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Kähler–Einstein metrics and Archimedean zeta functions

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Abstract. While the existence of a unique Kähler–Einstein metric on a canonically polarized manifold X was established by Aubin and Yau already in the 70s, there are only a few explicit formulas available. In a previous work, a probabilistic construction of the Kähler–Einstein metric was introduced – involving canonical random point processes on X – which yields canonical approximations of the Kähler–Einstein metric, expressed as explicit period integrals over a large number of products of X. Here it is shown that the conjectural extension to the case when X is a Fano variety suggests a zero-free property of the Archimedean zeta functions defined by the partition functions of the probabilistic model. A weaker zero-free property is also shown to be relevant for the Calabi–Yau equation. The convergence in the case of log Fano curves is settled, exploiting relations to the complex Selberg integral in the orbifold case. Some intriguing relations to the zero-free property of the local automorphic L-functions appearing in the Langlands program and arithmetic geometry are also pointed out. These relations also suggest a natural p-adic extension of the probabilistic approach.

1. Introduction

A metric ω on a compact complex manifold X is said to be *Kähler–Einstein* if it has constant Ricci curvature:

$$\operatorname{Ric}\omega = -\beta\omega$$

for some constant β and ω is Kähler (i.e., parallel translation preserves the complex structure on X). Such metrics play a prominent role in current complex differential geometry and the study of complex algebraic varieties, in particular in the context of the Yau–Tian–Donaldson conjecture [39] and the minimal model program (MMP) in birational algebraic geometry [61]. In [7, 8], a probabilistic construction of Kähler–Einstein metrics with negative Ricci curvature on a complex projective algebraic variety X was introduced, where the Kähler–Einstein metric emerges from a canonical random point process on X. The random point process is defined in terms of purely algebro-geometric data. Accordingly, one virtue of this approach is that it generates

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new links between differential geometry on the one hand and algebraic-geometry on the other. In the present work, it is, in particular, shown that the conjectural extension to Kähler–Einstein metrics with positive Ricci curvature suggests a zero-free property of the Archimedean zeta functions defined by the partition functions of the probabilistic model. The particular case of Kähler–Einstein metrics with conical singularities on the Riemann sphere is settled, which from the algebro-geometric perspective corresponds to the case of log Fano curves.

We start by providing some background on Kähler–Einstein metrics and recapitulating the probabilistic approach to Kähler–Einstein metrics; the reader is referred to the survey [9] for more background and [13] for relations to the Yau–Tian–Donaldson conjecture. See also [20] for connections to quantum gravity in the context of the AdS/CFT correspondence and [11, 41] for connections to polynomial approximation theory and pluripotential theory in $\mathbb{C}^n$.

1.1. Kähler–Einstein metrics

The existence of a Kähler–Einstein metric on X implies that the *canonical line bundle* K_X of X (i.e., the top exterior power of the cotangent bundle of X) has a definite sign:

$$\operatorname{sign}(K_X) = \operatorname{sign}(\beta). \tag{1.1}$$

We will be using the standard terminology of positivity in complex geometry: a line bundle L is said to be *positive*, $L > 0$, if it is ample and *negative*, $L < 0$, if its dual is positive. In analytic terms, $L > 0$ iff L carries some Hermitian metric with strictly positive curvature. The standard additive notation for tensor products of line bundles will be adopted. Accordingly, the dual of L is expressed as $-L$. We will focus on the cases when $\beta \neq 0$. Then X is automatically a complex projective algebraic manifold and after a rescaling of the metric we may as well assume that $\beta = \pm 1$. For example, in the case when X is a hypersurface in $\mathbb{P}^{n+1}$, cut out by a homogeneous polynomial of degree d,

$$K_X > 0 \Leftrightarrow d > n + 2, \quad -K_X > 0 \Leftrightarrow d < n + 2.$$

In the case when $K_X > 0$, the existence of a Kähler–Einstein metric was established in the late seventies [3, 82]. The opposite case $-K_X > 0$ is the subject of the Yau–Tian–Donaldson conjecture, which was settled only recently (see the survey [39]). However, these are abstract existence results and there are very few explicit formulas for Kähler–Einstein metrics on complex algebraic varieties available. For example, even in the simplest case when $K_X > 0$ and X is complex curve, $n = 1$, finding an explicit formula for the Kähler–Einstein metric is equivalent to finding an explicit uniformization map from the curve X to the quotient $\mathbb{H}/G$ of the upper half-plane by a discrete subgroup $G \subset \mathrm{SL}(2, \mathbb{R})$. This has only been achieved for very special curves (such as the Klein quartic and Fermat curves), using techniques originating in the classical works by Weierstrass, Riemann, Fuchs, Schwartz, Klein, Poincaré, etc. Thus one virtue of the probabilistic approach is that it yields canonical approximations of the Kähler–Einstein metric on X, expressed as essentially explicit period-type integrals formulas (see formula (1.4)). These are reminiscent of the aforementioned few explicit formulas for Kähler–Einstein metrics, involving hypergeometric integrals (see [9, Section 2.1]).

1.2. The probabilistic approach

First, recall that, in the case when $\beta \neq 0$, a Kähler–Einstein metric ω_{KE} on X can be readily recovered from its (normalized) volume form dV_{KE}:

$$\omega_{KE} = \frac{1}{\beta}\frac{i}{2\pi}\partial\bar{\partial}\log dV_{KE},$$

where we have identified the volume form dV with its local density, defined with respect to a choice of local holomorphic coordinates z. The strategy of the probabilistic approach is to construct the normalized volume form dV_{KE} by a canonical sampling procedure on X. In other words, after constructing a canonical symmetric probability measure $\mu^{(N)}$ on X^N, the goal is to show that the corresponding *empirical measure*

$$\delta_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i},$$

viewed as a random discrete measure on X, converges in probability as $N \to \infty$ to the volume form dV_{KE} of the Kähler–Einstein metric ω_{KE}.

1.2.1. The case $\beta > 0$. When $K_X > 0$, the canonical probability measure $\mu^{(N)}$ on X^N, introduced in [7], is defined for a specific subsequence of integers N_k tending to infinity, the *plurigenera* of X:

$$N_k := \dim H^0(X, kK_X),$$

where $H^0(X, kK_X)$ denotes the complex vector space of all global holomorphic sections $s^{(k)}$ of the kth tensor power of the canonical line bundle $K_X \to X$ (called pluricanonical forms). The assumption that $K_X > 0$ ensures that $N_k \to \infty$, as $k \to \infty$. In terms of local holomorphic coordinates $z \in \mathbb{C}^n$ on X, a section $s^{(k)}$ of $kK_X \to X$ may be represented by local holomorphic functions $s^{(k)}$ on X, such that $|s^{(k)}|^{2/k}$ transforms as a density on X, i.e., defines a measure on X. The canonical symmetric probability measure $\mu^{(N_k)}$ on X^{N_k} is concretely defined by

$$\mu^{(N_k)} := \frac{1}{\mathcal{Z}_{N_k}}|\det S^{(k)}|^{2/k}, \quad \mathcal{Z}_{N_k} := \int_{X^{N_k}}|\det S^{(k)}|^{2/k}, \tag{1.2}$$

where $\det S^{(k)}$ is the holomorphic section of the canonical line bundle $(kK_{X^{N_k}})$ over X^{N_k}, defined by the Slater determinant

$$(\det S^{(k)})(x_1, x_2, \ldots, x_{N_k}) := \det\left(s_i^{(k)}(x_j)\right), \tag{1.3}$$

in terms of a given basis $s_i^{(k)}$ in $H^0(X, kK_X)$. Under a change of bases, the section $\det S^{(k)}$ only changes by a multiplicative complex constant (the determinant of the change of bases matrix on $H^0(X, kK_X)$) and so does the normalizing constant $\mathcal{Z}_{N_k}$. As a result, $\mu^{(N_k)}$ is indeed canonical, i.e., independent of the choice of bases. Moreover, it is completely encoded by algebro-geometric data in the following sense: realizing X as projective algebraic subvariety, the section $\det S^{(k)}$ can be identified with a homogeneous polynomial, determined by the coordinate ring of X (or more precisely, the degree k component of the canonical ring of X).

The following convergence result was shown in [7].

Theorem 1.1. *Let X be a compact complex manifold with positive canonical line bundle K_X. Then the empirical measures δ_{N_k} of the corresponding canonical random point processes on X converge in probability, as $N_k \to \infty$, towards the normalized volume form dV_{KE} of the unique Kähler–Einstein metric ω_{KE} on X.*

In fact, the proof (discussed in Section 2.2) shows that the convergence holds at an exponential rate, in the sense of large deviation theory: for any given $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\operatorname{Prob}\left(d\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, dV_{KE}\right) > \varepsilon\right) \leq C_\varepsilon e^{-N\varepsilon},$$

where d denotes any metric on the space $\mathcal{P}(X)$ of probability measures on X compatible with the weak topology. The convergence in probability implies, in particular, that the measures dV_k on X, defined by the expectations $\mathbb{E}(\delta_{N_k})$ of the empirical measure δ_{N_k}, converge towards dV_{KE} in the weak topology of measures on X:

$$dV_k := \mathbb{E}(\delta_{N_k}) = \int_{X^{N_k-1}} \mu^{(N_k)} \to dV_{KE}, \quad k \to \infty.$$

For k sufficiently large (ensuring that kK_X is very ample), the measures dV_k are, in fact, volume forms on X and induce a sequence of canonical Kähler metrics ω_k on X, expressed in terms of period-type integrals:

$$\omega_k := \frac{i}{2\pi}\partial\bar{\partial}\log dV_k = \frac{i}{2\pi}\partial\bar{\partial}\log\int_{X^{N_k-1}} |\det S^{(k)}|^{2/k}, \tag{1.4}$$

whose integrands are encoded by the degree k component of the canonical ring of X. The convergence above also implies that the canonical Kähler metrics ω_k converge, as $k \to \infty$, towards the Kähler–Einstein metric ω_{KE} on X, in the weak topology.

1.2.2. The case $\beta < 0$. When $-K_X > 0$, i.e., X is a Fano manifold, there are obstructions to the existence of a Kähler–Einstein metric. According to the *Yau–Tian–Donaldson conjecture (YTD)*, X admits a Kähler–Einstein metric iff X is *K-polystable*. The non-singular case was settled in [31–33] and the singular case in [68–70], building on the proof of the uniform version of the YTD conjecture on Fano manifolds in [18] (the "only if" direction was previously shown in [6]). In the probabilistic approach, a different type of stability condition naturally appears, dubbed *Gibbs stability* (connections with the YTD conjecture are discussed in [13]). The starting point for the probabilistic approach on a Fano manifold, introduced in [8, Section 6], is the observation that when $-K_X > 0$, one can replace k with $-k$ in the previous constructions concerning the case $K_X > 0$. Thus, given a positive integer k, we set

$$N_k := \dim H^0(X, -kK_X)$$

(which tends to infinity as $k \to \infty$, since $-K_X$ is ample) and define a measure on X^{N_k} by

$$\mu^{(N_k)} := \frac{1}{\mathcal{Z}_{N_k}} |\det S^{(k)}|^{-2/k}, \quad \mathcal{Z}_{N_k} := \int_{X^{N_k}} |\det S^{(k)}|^{-2/k}. \tag{1.5}$$

However, in this case it may happen that the normalizing constant $\mathcal{Z}_{N_k}$ diverges, since the integrand of $\mathcal{Z}_{N_k}$ blows up along the zero-locus in X^{N_k} of $\det S^{(k)}$. Accordingly, a Fano manifold X is called *Gibbs stable at level* k if $Z_{N_k} < \infty$ and *Gibbs stable* if it is Gibbs stable at level k for k sufficiently large. For a Gibbs stable Fano manifold X, the measure $\mu^{(N_k)}$ in formula (1.5) defines a canonical symmetric probability measure on X^{N_k}. We thus arrive at the following probabilistic analog of the YTD conjecture posed in [8, Section 6]:

Conjecture 1.2. *Let X be Fano manifold. Then*

- *X admits a unique Kähler–Einstein metric ω_{KE} if and only if X is Gibbs stable;*
- *if X is Gibbs stable, the empirical measures δ_N of the corresponding canonical point processes converge in probability towards the normalized volume form of ω_{KE}.*

In order to briefly compare with the YTD conjecture, denote by $\mathrm{Aut}(X)_0$ the Lie group of automorphisms (biholomorphisms) of X homotopic to the identity I. Fano manifolds are divided into the two classes, according to whether $\mathrm{Aut}(X)_0$ is *trivial* or *non-trivial*,

$$\mathrm{Aut}(X)_0 = \{I\} \quad \text{or} \quad \mathrm{Aut}(X)_0 \neq \{I\}.$$

In the former case, the Kähler–Einstein metric is uniquely determined (when it exists), while in the latter case, it is only uniquely determined modulo the action of the group $\mathrm{Aut}(X)_0$. This dichotomy is also reflected in the difference between *K-polystability* and the stronger notion of *K-stability*, which implies that $\mathrm{Aut}(X)_0$ is trivial. Similarly, the Gibbs stability of X also implies that the group $\mathrm{Aut}(X)_0$ is trivial [14] and should thus be viewed as the analog of K-stability. Accordingly, we shall focus on the case when $\mathrm{Aut}(X)_0$ is trivial (but see [9, Conjecture 3.8] for a generalization of Conjecture 1.2 to the case when $\mathrm{Aut}(X)_0$ is non-trivial).

