Paris* **253** (1961), 1901–1903 Zbl [0099.31601](#) MR [131746](#)
- [6] G. Choquet and J. Deny, Sur l'équation de convolution  $\mu = \mu * \sigma$ . *C. R. Acad. Sci. Paris* **250** (1960), 799–801 Zbl [0093.12802](#) MR [119041](#)
- [7] P. Diaconis and R. Hough, Random walk on unipotent matrix groups. *Ann. Sci. Éc. Norm. Supér. (4)* **54** (2021), no. 3, 587–625 Zbl [07452920](#) MR [4311095](#)
- [8] Y. Guivarc'h, Groupes nilpotents et probabilité. *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), A997–A998 Zbl [0222.43004](#) MR [296986](#)
- [9] J. W. B. Hughes, Lie algebraic proofs of some theorems on partitions. In *Number Theory and Algebra*, pp. 135–155, Acad. Press, 1977 Zbl [0372.10010](#) MR [0491213](#)
- [10] G. A. Margulis, Positive harmonic functions on nilpotent groups. *Soviet Math. Dokl.* **7** (1966), 241–244 Zbl [0187.38902](#) MR [0222217](#)

- [11] S. Melczer, G. Panova, and R. Pemantle, Counting partitions inside a rectangle. *SIAM J. Discrete Math.* **34** (2020), no. 4, 2388–2410 Zbl [1453.05010](#) MR [4175829](#)
- [12] R. A. Proctor, Solution of two difficult combinatorial problems with linear algebra. *Amer. Math. Monthly* **89** (1982), no. 10, 721–734 Zbl [0509.05007](#) MR [683197](#)
- [13] D. Revuz, *Markov Chains*. 2nd edn., North-Holland Mathematical Library 11, North-Holland, Amsterdam, 1984 Zbl [0539.60073](#) MR [758799](#)
- [14] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph Theory and its Applications: East and West (Jinan, 1986)*, pp. 500–535, Ann. New York Acad. Sci. 576, New York Acad. Sci., New York, 1989 Zbl [0792.05008](#) MR [1110850](#)
- [15] L. Takács, Some asymptotic formulas for lattice paths. *J. Statist. Plann. Inference* **14** (1986), no. 1, 123–142 Zbl [0616.60016](#) MR [845921](#)

**Yves Benoist**

CNRS, Université Paris-Saclay, Bâtiment 307, 91405 Orsay, France; [yves.benoist@u-psud.fr](mailto:yves.benoist@u-psud.fr)