There is also a natural analog of the stronger notion of *uniform K-stability* (discussed in more detail in [13]). To see this, first recall that Gibbs stability can be given a purely algebro-geometric formulation, saying that the $\mathbb{Q}$-divisor $\mathcal{D}_{N_k}$ in X^{N_k} cut out by the (multi-valued) holomorphic section $(\det S^{(k)})^{1/k}$ of $-K_{X^{N_k}}$ has mild singularities in the sense of the MMP. More precisely, X is Gibbs stable at level k iff $\mathcal{D}_{N_k}$ is *Kawamata log terminal (klt)*. This means that the *log canonical threshold (lct)* of $\mathcal{D}_{N_k}$ satisfies

$$\mathrm{lct}(\mathcal{D}_{N_k}) > 1 \tag{1.6}$$

(as follows directly from the analytic representation of the lct of a $\mathbb{Q}$-divisor $\mathcal{D}$, recalled in the appendix). Accordingly, X is called *uniformly Gibbs stable* if there exists $\varepsilon > 0$ such that, for k sufficiently large,

$$\mathrm{lct}(\mathcal{D}_{N_k}) > 1 + \varepsilon. \tag{1.7}$$

One is thus led to pose the following purely algebro-geometric conjecture:

Conjecture 1.3. *Let X be a Fano manifold. Then X is (uniformly) K-stable iff X is (uniformly) Gibbs stable.*

The uniform version of the "if" direction was settled in [48], using algebro-geometric techniques (see also [12] for a different direct analytic proof that uniform Gibbs stability implies the existence of a unique Kähler–Einstein metric). However, the converse is still widely open. And even if confirmed, it is a separate analytic problem to prove the convergence towards the Kähler–Einstein metric in Conjecture 1.2. In [9, Section 7], a variational approach to the convergence problem was introduced, which reduces the proof of the convergence towards the volume form dV_{KE} of Kähler–Einstein metric to establishing the following convergence result for the normalization constants $\mathcal{Z}_{N_k}$:

$$\lim_{N_k\to\infty} -\frac{1}{N_k}\log \mathcal{Z}_{N_k} = \inf_{\mu\in\mathcal{P}(X)} F(\mu), \tag{1.8}$$

where $F(\mu)$ is a functional on the space $\mathcal{P}(X)$ of probability measures on X, minimized by dV_{KE}, which may be identified with the Mabuchi functional (see Section 2.2). This variational approach is inspired by a statistical mechanical formulation, where F appears as a free-energy type functional and β appears as the "inverse temperature". A central role is played by the *partition function*

$$\mathcal{Z}_{N_k}(\beta) := \int_{X^{N_k}} \| \det S^{(k)} \|^{2\beta/k} \, dV^{\otimes N_k}, \quad \beta \in [-1, \infty[\tag{1.9}$$

coinciding with the normalization constant $\mathcal{Z}_N$ when $\beta = -1$. However, for $\beta \neq -1$, $\mathcal{Z}_{N_k}(\beta)$ depends on the choice of a Hermitian metric $\| \cdot \|$ on $-K_X$, which, in turn, induces a volume form dV on X. In order to establish the convergence (1.8), two different approaches were put forth in [9, Section 7], which hinge on establishing either of the following two hypotheses:

- the "upper bound hypothesis" for the mean energy (discussed in Section 2.2),
- the "zero-free hypothesis" (discussed in Section 2.4):

$$\mathcal{Z}_{N_k}(\beta) \neq 0 \text{ on some } N_k\text{-independent neighborhood } \Omega \text{ of }]-1, 0] \text{ in } \mathbb{C}. \tag{1.10}$$

While originally defined for $\beta \in [-1, \infty[$, the partition function $\mathcal{Z}_{N_k}(\beta)$ extends to a meromorphic function of $\beta \in \mathbb{C}$, all of whose poles appear on the negative real axes. Indeed, by taking a covering of X, the function $\mathcal{Z}_{N_k}(\beta)$ may be expressed as a sum of functions of the form

$$Z(\beta) := \int_{\mathbb{C}^m} |f|^{2\beta} \Phi \, d\lambda, \tag{1.11}$$

for a holomorphic function f and a Schwartz function Φ on $\mathbb{C}^m$. One can then invoke classical general results of Atyiah and Bernstein for such meromorphic functions $Z(\beta)$ (recalled in Section A.2 of the appendix). The first negative pole of $\mathcal{Z}_{N_k}(\beta)$ is precisely the negative of the log canonical threshold $\operatorname{lct}(\mathcal{D}_{N_k})$. The zero-free hypothesis referred to above demands that there exists an N-independent neighborhood of $]-1, 0]$ in $\mathbb{C}$, where $\mathcal{Z}_{N_k}(\beta) \neq 0$. As shown in Section 2.4, the virtue of this hypothesis is that it allows one to prove the convergence in formula (1.8) by "analytically continuing" the convergence for $\beta > 0$ to $\beta = -1$. In the statistical mechanics literature, this line of argument goes back to the Lee–Yang theory of phase transitions (see Remark 2.7).

1.3. The partition function $\mathcal{Z}_{N_k}(\beta)$ viewed as local Archimedean zeta function

From an algebro-geometric perspective, the partition function $\mathcal{Z}_{N_k}(\beta)$ (formula (1.9)) is an instance of an *Archimedean zeta function*. More generally, replacing the local field $\mathbb{C}$ and its standard Archimedean absolute value $|\cdot|$ with a local field F and an absolute value $|\cdot|_F$ on F, meromorphic functions $Z(\beta)$ as in formula (1.11) can be attached to any polynomial f defined over the local field F. Such meromorphic functions are usually called *local Igusa zeta functions* [53]. This is briefly recalled in Section A.2 of the appendix. For example, the Riemann zeta function $\zeta(s)$ may be expressed as a Euler product over such local meromorphic functions $Z_p(s)$ as p

ranges over all primes p, i.e., all non-Archimedean places p of the global field $\mathbb{Q}$:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p} Z_p(s), \quad Z_p(s) = \int_{\mathbb{Q}_p^\times} |x|^s_{\mathbb{Q}_p} \Phi_p \, d^\times x = (1 - p^{-s})^{-1},$$

where $\mathbb{Q}_p$ is the localization of $\mathbb{Q}$ at p, i.e., the p-adic field $\mathbb{Q}_p$, endowed with its standard normalized non-Archimedean absolute value and multiplicative Haar measure $d^\times x$ on $\mathbb{Q}_p^\times$ and Φ_p denotes the p-adic Gaussian. This is explained in Tate's celebrated thesis [78], where it is shown that the classical procedure of completing the Riemann zeta function amounts to including a factor $Z_p(s)$ corresponding to the standard Archimedean absolute value on $\mathbb{R}$, which is proportional to the Gamma function.[1] In this case, all the local factors $Z_p(s)$ are manifestly non-zero (while the corresponding global zeta function $\zeta(s)$ does have zeros). It should, however, be stressed that it is rare that general local Igusa zeta functions of the form (1.11) and their zeros can be computed explicitly. Still, one might hope that the canonical nature of $\mathcal{Z}_{N_k}(\beta)$ may facilitate the situation. One small step in this direction is taken in Section 5, where some intriguing relations between the partition functions $\mathcal{Z}_{N_k}(\beta)$ and the local L-functions appearing in the Langlands program are pointed out (generalizing the local factors $Z_p(\beta)$ of the Riemann zeta function). In particular, it is shown that in the simplest case when X is n-dimensional complex projective space and N_k is minimal, i.e., $N_k = n + 1$, the partition function $\mathcal{Z}_{N_k}(\beta)$ can be identified with a standard local L-function L_p attached to the group $\mathrm{GL}(n + 1, \mathbb{Q})$ when the place p of the global field $\mathbb{Q}$ is taken to be the one defined by the complex Archimedean absolute. Accordingly, in this particular case, $\mathcal{Z}_{N_k}(\beta)$ has a strong zero-free property as a consequence of the standard zero-free property of local L-functions.

1.4. Main new results in the case of log Fano curves

Here it will be demonstrated that both approaches discussed above are successful in one complex dimension, $n = 1$. The only one-dimensional Fano manifold X is the complex projective line (the Riemann sphere) and its Kähler–Einstein metrics are all biholomorphically equivalent to the standard round metric on the two-sphere. But a geometrically richer situation appears when introducing weighted points (conical singularities) on the Riemann sphere. From the algebro-geometric point of view, this fits into the standard setting of *log pairs* (X, Δ), consisting of complex (normal) projective variety X (here assumed to be non-singular, for simplicity) endowed with a $\mathbb{Q}$-divisor Δ on X, i.e., a sum of irreducible subvarieties Δ_i of X of codimension one, with coefficients w_i in $\mathbb{Q}$. In this log setting, the role of the canonical line bundle

[1] Expressing $d^\times x = x^{-1} dx$ reveals that the role of β is played by $s - 1$; see Section 5.1.

K_X is placed by the *log canonical line bundle*

$$K_{(X,\Delta)} := K_X + \Delta$$

(viewed as a $\mathbb{Q}$-line bundle) and the role of the Ricci curvature $\operatorname{Ric}\omega$ of a metric ω is played by twisted Ricci curvature $\operatorname{Ric}\omega - [\Delta]$, where $[\Delta]$ denotes the current of integration defined by Δ. The corresponding *log Kähler–Einstein equation* thus reads

$$\operatorname{Ric}\omega - [\Delta] = \beta\omega, \quad \beta = \pm 1, \tag{1.12}$$

where $[\Delta]$ denotes the current of integration along Δ. When β is non-zero, existence of a solution ω_{KE} forces

$$\beta(K_X + \Delta) > 0.$$

In general, the equation (1.12) should be interpreted in the weak sense of pluripotential theory [16,42]. However, in case when (X, Δ) is *log smooth*, i.e., the components of Δ have simple normal crossings (which means that they intersect transversally), it follows from [52, 55] that a positive current ω solves the equation (1.12) iff ω is a bona fide Kähler–Einstein metric on $X - \Delta$ and ω has edge-cone singularities along Δ, with cone-angle $2\pi(1 - w_i)$, prescribed by the coefficients w_i of Δ. In particular, in the *orbifold case*

$$\Delta = \sum \left(1 - \frac{1}{m_i}\right)\Delta_i, \quad m_i \in \mathbb{Z}_+, \tag{1.13}$$

the log Kähler–Einstein metrics locally lift to a bona fide Kähler–Einstein metric on local coverings of X (branched along Δ and $K_X + \Delta$ may be identified with the orbifold canonical line bundle) [26, Section 2].

Example 1.4. Let X be the complex hypersurface of weighted projective space $\mathbb{P}(a_0, \dots, a_n)$, cut out by a quasi-homogeneous polynomial F on $\mathbb{C}^{n+1}$ of degree d, whose zero-locus $Y \subset \mathbb{C}^{n+1} - \{0\}$ is assumed non-singular. Then the orbifold (X, Δ) defined by the branching divisor Δ on X of the fibration $Y - \{0\} \to X$, induced by the natural quotient projection

$$\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}(a_0, \dots, a_n),$$

is a Fano orbifold (i.e., $-(K_X + \Delta) > 0$) iff $d < a_0 + a_1 + \dots + a_n$.

The probabilistic approach naturally extends to the setting of log pairs (X, Δ) satisfying $\beta(K_X + \Delta) > 0$ yielding a canonical probability measure on X^{N_k}, that we shall denote by $\mu_\Delta^{(N_k)}$. Indeed, one simply replaces the canonical line bundle K_X with the log canonical line bundle $K_{(X,\Delta)}$ in the previous constructions (cf. [8, Section 5] and [9, Section 3.2.4]).

1.4.1. Log Fano curves. Let now (X, Δ) be a log Fano curve (X, Δ), i.e., X is the complex projective line and

$$\Delta = \sum_{i=1}^{m} w_i p_i$$

for positive weights w_i satisfying $\sum_{i=1}^{m} w_i < 2$. In this case, it turns out that the "upper bound hypothesis" for the mean energy does hold, which leads to the following result announced in [9, Section 3.2.4]:

Theorem 1.5. *Let (X, Δ) be a log Fano curve. Then the following is equivalent:*

- *(X, Δ) is Gibbs stable;*
- *(X, Δ) is uniformly Gibbs stable;*
- *the following weight condition holds:*

$$w_i < \sum_{i \neq j} w_j, \quad \forall i; \tag{1.14}$$

- *there exists a unique Kähler–Einstein metric ω_{KE} for (X, Δ).*

Moreover, if any of the conditions above hold, then the laws of the corresponding empirical measures δ_N satisfy a large deviation principle (LDP) with speed N, whose rate functional has a unique minimizer, namely $\omega_{KE} / \int_X \omega_{KE}$. In particular, for any given $\varepsilon > 0$,

$$\text{Prob}\left(d\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \frac{\omega_{KE}}{\int_X \omega_{KE}} \right) > \varepsilon \right) \leq C_\varepsilon e^{-N\varepsilon}.$$

Existence of solutions to the log Kähler–Einstein equation (1.12) in the one-dimensional setting was first shown in [79], under the weight condition (1.14) and uniqueness in [71]. The weight condition (1.14) is also equivalent to uniform K-stability of (X, Δ) [47, Example 6.6] and thus the previous theorem confirms Conjecture 1.3 for log Fano curves.

We also show that in the case when the support of Δ consists of three points, the following variant of the "zero-free hypothesis" holds:

$$\mathcal{Z}_{N_k,\Delta} \neq 0,$$

when the coefficients of Δ are complexified, so that $\mathcal{Z}_{N_k,\Delta}$ is extended to a meromorphic function on $\mathbb{C}^3$ (the proof exploits that $\mathcal{Z}_{N_k,\Delta}$ can be expressed as the complex Selberg integral, which first appeared in the conformal field theory (CFT)). This leads to an alternative proof of the previous theorem, in this particular case, by "analytically continuing" the convergence result in the case $K_X + \Delta > 0$ to the log Fano case $K_X + \Delta < 0$.

Example 1.6. The case of three points includes, in particular, the case when X is a *Fano orbifold curve*. Such a curve may be embedded into a weighted $\mathbb{P}^2$ and is defined by the zero-locus of explicit quasi-homogeneous polynomial $F(X,Y,Z)$ in $\mathbb{C}^3$ (the du Val singularities). In the case of three orbifold points, there always exists a unique log Kähler–Einstein metric on X, concretely realized as the quotient $\mathbb{P}^1/G$ of the standard SU(2)-invariant metric on $\mathbb{P}^1$ under the action of a discrete subgroup G of SU(2) (branched over the three points in question).

1.5. Organization

In Section 2, conditional convergence results on log Fano varieties are obtained, formulated in terms of either the "upper bound hypothesis" on the mean-energy or the "zero-free hypothesis" of the partition function. Then – after a digression on the Calabi–Yau equation in Section 3 – in Section 4, the hypotheses in question are verified for log Fano curves and Fano orbifolds, respectively. Section 5 is of a speculative nature, comparing the strong form of the zero-free hypothesis with the standard zero-free property of the local L-functions appearing in the Langlands program. The paper is concluded with an appendix, providing background on lct's and Archimedean zeta functions.

2. Conditional convergence results on log Fano varieties

In this section, it is explained how to reduce the proof of the convergence on Fano manifolds X in Conjecture 1.2 to establishing either one of two different hypotheses, building on [9, Section 7]. More generally, we will consider the setup of log Fano varieties (X,Δ), discussed in Section 1.4. For simplicity, X will be assumed to be non-singular. We will be using the standard correspondence between metrics $\|\cdot\|$ on log canonical line bundles $-(K_X+\Delta)$ and volume forms dV_Δ on $X-\Delta$, which are singular when viewed as measures on X (see [9, Section 4.1.7] for background, where the measure dV_Δ is denoted by μ_0).

2.1. Setup

Let (X,Δ) be a log Fano variety. As recalled in Section 1.4, this means that Δ is a divisor with positive coefficients and that $-(K_X+\Delta)>0$. We will allow Δ to have real coefficients. Set

$$N_k := \dim H^0\big(X,-k(K_X+\Delta)\big),$$

where k ranges over the positive numbers with the property that $-k(K_X+\Delta)$ is a well-defined line bundle on X. To simplify the notation, we will often drop the subscript k in the notation for N_k. Since,

$$k \to \infty \Leftrightarrow N \to \infty,$$

this should, hopefully, not cause any confusion. As discussed in Section 1.4, assuming that (X, Δ) is Gibbs stable, we get a sequence of canonical probability measures $\mu_\Delta^{(N)}$ on X^N. Fixing a smooth Hermitian metric $\|\cdot\|$ on the $\mathbb{R}$-line bundle $-(K_X + \Delta)$ with positive curvature $\mu_\Delta^{(N)}$ may be expressed as

$$\mu_\Delta^{(N)} := \frac{1}{\mathcal{Z}_N} \| \det S^{(k)} \|^{2/k} dV_{(X,\Delta)}^{\otimes N}, \quad \mathcal{Z}_N := \int_{X^N} \| \det S^{(k)} \|^{2/k} dV_{(X,\Delta)}^{\otimes N}, \tag{2.1}$$

where $dV_{(X,\Delta)}$ is the singular volume form on X corresponding to the metric $\|\cdot\|$ on $-(K_X + \Delta)$ and $\det S^{(k)}$ is the Slater determinant of $H^0(X, -k(K_X + \Delta))$ induced by a choice of bases $s_1^{(k)}, \dots, s_N^{(k)}$ for $H^0(X, -k(K_X + \Delta))$, defined as in formula (1.3). Since $\mu_\Delta^{(N)}$ is independent of the choice of bases, we may as well assume that the basis is orthonormal with respect to the Hermitian product induced by $(\|\cdot\|, dV)$. The condition that (X, Δ) is Gibbs stable means that the normalization constant $\mathcal{Z}_N$ is finite. Hence, it implies that the local densities of dV are in L^1_{loc} (which in algebraic terms means that Δ is klt divisor).

From a statistical mechanical point of view, the probability measure $\mu_\Delta^{(N)}$ on X^N may be expressed as the *Gibbs measure*

$$\mu_\beta^{(N)} = \frac{e^{-\beta N E^{(N)}}}{\mathcal{Z}_N(\beta)} dV_\Delta^{\otimes N},$$

$$E^{(N)}(x_1, \dots, x_N) := -\frac{1}{kN} \log \left(\| \det S^{(k)}(x_1, \dots, x_N) \|^2 \right) \tag{2.2}$$

with $\beta = -1$. In physical terms, the Gibbs measure represents the microscopic state of N interacting particles in thermal equilibrium at inverse temperature β, with $E^{(N)}(x_1, \dots, x_N)$ playing the role of the *energy per particle and* the normalizing constant

$$\mathcal{Z}_N(\beta) = \int_{X^N} e^{-\beta N E^{(N)}} dV_{(X,\Delta)}^{\otimes N} = \int_{X^N} \| \det S^{(k)} \|^{2\beta/k} dV_{(X,\Delta)}^{\otimes N} \tag{2.3}$$

is called the *partition function*. It should, however, be stressed that, while the probability measure $\mu_\Delta^{(N)}$ is canonical, i.e., independent of the choice of metric $\|\cdot\|$, this is not so when $\beta \neq -1$. But one advantage of introducing the parameter β is that $\mu_\beta^{(N_k)}$ is a well-defined probability measure as long as $\beta > -\operatorname{lct}(X, \Delta)$, where $\operatorname{lct}(X, \Delta)$ denotes the global lct of (X, Δ) (whose definition is recalled in the appendix). In particular, it is, trivially, well defined when $\beta > 0$.

Fixing $\beta \in [-1, \infty[$, we can view the empirical measure

$$\delta_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} : \quad X^N \to \mathcal{P}(X)$$

as a random discrete measure on X. To be more precise, δ_N is a random variable on the ensemble $(X^N, \mu_\beta^{(N)})$, taking values in the space $\mathcal{P}(X)$ of probability measures on X. Accordingly, the *law* of δ_N is the probability measure

$$\Gamma_{N,\beta} := (\delta_N)_* \mu_\beta^{(N)} \in \mathcal{P}\big(\mathcal{P}(X)\big)$$

on $\mathcal{P}(X)$, defined as the push-forward of the probability measure $\mu_\beta^{(N)}$ on X^N to $\mathcal{P}(X)$ under the map δ_N.

2.2. The case $\beta > 0$

The following result, which is a special case of [7, Theorem 5.7] (when Δ is trivial) and [8, Theorem 4.3] (when Δ is non-trivial), establishes an LDP for the laws $\Gamma_{N,\beta}$ of δ_N as $N \to \infty$, which may be symbolically expressed as

$$\Gamma_{N,\beta} := (\delta_N)_* \mu_\beta^{(N)} \sim e^{-N(F(\mu) - F(\beta))}, \quad N \to \infty$$

(formally viewing the right-hand side as a density on the infinite dimensional space $\mathcal{P}(X)$; the precise meaning of the LDP is recalled below).

Theorem 2.1. *Let (X, Δ) be a log Fano variety. For $\beta > 0$, the sequence $\Gamma_{N,\beta}$ of probability measures on $\mathcal{P}(X)$ satisfies an LDP speed N and rate functional*

$$F_\beta(\mu) - F(\beta), \quad F(\mu) := \beta E(\mu) + \mathrm{Ent}(\mu), \quad F(\beta) := \inf_{\mathcal{P}(X)} F_\beta(\mu), \tag{2.4}$$

where $E(\mu)$ is the pluricomplex energy of μ relative to the Kähler form ω defined by the curvature of the metric $\|\cdot\|$ on $-(K_X + \Delta)$ and $\mathrm{Ent}(\mu)$ *is the entropy of μ relative to dV_Δ. In particular, the random measure δ_N converges in probability, as $N \to \infty$, to the unique minimizer μ_β of F_β in $\mathcal{P}(X)$, i.e.,*

$$\lim_{N\to\infty} \Gamma_{N,\beta} = \delta_{\mu_\beta} \quad \text{in } \mathcal{P}\big(\mathcal{P}(X)\big) \tag{2.5}$$

and the following convergence of the partition functions $\mathcal{Z}_N(\beta)$ holds:

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathcal{Z}_N(\beta) = F(\beta). \tag{2.6}$$

We recall that the *entropy* $\mathrm{Ent}(\mu)$ of μ relative to a given measure ν is defined by

$$\mathrm{Ent}(\mu) = \int_X \log \frac{\mu}{\nu} \mu$$

when μ has a density with respect to ν and otherwise $\mathrm{Ent}(\mu) := \infty$.[2] As for the pluricomplex energy $E(\mu)$ of a measure μ on X, relative to a reference form ω_0, it was first introduced in [17, Theorem 4.3]. From a thermodynamical point of view, the functional $F_\beta(\mu)$, introduced in [4, Theorem 4.3], can be viewed as the *free energy*.[3] The pluricomplex $E(\mu)$ may be defined as the greatest lsc extension to $\mathcal{P}(X)$ of the functional $E(\mu)$ on the space of volume forms μ in $\mathcal{P}(X)$ whose first variation is given by

$$dE(\mu) = -\varphi_\mu, \tag{2.7}$$

where φ_μ is a smooth solution to the complex Monge–Ampère equation (also known as the Calabi–Yau equation):

$$\frac{1}{V}\Big(\omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\beta\Big)^n = \mu, \quad V := \int_X \omega^n.$$

This property determines the functional $E(\mu)$ up to an additive constant which is fixed by imposing the normalization condition

$$E(\omega_0^n/V) = 0, \tag{2.8}$$

in the case when the reference form ω_0 is Kähler. Using the property (2.7), it is shown in [9, Proposition 4.1] that the minimizer μ_β of $F_\beta(\mu)$ is the normalized volume form on $X - \Delta$ uniquely determined by the property that

$$\mu_\beta = e^{\beta\varphi_\beta} dV_\Delta,$$

where the function φ_β is the unique smooth bounded Kähler potential on $X - \Delta$ solving the complex Monge–Ampère equation

$$\frac{1}{V}\Big(\omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\beta\Big)^n = e^{\beta\varphi_\beta} dV_\Delta. \tag{2.9}$$

It follows that the corresponding Kähler form

$$\omega_\beta := \omega + \frac{1}{\beta}\frac{i}{2\pi}\partial\bar{\partial}\log\frac{\mu_\beta}{dV_\Delta}\Big(= \omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\beta\Big)$$

---

[2] We are using the "mathematical" sign convention for the entropy, which renders $\mathrm{Ent}(\mu)$ non-negative when the reference measure ν is a probability measure and thus $\mathrm{Ent}(\mu)$ coincides with the *Kullback–Leibler divergence* in information theory.

[3] Strictly speaking, it is F_β/β which plays the role of free energy in thermodynamics.

satisfies the twisted Kähler–Einstein equation

$$\operatorname{Ric}\omega_\beta - [\Delta] = -\beta\omega_\beta + (\beta+1)\omega_0, \tag{2.10}$$

on X, coinciding with the (log) Kähler–Einstein equation (1.12) when $\beta = -1$.

Remark 2.2. Incidentally, the functional

$$\mathcal{M}(\varphi) := F_{-1}\Big(\frac{1}{V}\Big(\omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\beta\Big)^n\Big)$$

coincides with the *Mabuchi functional* for the log Fano variety (X, Δ), as explained in [9, Section 5.3]. Moreover, the twisted Kähler–Einstein equation (2.10) coincides with the logarithmic version of Aubin's continuity equation with "time-parameter" $t := -\beta$.

The precise definition of an LDP, which goes back to Cramér and Varadhan [37], is recalled in [9, Proposition 4.1]. For the purpose of the present paper, it will be convenient to use the following equivalent ("dual") characterization of the LDP in the previous theorem: for any continuous function $\Phi(\mu)$ on $\mathcal{P}(X)$:

$$\lim_{N\to\infty} -\frac{1}{N}\log\int_{X^N} e^{-N\beta E^{(N)}} e^{-N\Phi(\delta_N)} = \inf_{\mathcal{P}(X)}\big(F(\mu)+\Phi(\mu)\big) \tag{2.11}$$

(as follows from well-known general results of Varadhan and Bryc [37, Theorem 4.4.2]).

2.2.1. Outline of the proof. Before turning to the case when $\beta < 0$, we briefly recall that a key ingredient in the proof of the previous theorem is the convergence

$$E^{(N)}(x_1,\dots,x_N) \to E(\mu), \quad N \to \infty, \tag{2.12}$$

which holds in the sense of Gamma-convergence (deduced from the convergence and differentiability of weighted transfinite diameters in [15, Theorems A and B]). Combining this convergence with some heuristics going back to Boltzmann suggests that the contribution of the volume form $dV^{\otimes N}$ in the Gibbs measure (2.2) should give rise to the additional entropy term appearing in the rate functional:

$$(\delta_N)_*(e^{-\beta N E^{(N)}} dV^{\otimes N}) \sim e^{-NE(\mu)}(\delta_N)_*(dV^{\otimes N}) \sim e^{-N\beta E(\mu)} e^{-N\operatorname{Ent}(\mu)}.$$

This is made rigorous in [7] using an effective submean property of the density of $\mu_\beta^{(N)}$ on the N-fold symmetric product of X, viewed as a Riemannian orbifold (leveraging results in geometric analysis).

2.3. The case $\beta < 0$

In the case when $\beta < 0$, we may define the free energy functional $F_\beta(\mu)$ by the same expression as in formula (2.4), $F_\beta = \beta E + \mathrm{Ent}(\mu)$, when $E_{\omega_0}(\mu) < \infty$ and otherwise we set $F_\beta(\mu) = \infty$. The definition is made so that we still have $F_\mu(\mu) \in]-\infty, \infty]$ with $F_\mu(\mu) < \infty$ iff both $E(\mu) < \infty$ and $\mathrm{Ent}(\mu) < \infty$.

In order to handle the large N-limit in the case when $\beta < 0$, a variational approach was introduced in [9, Section 7], which reduces the problem to establishing the following *"upper bound hypothesis"* for the mean energy:

$$\limsup_{N\to\infty} \int_{X^N} E^{(N)} \mu^{(N)}_{\Delta,\beta} \leq E(\Gamma_\beta) := \int_{\mathcal{P}(X)} E(\mu)\Gamma_\beta(\mu) \tag{2.13}$$

for any large N-limit point Γ of $\Gamma_{N,\beta}$ in $\mathcal{X}$. This property is independent of the choice of metric $\|\cdot\|$ on $-(K_X + \Delta)$. Moreover, the corresponding lower bound always holds (as follows from the convergence (2.12)). The following theorem is an extension of the results in [9, Section 7] to the case when Δ is non-trivial.

Theorem 2.3. *Let (X, Δ) be a log Fano variety. Assume that (X, Δ) is uniformly Gibbs stable. Then (X, Δ) admits a unique Kähler–Einstein metric ω_{KE}. Moreover, in the following list each statement implies the next one:*

(1) *the "upper bound hypothesis" (2.13) for the mean energy holds when $\beta = -1$;*

(2) *the convergence (2.6) for the partition functions holds when $\beta = -1$;*

(3) *the empirical measures δ_N of the canonical random point process on X converge in law towards the normalized volume form dV_{KE} of ω_{KE}; i.e., the convergence (2.5) holds when $\beta = -1$.*

Furthermore, if the "upper bound hypothesis" (2.13) is replaced by the stronger hypothesis that the convergence holds when $E^{(N)}$ is replaced by $E^{(N)} + \Phi(\delta_N)$ for any continuous functional Φ on $\mathcal{P}(X)$ (and E is replaced by $E + \Phi$), then the LDP in Theorem 2.1 holds for $\beta = -1$.

Proof. The proof in the general case is similar to the case when Δ is trivial. Indeed, the assumption that (X, Δ) is uniformly Gibbs stable implies, by a simple modification of the proof of [48, Theorem 2.5] (concerning the case when Δ is trivial) that $\delta(X, \Delta) > 1$, which by [47] is equivalent to (X, Δ) being uniformly K-stable. Hence, by the solution of the uniform version of the YTD conjecture for log Fano varieties (X, Δ) with X non-singular in [18] (extended to general log Fano varieties in [68, 69]), it follows that (X, Δ) admits a unique Kähler–Einstein metric. Next, we summarize the proof of the convergence in [9, Section 7]; all steps are essentially the same in the case when Δ is non-trivial. Set

$$F_N(\beta) := -\frac{1}{N}\log \mathcal{Z}_N(\beta), \quad F(\beta) := \inf_{\in\mathcal{P}(X)} F_\beta(\mu) \tag{2.14}$$

and consider the mean free energy functional on $\mathcal{P}(X^N)$ defined by

$$F_N(\mu_N) := \beta \int_{X^N} E^{(N)}\mu_N + \frac{1}{N}\,\mathrm{Ent}(\mu_N),$$

where $\mathrm{Ent}(\mu_N)$ denotes the entropy of μ_N relative to $(dV_\Delta)^{\otimes N}$. By Gibbs variational principle (or Jensen's inequality),

$$F_N(\beta) = \inf_{\mu_N \in \mathcal{P}(X^N)} F_{N,\beta}(\mu_N) = F_{N,\beta}(\mu_{N,\beta}). \tag{2.15}$$

Moreover,

$$F(\beta) = \inf_{\mathcal{P}(\mathcal{P}(X))} F_\beta(\Gamma) = F_\beta(\delta_{\mu_\beta}), \tag{2.16}$$

where $F_\beta(\Gamma)$ denotes the following functional on $\mathcal{P}(\mathcal{P}(X))$:

$$F_\beta(\Gamma) := \int_{\mathcal{P}(X)} F_\beta(\mu)\Gamma$$

and δ_{μ_β} is the unique minimizer of $F(\Gamma)$ in $\mathcal{P}(\mathcal{P}(X))$ (using that $F(\mu)$ is lsc, thanks to the energy/entropy compactness theorem in [16] and hence $F(\Gamma)$ is lsc and linear on $\mathcal{P}(\mathcal{P}(X))$). Now, as shown in the course of the proof of [8, Theorem 6.7] (and refined in Step 1 in the proof of [9, Theorem 7.6]) for *any* β, the following inequality holds:

$$\limsup_{N\to\infty} F_N(\beta) \leq F(\beta) \tag{2.17}$$

(as follows from combining Gibbs variational principle with the Gamma-convergence (2.12) of $E^{(N)}$ towards $E(\mu)$). Combining Gibbs variational principle (2.15) with the variational principle (2.16) for $F(\beta)$, this means that

$$\limsup_{N\to\infty}\Big(\inf_{\mu_N \in \mathcal{P}(X^N)} F_{N,\beta}(\mu_N)\Big) \leq \inf_{\in \mathcal{P}(X)} F_\beta(\mu).$$

Moreover, as shown in [9, Section 7], if the "upper bound hypothesis" on the mean energy holds, then the corresponding lower bound also holds; i.e., the convergence (2.6) of the partition functions holds:

$$\lim_{N\to\infty} F_N(\beta) = F(\beta). \tag{2.18}$$

Indeed, combining the "upper bound hypothesis" with the well-known sub-additivity property of the mean entropy yields

$$F_\beta(\Gamma_\beta) \leq \liminf_{N\to\infty} F_{N,\beta}(\mu_{N,\beta})$$

for any limit point Γ_β of $\Gamma_{N,\beta}$, in the case $\beta = -1$. Combined with the upper bound

(2.17) and formula (2.16) for $F(\beta)$, it then follows that Γ_β minimizes $F_{-1}(\Gamma)$ and hence, by the uniqueness of minimizer, $\Gamma = \delta_{\mu_{-1}}$, as desired. All in all, this shows that "(1)⇒(2)⇒(3)" in the theorem.

Finally, to prove the LDP stated in the theorem, one just repeats the previous argument with $E^{(N)}$ replaced by $E^{(N)}_\Phi := E^{(N)} + \Phi(\delta_N)$. Then $\mathcal{Z}_N(\beta)$ gets replaced with $\int_{X^N} e^{-NE^{(N)}_\Phi} dV^{\otimes N}$ and hence the convergence (2.11) follows, as before, from the implication (1)⇒(2), now applied to $E^{(N)}_\Phi$.

In fact, the implications in the previous theorem may "almost" be reversed, by exploiting that the mean N-particular energy at inverse temperature β is proportional to the logarithmic derivative of $Z_N(\beta)$. More precisely, the following theorem holds, where it is assumed, for technical reasons, that X is a Fano orbifold. ■

Theorem 2.4. *Let (X, Δ) be a Fano orbifold and assume that (X, Δ) is uniformly Gibbs stable. Then there exists $\varepsilon > 0$ such that F_β admits a unique minimizer μ_β for any $\beta \in]-1-\varepsilon, 0[$. Moreover, the following is equivalent:*

(1) *the "upper bound hypothesis" for the mean energy* (2.13) *holds for any $\beta \in]-1-\varepsilon, 0[$;*
(2) *the convergence* (2.6) *for the partition functions holds for any $\beta \in]-1-\varepsilon, 0[$;*
(3) *the convergence* (2.6) *for the partition functions holds* and *the convergence* (2.5) *of the laws of δ_N holds for any $\beta \in]-1-\varepsilon, 0[$.*

Furthermore, If (1), (2) or (3) holds, then

$$\lim_{N\to\infty} \int_{X^N} E^{(N)} \mu^{(N)}_{\Delta,\beta} = E(\mu_\beta). \tag{2.19}$$

Proof. First, assume that (X, Δ) is a log Fano variety. As explained in the proof of the previous theorem, X admits a unique Kähler–Einstein metric. Hence, it follows from [34] (and [18]) that $F_{-1}(\mu)$ is coercive with respect to E; i.e., there exists $\varepsilon > 0$ such that

$$F_{-1} \geq \varepsilon E - 1/\varepsilon$$

on $\mathcal{P}(X)$. Thus F_β is also coercive with respect to E for any $\beta > -1-\varepsilon$. In particular, it follows from the energy-entropy compactness theorem in [16] that F_β admits a minimizer. Moreover, as shown in [16], any minimizer has the property that the corresponding function φ_β satisfies the complex Monge–Ampère equation (2.9). Next assume that (X, Δ) is a Fano orbifold. Then, for β sufficiently close to -1, the equation (2.9) has a unique solution. Indeed, since the Kähler–Einstein metric is unique, the orbifold X admits no non-trivial orbifold holomorphic vector fields, which, in turn, implies that the linearization of the equation (2.9) has a unique solution, defining a smooth function in the orbifold sense (see [36]). It then follows from a standard application of the implicit function theorem on orbifolds that the solution ϕ_β is uniquely determined for β sufficiently close to -1.

By the previous theorem (and its proof), it will be enough to show that (2)⇒(1). Since, trivially, (2)⇒(3), we have that $\Gamma_\beta = \delta_{\mu_\beta}$ and hence it will be enough to show the convergence in formula (2.19). To this end, first note that the functions $F_N(\beta)$ and $F(\beta)$ (defined in formula (2.14)) are concave in β, as follows readily from the definitions. Moreover, $F_N(\beta)$ and $F(\beta)$ are differentiable on $]-1-\varepsilon, 0[$ and

$$\frac{dF_N(\beta)}{d\beta} = \int_{X^N} E^{(N)} \mu^{(N)}_{\Delta,\beta}, \qquad \frac{dF(\beta)}{d\beta} = E(\mu_\beta), \tag{2.20}$$

using that μ_β is the unique minimizer of F_β. Hence, if the convergence in item (2) of the theorem holds, then it follows from basic properties of concave functions that the derivative of $F_N(\beta)$ converges towards the derivative of $F(\beta)$ at $\beta = -1$ (see [19, Lemma 3.1]). Applying formula (2.20) thus concludes the proof of the convergence (2.19). ■

Remark 2.5. The reason that we have assumed that (X, Δ) is a Fano *orbifold* is that the proof involves the implicit function theorem in Banach spaces and thus relies on analytic properties of the linearized log Kähler–Einstein equation. We will come back to this point in Section 2.4.3.

2.4. The zero-free hypothesis

An alternative approach towards the case $\beta < 0$ was also introduced in [9, Section 7.1]. In a nutshell, it aims to "analytically continue" the convergence when $\beta > 0$ to $\beta < 0$. Here we formulate the approach in terms of the following *zero-free hypothesis* on the partition function $\mathcal{Z}_N(\beta)$ (defined in formula (2.3)):

$$\mathcal{Z}_N(\beta) \neq 0 \text{ on some } N\text{-independent neighborhood } \Omega \text{ of }]-1, 0] \text{ in } \mathbb{C}. \tag{2.21}$$

We also need to assume that $\mathcal{Z}_N(\beta)$ is finite on a neighborhood of $[-1, 0]$ in $\mathbb{R}$ in a quantitative manner depending on N. This is made precise in the following result, which is a refinement of [9, Theorem 7.9]:

Theorem 2.6. *Let (X, Δ) be a Fano orbifold. Assume that there exists $\varepsilon > 0$ such that*

- $\mathcal{Z}_N(\beta) \leq C^N$ *for* $\beta = -(1+\varepsilon)$,
- *the zero-free hypothesis* (2.21) *holds.*

Then (X, Δ) admits a Kähler–Einstein metric ω_{KE} and δ_N converges in law towards the normalized volume form dV_{KE} of ω_{KE}. More precisely, the convergence (2.5) *of laws holds and $-\frac{1}{N}\log \mathcal{Z}_N(\beta)$ converges towards $F(\beta)$ in the C^∞-topology on a* *neighborhood of* $]-1,0]$. *Moreover, if* $[-1,0] \Subset \Omega$, *then the convergence holds on a neighborhood of* $[-1,0]$.

Proof. First, assume that (X,Δ) is a log Fano variety. Then the first point in the theorem implies that F admits a minimizer μ_β for any $\beta \in]-1-\varepsilon,0[$. Indeed, by the bound (2.17), $F(\beta)$ is bounded from below for any $\beta \in]-1-\varepsilon,0]$. Thus, for any $\beta \in]-1-\varepsilon,0[$, there exists $\delta>0$ such that $F_\beta \geq \delta E - \delta^{-1}$, which implies the existence of μ_β (as recalled in the proof of Theorem 2.4). In particular, taking $\beta=-1$ shows that X admits a unique Kähler–Einstein metric. Next, assume that X is a Fano orbifold. Then the argument using the implicit function, employed in the proof of Theorem 2.4, shows that after perhaps replacing ε with a small positive number there exists a unique solution φ_β to the equation (2.9), in the orbifold sense. In the case when X is a Fano manifold, it was shown in the proof of [9, Theorem 7.9] that $F(\beta)$ $(=F(\mu_\beta))$ defines a real-analytic function on $]-(1+\varepsilon),\infty[$. Since the proof only employs the implicit function theorem, it applies more generally when (X,Δ) is a Fano orbifold. Next, first consider the case when $\mathcal{Z}_N(\beta)$ is zero-free on an N-independent neighborhood Ω of $[-1,0]$ in $\mathbb{C}$. By Theorem 2.3, it will be enough to show that $\mathcal{Z}_N(\beta)^{1/N} \to e^{-F(\beta)}$ point-wise on $]-(1+\varepsilon),\varepsilon[$. To this end, first recall that, by Theorem 2.1, the convergence holds when $\beta \geq 0$. Next, by the zero-free hypothesis, $\mathcal{Z}_N(\beta)^{1/N}$ extends from $[-1,0]$ to a holomorphic function defined on a neighborhood Ω of $[-1,0]$ in $\mathbb{C}$. Moreover, by the first point,

$$\left|\mathcal{Z}_N(\beta)^{1/N}\right| \leq C \quad \text{on } \Omega \tag{2.22}$$

(using that $|\mathcal{Z}_N(\beta)^{1/N}| \leq \mathcal{Z}_N(\Re\beta)^{1/N} \leq \mathcal{Z}_N(-1-\varepsilon)^{1/N}$, which is uniformly bounded, by assumption). Hence, after perhaps passing to a subsequence, we may assume that $\mathcal{Z}_{N_j}(\beta)^{1/N_j}$ converges uniformly in the C^∞-topology on any compact subset of Ω to a holomorphic function $\mathcal{Z}(\beta)$, which, in particular, defines a real-analytic function on $]-1-\varepsilon,\varepsilon[$. But when $\beta \geq 0$, we have, as explained above, that $\mathcal{Z}(\beta)=e^{-F(\beta)}$ which extends to a real-analytic function on $]-1-\varepsilon,\varepsilon[$. By the identity principle for real-analytic functions, it thus follows that $\mathcal{Z}_{N_j}(\beta)^{1/N_j} \to e^{-F(\beta)}$ for any β in $]-1-\varepsilon,\varepsilon[$, in the C^∞-topology. Since the limit is uniquely determined, it thus follows that the whole sequence $\mathcal{Z}_N(\beta)^{1/N}$ converges towards $e^{-F(\beta)}$, as desired.

Finally, consider the case when it is only assumed that Ω is a neighborhood of $]-1,0]$ in $\mathbb{C}$. By assumption, the sequence of functions

$$F_N(\beta) := -\log\left(\mathcal{Z}_N(\beta)^{1/N}\right)$$

is uniformly bounded on $[-1-\varepsilon,\varepsilon]$. Since $F_N(\beta)$ is concave in β, it thus follows that $F_N(\beta)$ is uniformly Lipschitz continuous on $[-1,0]$. Hence, by the Arzela–Ascoli theorem, we may, after perhaps passing to a subsequence, assume that $F_N(\beta)$ converges uniformly to continuous function $F_\infty(\beta)$ on $[-1,0]$. By the previous argument, $F_\infty(\beta) = F(\beta)$ on $]-1,0]$. But since F_∞ and F are both continuous on $[-1,0]$, it follows that they also coincide at $\beta = -1$, as desired. ∎

Remark 2.7. In statistical mechanical terms, the C^∞-convergence of $N^{-1}\log \mathcal{Z}_N(\beta)$ amounts to the absence of phase transitions [75, Chapter 5]. It seems natural to expect that the zero-free hypothesis (2.21) is satisfied as soon as X admits a Kähler–Einstein metric. Indeed, it can be viewed as a strengthening of the real-analyticity of free energy $F(\beta)$ in some neighborhood of $]0,1]$ in $\mathbb{C}$ (discussed in the proof of the previous theorem). The zero-free hypothesis for general statistical mechanical partition functions was introduced in the Lee–Yang theory of phase transitions (and has been verified for some spin systems and lattice gases [67, 81]). More precisely, originally Lee–Yang considered zeros in the complexified field parameter h called *Lee–Yang zeros*, while zeros with respect to the complexified inverse temperature β are called *Fisher zeros* [43]. The role of h in the present complex geometric setup is discussed in Remark 3.4.

As discussed in [8, Section 6], the bound in first point in the previous theorem – which is independent of the choice of metric $\|\cdot\|$ (up to changing the constant C) – can be viewed as an analytic (stronger) version of uniform Gibbs stability (cf. [8, Theorem 6.7]). As shown in [9, Lemma 7.1], the bound always holds for β sufficiently close to 0. More precisely,

$$\beta > -\operatorname{lct}(-K_X) \Rightarrow \mathcal{Z}_N(\beta) \leq C_\beta^N \tag{2.23}$$

for any N $(= N_k)$, where $\operatorname{lct}(L)$ denotes the global lct of a line bundle L (whose definition is recalled in the appendix). The proof exploits that $\operatorname{lct}(-K_X)$ coincides with Tian's analytically defined α-invariant $\alpha(-K_X)$. Accordingly, under the weaker hypothesis that $\mathcal{Z}_N(\beta)$ is zero-free, for β in some ε-neighborhood of $]-\operatorname{lct}(X),0]$ in $\mathbb{C}$, the convergence statements in the theorem hold when $\beta \in\,]-\operatorname{lct}(X),0]$.

Remark 2.8. If $\operatorname{lct}(X) > 1$, the first assumption in Theorem 2.6 is automatically satisfied. Such Fano orbifolds are called *exceptional* (see [30], where two-dimensional exceptional hypersurfaces in three-dimensional weighted projective space are classified). Exceptional Fano orbifolds appear naturally in the MMP as the base of exceptional isolated affine singularities [76].

2.4.1. The strong zero-free hypothesis. The zero-free hypothesis is independent of the choice of basis in $H^0(X,-kK_X)$. Indeed, under a change of basis, $\det S^{(k)}$ gets multiplied by a non-zero scalar $c \in \mathbb{C}$ and hence $\mathcal{Z}_{N_k}(\beta)$ gets multiplied by $c^{\beta/k}$. However, it should be stressed that the zero-free hypothesis depends, a priori, on the choice of metric $\|\cdot\|$. For example, there are reasons to expect that it fails unless

$\|\cdot\|$ has positive curvature. Accordingly, the zero-free hypothesis might be more accessible for special/canonical choices of positively curved metrics, such as the Kähler–Einstein metric itself. This is illustrated by the following example, where $\mathcal{Z}_{N_k}(\beta)$ can be explicitly computed:

Example 2.9. When $X = \mathbb{P}^n_{\mathbb{C}}$ we have that $-K_X = \mathcal{O}(n+1)$ and hence the minimal value for k is $k = 1/(n+1)$, which means that the minimal value for N_k is $N_k = n+1$. Taking $\|\cdot\|$ to be the Fubini–Study metric (which is Kähler–Einstein) the following formula holds in the minimal case $N = n+1$ (where c_n is a computable positive constant), proved in the appendix (see Proposition A.3):

$$\mathcal{Z}_{n+1}(\beta) = c_n \frac{\prod_{j=1}^{n} \Gamma\big(\beta(n+1)+j\big)}{\big(\Gamma\big(\beta(n+1)+n+1\big)\big)^n}, \quad \Gamma(a) := \int_0^\infty t^a e^{-t} \frac{dt}{t} \tag{2.24}$$

where $\Gamma(a)$ denotes the classical Γ-function, which defines a meromorphic function on $\mathbb{C}$ whose poles are located at $0, -1, -2, \ldots$ (as follows from the functional relation $\Gamma(a+1) = a\Gamma(a)$). Thus the first negative pole of $\mathcal{Z}_N(\beta)$ comes from the first pole of the factor corresponding to $j = 1$ in the nominator above, i.e., when $\beta = -1(n+1)$. Moreover, since $\Gamma(a)$ is zero-free on all of $\mathbb{C}$, $\mathcal{Z}_N(\beta)$ is zero-free in the maximal strip $\{\Re\beta > -1/(n+1)\}$ of holomorphicity (but the meromorphic continuation $\mathcal{Z}_N(\beta)$ does have zeros in $\mathbb{C}$, coming from the poles of the denominator).

In the light of this example, it is tempting to speculate that the following *strong zero-free hypothesis* holds for Kähler–Einstein metrics:

$$\mathcal{Z}(\beta) \neq 0, \quad \text{when } \Re\beta > \max\big\{-\operatorname{lct}(\mathcal{D}_N), -1\big\}.$$

In other words, this means that $\mathcal{Z}_N(\beta)$ is zero-free in the maximal strip inside $\{\Re\beta > -1\}$, where it is holomorphic. To provide some further evidence for the strong zero-free property, we note that if its holds, then the bound (2.23), combined with the proof of Theorem 2.6, shows that, for any given $\varepsilon > 0$, the function $F(\beta)$ on $]-\operatorname{lct}(-K_X)+\varepsilon, \varepsilon[\ \subset \mathbb{R}$, induced by the Kähler–Einstein metric, is "strongly real-analytic" in the following sense: $F(\beta)$ extends to a bounded holomorphic function on the infinity strip $]-\operatorname{lct}(-K_X)+\varepsilon, \varepsilon[+i\mathbb{R} \subset \mathbb{C}$. This condition is much stronger than ordinary real-analyticity (which only implies holomorphic extension to a finite strip). But it does hold for the Kähler–Einstein metric. Indeed, in this case,

$$F(\beta) \equiv 0, \quad \beta \in\ ]-1, \infty[,$$

which trivially extends to a bounded holomorphic function on the infinity strip. To prove the identity above, first observe that when $\omega_0 = \omega_{KE}$, the twisted Kähler–Einstein equation (2.10) is solved by $\omega_\beta = \omega_{KE}$ for *any* β (equivalently, in the case when $\omega_0 = \omega_{KE}$, we have $\omega_0^n/V = dV_{(X,\Delta)}$ and hence the complex Monge–Ampère equation (2.9) is solved by $\varphi_\beta = 0$). But, as recalled above, for $\beta > -1$, the equation (2.9) admits a unique solution and hence

$$F(\beta) = F_\beta(dV_{KE}) = 0$$

(using the vanishing (2.8) combined with the vanishing $\mathrm{Ent}(\mu) = 0$ when $\mu = dV_{KE} = dV_\Delta$). In fact, this argument shows that $F(\beta) \equiv 0$ on all of $[-1, \infty[$. Moreover, if $\mathrm{Aut}(X)_0$ is trivial, then there exists an $\varepsilon > 0$ such that $F(\beta) \equiv 0$ on all of $]-1-\varepsilon, \infty[$, as follows from the argument using the implicit function theorem, employed in the proof of Theorem 2.4. This argument suggests that when $\mathrm{Aut}(X)_0$ is trivial, one can, perhaps, expect the strong zero-free property to even hold in the larger region where $\Re\beta > \max\{-\mathrm{lct}(\mathcal{D}_N), -1-\varepsilon\}$ for some $\varepsilon > 0$.

Remark 2.10. Coming back to Example 2.9, it is natural to ask if there exists an explicit formula for $\mathcal{Z}_N(\beta)$ when $X = \mathbb{P}^n_{\mathbb{C}}$ for general N, generalizing formula (2.24) (or, more precisely, for any N of the form $N = N_k$). However, as discussed in Remark A.4, this problem appears to be open even when $n = 1$. But one interesting consequence of formula (2.24) is that it reveals that in the case when $X = \mathbb{P}^n_{\mathbb{C}}$ and N is minimal,

$$\mathrm{lct}(\mathcal{D}_N) = \mathrm{lct}(-K_X)$$

since $\mathrm{lct}(-K_X) = 1/(n+1)$. This shows that the estimate in formula (2.23) is sharp (in the sense that there are cases where it fails for $\beta \leq -\mathrm{lct}(-K_X)$). The point of Conjecture 1.2, however, is that it only requires that $\mathrm{lct}(\mathcal{D}_{N_k}) > 1$ when N_k is sufficiently large. Similarly, in the case of $\mathbb{P}^n_{\mathbb{C}}$, where $\mathrm{Aut}(X)_0 \neq \{I\}$, the corresponding conjecture only requires that $\mathrm{lct}(\mathcal{D}_{N_k}) \to 1$, when $N_k \to \infty$ (see [9, Conjecture 3.8]). For example, when $X = \mathbb{P}^1_{\mathbb{C}}$, one has $\mathrm{lct}(\mathcal{D}_N) = (N-1)/N$ (by Theorem 4.5) which indeed tends to 1 as $N \to \infty$ (and equals $1/2$ when $N = 2$, which is the minimal case).

2.4.2. Allowing singular metrics $\|\cdot\|$. Alternatively, when X is a Fano manifold, one can take $\|\cdot\|$ to be the singular metric induced by the anti-canonical $\mathbb{Q}$-divisor Δ_m defined by the zero-locus of a holomorphic section of $-mK_X$, assuming that $m > 0$ and the zero-locus is non-singular (which ensures that the corresponding singular volume form dV has a density in L^p_{loc} for some $p > 1$). In other words, the curvature of $\|\cdot\|$ is given by the positive current $[\Delta_m]$ supported on Δ_m. Then Theorem 2.6 still applies. Indeed, in the proof one can apply the implicit function to the wedge-Hölder spaces appearing in [38, 55], which are independent of β (see, in particular, [55, Corollary 3.5]). In this singular setup, the corresponding equations (2.10) become Donaldson's variant of Aubin's continuity equations

$$\mathrm{Ric}\,\omega_\beta = t\omega_\beta + (1-t)[\Delta_m], \quad t = -\beta, \tag{2.25}$$

that were used in the proof of the YTD conjecture in [31–33], by deforming t from an initial small value, where there always exists a solution (by [4, Theorem 1.5]) to $t = 1$, assuming that X is K-stable. In other words, β is deformed down to -1. In the present probabilistic approach, the (potential) advantage of employing the singular metric on $-K_X$ induced by the $\mathbb{Q}$-divisor Δ_m is that the corresponding partition function $\mathcal{Z}_N(\beta)$ is encoded by purely algebraic data: the divisors $\mathcal{D}_N$ and Δ_m on X^N and X, respectively. In this case, combining [4, Proposition 6.2] with [9, Lemma 7.1] gives

$$\beta > -\min\left\{\operatorname{lct}(-K_X), \operatorname{lct}(-K_{X|\Delta_m})\right\} \Rightarrow \mathcal{Z}_N(\beta) \leq C_\beta^N,$$

where $-K_{X|\Delta_m}$ denotes the restriction of $-K_X$ to the support of Δ_m. More generally, it seems natural to expect that Theorem 2.6 holds for *any* log Fano variety (X, Δ) (when $\|\cdot\|$ is either a smooth metric on $K_X + \Delta$ with positive curvature or the singular metric defined by *any* klt $\mathbb{Q}$-divisor in $-(K_X + \Delta)$). In the case when $\Delta + \Delta_m$ defines a divisor whose components are non-singular and mutually non-intersecting, the aforementioned results in [38, 55] still apply.

2.4.3. Deforming the divisor Δ. Sometimes, it is advantageous to keep $\beta = -1$ and instead deform the divisor Δ as follows. Given a log Fano variety (X, Δ) and a positive real number k such that $-k(K_X + \Delta)$ is a well-defined line bundle $\mathcal{L}$, i.e., defines an element in the integral lattice $H^2(X, \mathbb{Z})$ of $H^2(X, \mathbb{R})$, consider the affine subspace $\mathcal{A}$ of $\mathbb{R}^{M+1}$ of all $(\boldsymbol{w}, s)$ which are "admissible" in the sense that

$$-\big(K_X + \Delta(\boldsymbol{w})\big) = s\mathcal{L}, \tag{2.26}$$

where $\Delta(\boldsymbol{w})$ denotes the divisor with the same M irreducible components as the given divisor Δ and coefficients $\boldsymbol{w} \in \mathbb{R}^M$. In particular, $(\boldsymbol{w}_0, k^{-1})$ is "admissible", where $\boldsymbol{w}_0 \in \mathbb{R}^M$ denotes the coefficients of the initial divisor Δ. If there exists $(\boldsymbol{w}_1, s_1) \in \mathcal{A}$ such that $K_X + \Delta(\boldsymbol{w}_1) > 0$ (and hence $s_1 < 0$), the conclusion of Theorem 2.6 still applies if the corresponding partition function $\mathcal{Z}_N$, viewed as a meromorphic function on $\mathbb{C}^{M+1}$, satisfies

- $\mathcal{Z}_N \leq C_0^N$ in a neighborhood in $\mathbb{R}^{M+1}$ of $(\boldsymbol{w}_0, k^{-1})$,
- $\mathcal{Z}_N \neq 0$ in an N-independent neighborhood of the line-segment in $\mathbb{C}^{M+1}$ connecting $(\boldsymbol{w}_0, k^{-1})$ and $(\boldsymbol{w}_1, s_1)$.

More precisely, as discussed in the previous section, in order to apply the implicit function theorem in Banach spaces, the appropriate linear PDE-theory needs to be in place. For example, by [38, 55], this is the case when the components of Δ are non-singular and mutually non-intersecting (results concerning the case when (X, Δ) is log smooth are announced in [73]). The previous proof can then by applied to the meromorphic function $\mathcal{Z}_N(t)$ on $\mathbb{C}$ defined by the partition functions associated to the line-segment $I \Subset \mathbb{C}^{m+1}$ connecting the initial $(\boldsymbol{w}_0, k^{-1})$ with $(\boldsymbol{w}_1, s_1)$ (where t

denotes the complexification of the standard parametrization of I). In this situation, the estimate (2.22) still holds, i.e., $|\mathcal{Z}_N(t)^{1/N}| \leq C$ on some N-independent neighborhood Ω of $[0, 1]$ in $\mathbb{C}$. Indeed, by assumption, the estimate holds with constant C_0 in a neighborhood of $t = 0$ and, moreover, it trivially holds with a constant C_1 when t is close to $t = 1$. Since $\log \mathcal{Z}_N(t)$ is convex with respect to $t \in [0, 1]$, one can thus take $C = \max\{C_0, C_1\}$.

3. Intermezzo: A zero-free hypothesis for polarized manifolds (X, L) and the Calabi–Yau equation

Before turning to the case of log Fano curves, we make a digression on general polarized manifolds (X, L), i.e., a compact complex manifold X endowed with an ample line bundle L. To a metric $\|\cdot\|$ on L and a volume form dV on X, we may attach partition functions $\mathcal{Z}_N(\beta)$, by replacing the log canonical line bundle $-(K_X + \Delta)$ with L and dV_Δ with dV in formula (2.3):

$$\mathcal{Z}_N(\beta) := \int_{X^N} \|\det S^{(k)}\|^{2\beta/k} dV^{\otimes N}, \tag{3.1}$$

where k is a given positive integer and N denotes the dimension of $H^0(X, kL)$. This is the general setup considered in [7], where the corresponding free energy functional is of the form

$$F_\beta(\mu) := \beta E(\mu) + \mathrm{End}(\mu),$$

where $E(\mu)$ denotes the pluricomplex energy of μ with respect to the normalized curvature form ω of the metric $\|\cdot\|$ on L and $\mathrm{Ent}(\mu)$ denotes the entropy of μ relative to dV. The minimizers μ_β of $F_\beta(\mu)$ are of the form

$$\mu_\beta = e^{\beta\varphi_\beta} dV$$

for a smooth solution φ_β of the complex Monge–Ampère equation

$$\frac{1}{V}\left(\omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi_\beta\right)^n = e^{\beta\varphi_\beta} dV. \tag{3.2}$$

Remark 3.1. In the case when $\beta = k$ and X is a Riemann surface, the corresponding partition function $\mathcal{Z}_N(\beta)$ coincides with the L^2-norm of the Laughlin wave function for the (integer) Quantum Hall state on X, subject to the magnetic two-form $ik\omega$ [58]. Accordingly, as shown in [5], in this case (and for any dimension of X) the corresponding large N-limit is described by the minimizers $F_\beta(\mu)/\beta$, as $\beta \to \infty$, i.e., of $E(\mu)$. However, here we are concerned with the case when β is fixed, where entropy enters the picture and dominates when β is close to 0.

Consider, in this general setup, the following *weak zero-free hypothesis*:

$$\mathcal{Z}_N(\beta) \neq 0 \text{ on some } N\text{-independent neighborhood } \Omega \text{ of } 0 \text{ in } \mathbb{C}. \tag{3.3}$$

It implies a weaker form of the upper bound hypothesis (2.13) on the mean energy:

Theorem 3.2. *Let (X, L) be a polarized manifold. Given a metric $\|\cdot\|$ on L and a volume form dV on X, assume that the corresponding partition functions $\mathcal{Z}_N(\beta)$ satisfy the weak zero-free hypothesis above. Then $-\frac{1}{N}\log \mathcal{Z}_N(\beta)$ converges towards $F(\beta)$ in the C^∞-topology on a neighborhood of 0 in $\mathbb{R}$. In particular, the mean energy of $dV^{\otimes N}$ converges towards the pluricomplex energy $E(dV)$ of dV:*

$$\begin{aligned} \lim_{N\to\infty}\int_{X^N} E^{(N)} dV^{\otimes N} &= E(dV), \\ E^{(N)}(x_1,\dots,x_N) &:= -\frac{1}{kN}\log\left(\left\| \det S^{(k)}(x_1,\dots,x_N)\right\|^2\right). \end{aligned} \tag{3.4}$$

Proof. In general, given a metric $\|\cdot\|$ on L and a volume form dV on X, there exists $\varepsilon > 0$ such that $F(\beta)$ is real-analytic on $]-\varepsilon, \varepsilon[$. Indeed, this follows, as before, from an application of the implicit function theorem at $\beta = 0$. Moreover, by the argument discussed in connection to formula (2.23),

$$\beta > -\operatorname{lct}(L) \Rightarrow \mathcal{Z}_N(\beta) \leq C_\beta^N. \tag{3.5}$$

In particular, the estimate holds when $\beta > -\varepsilon$ for ε sufficiently small. The C^∞-convergence of $-\frac{1}{N}\log \mathcal{Z}_N(\beta)$ towards $F(\beta)$ then follows exactly as in the proof of Theorem 2.6. Finally, the convergence of the first derivatives at $\beta = 0$ yields the convergence (3.4). ∎

We next show that a variant of the weak zero-free hypothesis yields canonical approximations φ_N of the solution of the *Calabi–Yau equation*, i.e., the equation obtained by setting $\beta = 0$ in equation (3.2):

$$\frac{1}{V}\left(\omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi\right)^n = dV \tag{3.6}$$

for a smooth function φ on X. By Yau's theorem [82], there exists a unique smooth solution φ with vanishing average on (X, dV). Given a volume form dV with unit total volume, the canonical approximation φ_N in question is defined by the integral formula

$$\varphi_N(x) := \int \frac{1}{k}\log\left(\left\| \det S^{(k)}(x, x_2,\dots,x_N)\right\|^2\right) dV^{\otimes N-1} - c_N, \tag{3.7}$$

where c_N is the constant ensuring that the average of φ_N on (X, dV) vanishes:

$$c_N := \int_{X^N} \frac{1}{k}\log\left(\left\| \det S^{(k)}(x, x_2,\dots,x_N)\right\|^2\right) dV^{\otimes N}.$$

For a given smooth function u on X, denote by $\mathcal{Z}_N(\beta, h)$ the function on $\mathbb{R}^2$ obtained by replacing dV in formula (3.1) with $e^{hu}dV$:

$$\mathcal{Z}_N(\beta, h) := \int_{X^N} \| \det S^{(k)} \|^{2\beta/k} (e^{hu} dV)^{\otimes N}. \tag{3.8}$$

Theorem 3.3. *Let (X, L) be a polarized manifold and $\| \cdot \|$ a metric on L. Given a volume form dV on X with unit total volume, assume that*

$$\mathcal{Z}_N(\beta, h) \neq 0 \text{ on some } N\text{-independent neighborhood } \Omega \text{ of } (0,0) \text{ in } \mathbb{C}^2, \tag{3.9}$$

for any smooth function u on X (where Ω depends on u). Then the functions φ_N, defined by formula (3.7), converge in $L^1(X)$, as $N \to \infty$, to the unique smooth solution φ of the Calabi–Yau equation (3.6) satisfying $\int_X \varphi dV = 0$.

Proof. First, observe that $\varphi_N(x)$ is ω-psh, since it is a superposition of the ω-psh functions $\log(\| \det S^{(k)}(x, x_2, \dots, x_N) \|^2)$. Hence, by standard properties of ω-psh functions, the L^1-convergence in question is equivalent to weak convergence. In other words, it is equivalent to proving that for any given smooth function $u \in C^\infty(X)$

$$\lim_{N \to \infty} \int \varphi_N u \, dV = \int \varphi \, dV.$$

Moreover, since the integrals on both sides of the previous equality vanish for $u = 1$, it is enough to prove the convergence for any $u \in C^\infty(X)$ satisfying $\int u dV = 0$. To this end, fix such a function u and consider the corresponding partition functions $\mathcal{Z}_N(\beta, h)$, defined by formula (3.8). A direct calculation reveals that

$$\int \varphi_N u \, dV = \frac{\partial}{\partial h} \frac{\partial}{\partial \beta} N^{-1} \log \mathcal{Z}_N(\beta, h), \quad \text{at } (\beta, h) = (0,0). \tag{3.10}$$

By assumption, there exists a neighborhood Ω of $(0,0)$ in $\mathbb{C}^2$, where $\log \mathcal{Z}_N(\beta, h)$ is holomorphic. Moreover, by Theorem 3.2,

$$-N^{-1} \log \mathcal{Z}_N(\beta, h) \to F(\beta, h) := \inf_{\in \mathcal{P}(X)} \left(\beta E(\mu) - h \int_X u \, dV + \mathrm{Ent}(\mu) \right)$$

in the C^∞_{loc}-topology on Ω, where $\mathrm{Ent}(\mu)$ denotes the entropy of μ relative to dV. In particular, the convergence of the second derivatives at $(0,0)$ yields, by formula (3.10),

$$\lim_{N \to \infty} \int \varphi_N u \, dV = -\frac{\partial}{\partial h} \frac{\partial F(\beta, h)}{\partial \beta} \quad \text{at } (\beta, h) = (0,0).$$

Since dV is the unique minimizer of F_β when $\beta = 0$,

$$\frac{\partial F(\beta, h)}{\partial \beta} = E(dV_h), \quad dV_h := dVe^{hu} \Big/ \int_X dVe^{hu}.$$

The proof is thus concluded by invoking the property (2.7) of the functional E, which gives

$$-\frac{E(dV_h)}{\partial h}_{|u=0} = \int_X \varphi u \, dV. \qquad \blacksquare$$

In the particular case when X is a Calabi–Yau manifold – i.e., when some power of K_X is trivial – we can apply the previous theorem to the canonical normalized volume form dV on X,

$$dV := \frac{(s_m \wedge \bar{s}_m)^{1/m}}{\int_X (s_m \wedge \bar{s}_m)^{1/m}},$$

where s_m trivializes mK_X for some positive integer m. Then the corresponding convergence implies that the positive $(1,1)$-currents

$$\omega_N := \frac{i}{2\pi k} \int \partial\bar{\partial} \log \left(\left| \det S^{(k)}(\cdot, x_2, \ldots, x_N) \right|^2 \right) dV^{\otimes N-1}$$

converge weakly towards the unique Calabi–Yau metric ω_{CY} on X in $c_1(L)$, i.e., towards the unique Ricci flat Kähler metric in $c_1(L)$. Note that, by the Poincaré–Lelong formula, ω_N is the average over X^{N-1} of the currents of integration defined by the zero-loci in X of the holomorphic sections $\det S^{(k)}(\cdot, x_2, \ldots, x_N)$.

Remark 3.4. It seems natural to expect that the zero-free hypothesis (3.9) is always satisfied. Indeed, it can be viewed as a strengthening of the real-analyticity of the free energy $F(\beta, h)$ in some neighborhood of $(0,0)$ in $\mathbb{C}^2$ (discussed in the proof of the previous theorem). This expectation is in line with corresponding expectations in the Lee–Yang theory of phase transitions [67, 81], where the role of β and h/β is played by the inverse temperature and the field strength, respectively (see the discussion in the introduction of [64]).

When X is a compact complex curve, i.e., $n = 1$, the convergence in Theorem 3.2 and Theorem 3.3 can, unconditionally, be deduced from the bosonization formula for $\det S^{(k)}(x_1, \ldots, x_N)$ [1]. To the leading order, this formula expresses $\| \det S^{(k)}(x, x_2, \ldots, x_N) \|$ as a product of $G(x_i, x_j)$, where G is Green's function for the Laplacian $i\partial\bar{\partial}$ (see Lemma 4.3 for the case when $X = \mathbb{P}^1_{\mathbb{C}}$).

4. The case of log Fano curves

Let X be the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. Fix an $\mathbb{R}$-divisor Δ on X, i.e.,

$$\Delta := \sum_{1=1}^{m} p_i w_i$$

for given points $p_1, \dots, p_m$ on X and with real coefficients/weights w_i and assume that

$$w_i < 1.$$

In contrast to Section 1.4, we thus allow w_i to be negative. Assume that (X, Δ) is a log Fano manifold, i.e., the anti-canonical line bundle of (X, Δ) is positive:

$$L := -(K_X + \Delta) > 0.$$

Since X is a complex curve, the assumption that L is positive simply means that its degree d_L is positive:

$$d_L = 2 - \sum w_i > 0. \tag{4.1}$$

Given a positive real number k and assuming that kL defines a line bundle, i.e., kd_L is an integer set,

$$N_k := \dim H^0(X, kL).$$

To the log Fano curve (X, Δ) we attach (as in the beginning of Section 2) the following symmetric probability measure on X^{N_k}:

$$\mu_\Delta^{(N_k)} = \frac{1}{\mathcal{Z}_{N_k}} \left| \det S^{(k)}(z_1, \dots, z_N) \right|^{-2/k} |s_\Delta|^{-2}(z_1) \cdots |s_\Delta|^{-2}(z_{N_k}),$$

which is well defined precisely when $\mathcal{Z}_{N_k} < \infty$. The following result implies Theorem 1.5 (concerning the case when $w_i > 0$):

Theorem 4.1. *Let (X, Δ) be a log Fano curve. Then the following is equivalent:*

- $\mathcal{Z}_{N_k} < \infty$ *for k sufficiently large;*
- *the following weight condition holds:*

$$w_i < \sum_{i \neq j} w_j, \quad \forall i. \tag{4.2}$$

Moreover, if any of the conditions above hold, then the law of the empirical measure δ_N on $(X^{N_k}, \mu_\Delta^{(N_k)})$ satisfies an LDP with speed N and rate functional $F_{-1} - \inf_{\mathcal{P}(X)} F_{-1}$ (where F_{-1} is the free energy functional on $\mathcal{P}(X)$ defined in Section 2.3, which coincides with the Mabuchi functional for (X, Δ)).

Remark 4.2. In particular, if the weight condition above holds, then F_{-1} is lsc on $\mathcal{P}(X)$ (since, in general, any rate functional for an LDP is lsc) and thus admits a minimizer. The existence of a minimizer was first shown in [79] using a different variational argument. By the general results for log Fano varieties (X, Δ) in [16], any minimizer satisfies the Kähler–Einstein equation for (X, Δ). In general, a solution is not uniquely determined (see [71, Remark 2]). However, when $w_i > 0$, the uniqueness in the case of the Riemann sphere was shown in [71] (see [16, 31–33] for the general higher dimensional log Fano case).

To prove the previous theorem, we first recall some standard identifications (see [11, Section 3.7]). Fixing a point p_∞, we identify $X-\{p_\infty\}$ with $\mathbb{C}$. The point p_∞ induces a trivialization e_∞ of the restriction of the hyperplane line bundle $\mathcal{O}(1)\to\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{C}$ (vanishing at p_∞) and thus the space $H^0(X,d\mathcal{O}(1))$ of all global holomorphic sections of the dth tensor power of the hyperplane line bundle $\mathcal{O}(1)\to X$ may be identified with the space of all polynomials in z of degree at most d. Moreover, the anti-canonical line bundle $-K_X$ of X may be identified with $2\mathcal{O}(1)$ and s_Δ with a (multivalued) holomorphic section of $\sum w_i\mathcal{O}(1)$. In particular, we identify

$$kL\leftrightarrow kd_L\mathcal{O}(1)=k\left(2-\sum_{i=1}^m w_i\right)\mathcal{O}(1),$$

(recall that we are assuming that kd_L is an integer). Thus $H^0(X,kL)$ gets identified with the space of all polynomials in z of degree at most $k(2-\sum_{i=1}^m w_i)$. This identification reveals that

$$N_k=kd_L+1. \tag{4.3}$$

Fix the standard basis of monomials $1,z,z^2,\dots$ in $H^0(X,kL)$. Then the corresponding section $\det S^{(k)}$ over X^{N_k} gets identified with the usual Vandermonde determinant on $\mathbb{C}^{N_k}$:

$$\det S^{(k)}\leftrightarrow D(z_1,\dots,z_{N_k}):=\det_{i,j\le N_k}(z_i^j). \tag{4.4}$$

Next, we identify X with the unit-sphere S^2 in $\mathbb{R}^3$, using the standard stereographic projection, so that the fixed point $p_\infty\in X$ corresponds to the "north-pole" $(0,0,1)$ in S^2:

$$z\mapsto x:=\left(\frac{z+\bar z}{1+|z|^2},\frac{z-\bar z}{1+|z|^2},\frac{-1+|z|^2}{1+|z|^2}\right),\quad \mathbb{C}\to\mathbb{R}^3.$$

Denote by dV_X the area form of the standard round metric on S^2 and by G the following lsc function on X:

$$G(x,y):=-\log\|x-y\|,$$

expressed in terms of the Euclidean norm on $\mathbb{R}^3$.

Lemma 4.3. *In terms of the standard identifications over $\mathbb{C}$,*

$$\begin{aligned}&\left|\det S^{(k)}(z_1,\dots,z_N)\right|^{-2/k}|s_\Delta|^{-2}(z_1)\cdots|s_\Delta|^{-2}(z_{N_k})\\ &\quad=\frac{1}{\left(\prod_{i\neq j}|z_i-z_j|\right)^{\frac{d_L}{N-1}}}\frac{1}{\prod_i}\frac{1}{|z_i-p_j|^{2w_j}}\end{aligned}$$

(where d_L is defined in formula (4.1)). As a consequence, on $X:=\mathbb{P}^1_{\mathbb{C}}$ the probability

measure $\mu_\Delta^{(N)}$ *may be expressed as*

$$\mu_\Delta^{(N)} = \frac{1}{Z_N} e^{\frac{d_L}{N-1}\sum_{i\neq j\leq N} G(x_i,x_j)} dV^{\otimes N}, \quad dV := e^{\sum_{i\leq m} w_i G(x,p_i)} dV_X. \quad (4.5)$$

Proof. First, factorizing the Vandermonde determinant $D(z_1,\ldots,z_{N_k})$ on $\mathbb{C}^N$ reveals that $D(z_1,\ldots,z_{N_k})$ is the product of $(z_i - z_j)$ over all i, j in $\{1,\ldots,N\}$ such that $i < j$. Hence,

$$\left|D(z_1,\ldots,z_{N_k})\right|^2 = \prod_{i\neq j} |z_i - z_j|. \quad (4.6)$$

Since $N_k = kd_L + 1$, we have that $k = (N-1)/d_L$ and hence the first formula of the lemma follows. To prove the second one, first recall that in the general setting of log Fano manifolds (X,Δ), the measure $\mu_\Delta^{(N)}$ may be expressed as in formula (2.1). In the present case, we take $\|\cdot\|$ to be the metric on L induced from the Fubini–Study metric $\|\cdot\|_{FS}$ on $\mathcal{O}(1)$ under the identification of L with $d_L\mathcal{O}(1)$. Recall that

$$\|e_\infty\|_{FS}^2 = e^{-\phi_{FS}(z)}, \quad \phi_{FS}(z) := \log\left(1 + |z|^2\right).$$

Hence, formula (4.5) follows from the following two facts: first,

$$\|z - w\|_{FS}^2 := |z-w|^2 e^{-\phi_{FS}(z)} e^{-\phi_{FS}(w)} \quad (4.7)$$

is proportional to the squared norm in $\mathbb{R}^3$ under stereographic projection and second,

$$\|dz\|_{FS}^2 := \|e_\infty^{\otimes 2}\|_{FS}^2 := e^{-2\phi_{FS}}$$

is proportional to the density of dV_X. These are well-known relations that can be checked explicitly, but they also follow readily from their invariance under the isometry group of S^2. ∎

Next, we recall the following general LDP [10, Theorem 1.5], generalizing the convergence in probability established in [27, 57] for the point-vortex model in a planar compact domain. Given a symmetric function W on a compact metric space X, a measure μ_0 on X, and $p \in \mathbb{R}$, set

$$\mu^{(N)}[p] = \frac{1}{Z_{N[p]}} e^{-p\frac{1}{N}\sum_{x_i\neq x_j} W(x_i,x_j)} \mu_0^{\otimes N},$$

$$Z_N[p] := \int_{X^N} e^{-p\frac{1}{N}\sum_{x_i\neq x_j} W(x_i,x_j)} \mu_0^{\otimes N},$$

assuming that $Z_N[p] < \infty$.

Theorem 4.4. *Let X be a compact metric space, μ_0 a measure on X, W a lower semi-continuous symmetric measurable function on X^2, and p_0 a negative number such that*

$$\sup_{x \in X} \int_X e^{-p_0 W(x,y)} \mu_0(y) < \infty. \tag{4.8}$$

Then, for any $p > p_0$, the normalizing constant $Z_N[p]$ is finite and the law of the empirical measure δ_N on $(X^N, \mu^{(N)}[p])$ satisfies an LDP with a rate functional

$$F_p - \inf_{\mathcal{P}(X)} F_p, \quad F_p(\mu) := p \int_{X \times X} W\mu \otimes \mu + \mathrm{Ent}_{\mu_0}(\mu).$$

Proof. It may be illuminating to reformulate the proof given in [10] in terms of the conditional convergence result in Theorem 2.3. First, the finiteness of $Z_N[p]$ follows readily from the arithmetic-geometric means inequality, using the integrability condition (4.8). A refinement of this argument also yields a priori estimates on each j-point correlation measure on X^j, building on [9, Section 3.2.4], showing that its density is uniformly bounded in $L^p(\mu_0^{\otimes j})$ for any $p > 1$. Applying this estimate to $j \leq 2$ shows that the "upper bound hypothesis" (2.13) of the energy is satisfied. A twist of this argument also yields the stronger form of the upper bound hypothesis with respect to any given continuous function $\Phi(\mu)$, as formulated in Theorem 2.3, and thus also the LDP. ∎

In the present case, we thus have

$$W(z, w) = -d_L \log \|z - w\|, \quad p = \beta \frac{N-1}{N}.$$

Moreover,

$$\int_X W\mu \otimes \mu = E(\mu) + C \tag{4.9}$$

for some constant C. Indeed, by a simple scaling argument, it is enough to consider the case when $d_L = 1$. Then we can write $W(x, y) = G(x, y)/2$, where $G(x, y) = -\log(\|z - w\|^2)$ has the property that $-\frac{i}{2\pi}\partial\bar{\partial}G(x, \cdot) = \delta_x - \omega_0$, where ω_0 is the normalized curvature of the Fubini–Study metric. Hence, the first variation of the functional $\mu \mapsto \int_X W\mu \otimes \mu$ on $\mathcal{P}(X)$ coincides with the first variation of $E(\mu)$ (formula (2.7)), which proves formula (4.9).

4.1. Conclusion of the proof of Theorem 4.1

Set $p = -t$ and observe that

$$\int_X e^{-pW(x,y)} \mu_0(y) = \int_X e^{-(t d_L \log \|x - y\| + \sum_i w_i \log \|x - p_i\|^2)} \, dV_X.$$

For any given $y \in X$, the function $e^{-c \log \|x-y\|^2}$ is locally integrable on X iff $c < 1$. Hence, the right-hand side above is integrable iff for any fixed index i

$$td_L/2 + w_i < 1, \quad \forall i.$$

But this condition holds for some $t > 1$ iff

$$d_L/2 + w_i < 1, \quad \forall i,$$

i.e., iff $1 - \sum w_j/2 + w_i < 1$ for all i, that is, $w_i < \sum_{j \neq i} w_j$, which is equivalent to the weight condition (4.2). Hence, if the weight condition holds, then by Theorem 4.4, the desired LDP follows.

Next, assume that the weight condition is violated. Without loss of generality we may assume that it is violated for the index $i = 1$, which equivalently means that

$$-d_L + 2(1 - w_1) = 0.$$

Set $B_R := \{\|x - p_1\| \leq R\}$. Since $e^{-\log \|x-y\|} \geq R^{-1}$ on B_R, we have

$$\int_{B_R^N} e^{W(x,y)} \mu_0(y) \geq (R^{-1})^{d_L N} \int_{B_R^N} \mu_0^{\otimes R}.$$

Using $\int_{|z| \leq R} e^{-w \log |z|^2} d(r^2) \wedge d\theta = \frac{1}{1-w}(R^2)^{1-w}$, we thus get

$$\int_{B_R} \mu_0 \geq \int e^{-(w_1 \log \|x-p_1\|^2)} \, dV_X \geq C(R^2)^{(1-w_1)}$$

for some constant independent of R. All in all, this means that

$$\left(\int_{B_R^N} e^{W(x,y)} \mu_0(y) \right)^{1/N} \geq CR^{-d_L + 2(1-w_1)} \geq CR^0 \geq C > 0.$$

But the right-hand side is independent of R. Hence, letting $R \to 0$ shows that the density $e^{W(x,y)}$ cannot be in $L^1(X^N \mu_0^{\otimes N})$, which means that $Z_{N,-1} = \infty$, as desired.

4.2. The case of a general divisor Δ

Now consider the case of general coefficients $w_i \in]-\infty, 1[$. By the previous theorem, $Z_{N,-1}$ diverges for large N, unless the weight condition (4.2) holds. But fixing any continuous metric $\|\cdot\|$ on L, we can consider the corresponding probability measures $\mu_{\Delta,\beta}^{(N)}$, defined by formula (2.2), which are well defined when $-\beta$ is sufficiently small.

Theorem 4.5. *$\mathcal{Z}_N(\beta) < \infty$ iff $\beta > -\gamma_N$, where*

$$\gamma_N = \frac{N-1}{N} 2 \frac{1 - \max_i w_i}{2 - \sum_i w_i}.$$

Moreover, if $\mathcal{Z}_N(\beta) < \infty$, then the law of the random variable δ_N on $(X^N, \mu_{\Delta,\beta}^{(N)})$ satisfies an LDP with speed N and rate functional $F_\beta - \inf_{\mathcal{P}(X)} F_\beta$.

Proof. First, consider the case when $\|\cdot\|$ is the metric $\|\cdot\|_{FS}$ induced from the Fubini–Study metric on $\mathcal{O}(1)$. Then we get, as above, that $\mu_\beta^{(N)} = \mu^{(N)}[p]$ for $p = \beta\frac{N-1}{N}$. Hence, by the argument in the beginning of the previous section, the integrability threshold is given by

$$\gamma_N = \frac{N-1}{N}\gamma, \quad \gamma = \sup\{t : td_L/2 + w_i < 1, \ \forall i\} = 2\frac{1-\max_i w_i}{2-\sum_i w_i},$$

and the LDP follows from the general LDP in Theorem 4.4. Finally, writing a general continuous metric $\|\cdot\|$ as $e^{-u/2}\|\cdot\|_{FS}$ for a continuous function u on X, we can express $\mu_\beta^{(N)} = \mu^{(N)}[p]$, where $\mu_0 = e^{-(\beta+1)u}dV$, and again apply Theorem 4.4. ■

As recalled in Section 2.3, any minimizer ω_β of F_β satisfies the twisted Kähler–Einstein equation (2.10) with ω_0 equal to the normalized curvature form of the metric $\|\cdot\|$ on L.

Remark 4.6. In the case when Δ is trivial (i.e., $w_i = 0$), the formula for γ_N in the previous theorem was shown in [46, Section 3], using a different algebro-geometric argument.

4.3. The zero-free hypothesis in the case of three points and the complex Selberg integral

We will next give an alternative proof of Theorem 4.1 in the case when $m = 3$ using the approach in Section 2.4.3. To simplify the notation, we will drop the subscript k in the notation N_k in formula (4.3). In other words, as our data we take a divisor Δ on $\mathbb{P}^1_{\mathbb{C}}$ and an integer N which is strictly greater than one (k can then be recovered from formula (4.3)). First, recall that, by Lemma 4.3, the normalizing constant $\mathcal{Z}_N$ – that we will write as $\mathcal{Z}_N(\Delta)$ to indicate the dependence on Δ – may be expressed by

$$\mathcal{Z}_N(\Delta) = \int_{\mathbb{C}^N}\Big(\prod_{i\neq j}|z_i - z_j|\Big)^{-\frac{d_L}{N-1}} \prod_{i\leq N, j\leq m}|z_i - p_j|^{-2w_i}\prod_i \frac{i}{2}dz_i\wedge d\bar{z}_i.$$

Now specialize to $m = 3$. Then we may, after perhaps applying an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, assume that the points p_1, p_2, and p_3 are given by the points 0, 1, and ∞. Hence,

$$\mathcal{Z}_N(\Delta) = \int_{\mathbb{C}^N}\Big(\prod_{i\neq j}|z_i - z_j|\Big)^{-\frac{d}{N-1}}\prod_i |z_i|^{-2w_0}\prod_i |z_i - 1|^{-2w_1}\prod_i \frac{i}{2}dz_i\wedge d\bar{z}_i,$$

$$d = 2 - (w_0 + w_1 + w_2).$$

This integral is known as the *complex Selberg integral* (when expressed in terms of the parameters w_0, w_1, and $d/(N-1)$). The original Selberg integral is the integral obtained by replacing $\mathbb{C}^N$ with $[0,1]^N$ and generalizes Euler's classical Beta-function to $N > 1$ (see the survey [44]). Its complex version above seems to first have appeared in the CFT, in the context of minimal CFTs, where it is known as one of the *Dotsenko–Fateev integrals* [40] (an equivalent formula was also established in [2], expressed in terms of the original Selberg integral). By [40, formula (B.9)], the integral $\mathcal{Z}_N(\Delta)$ is explicitly given by the following remarkable formula involving the classical Γ-function:

$$\mathcal{Z}_N(\Delta) = N!\left(\frac{\pi}{l\left(-\frac{1}{2}\frac{d}{N-1}\right)}\right)^N \prod_{j=1}^{N} \frac{l\left(-\frac{j}{2}\frac{d}{N-1}\right)}{l\left(w_1+\frac{j}{2}\frac{d}{N-1}\right)l\left(w_2+\frac{j}{2}\frac{d}{N-1}\right)l\left(w_3+\frac{j}{2}\frac{d}{N-1}\right)},$$
$$l(x) := \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{4.10}$$

Remark 4.7. The integral $\mathcal{Z}_N(\Delta)$ also appears in connection to the DOZZ formula of Dorn–Otto and Zamolodchikov–Zamolodchikov for the 3-point structure constants $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ in Liouville CFT, which has recently been given a rigorous proof in [63] (see also the exposition in [80, Section 2.3]). A general formula for Selberg-type integrals over a local field F of characteristic zero was recently established in [45] (specializing to Selberg's original integral when $F = \mathbb{R}_{>0}$ and its complex generalization when $F = \mathbb{C}$).

We next observe that for any given $\varepsilon \in]0,1[$, $\mathcal{Z}_N(\Delta)$ is zero-free in the convex tube domain Ω in $\mathbb{C}^3$ defined by

$$\Omega = \{\boldsymbol{w} \in \mathbb{C}^3 : \Re w_i < 1,\ \Re w_1 + \Re w_2 + \Re w_3 > 0\}. \tag{4.11}$$

Indeed, by formula (4.10),

$$\mathcal{Z}_N(\Delta) = N!\pi^N\left(\frac{\Gamma\left(1+\frac{1}{2}\frac{d}{N-1}\right)}{\Gamma\left(-\frac{1}{2}\frac{d}{N-1}\right)}\right)^N \prod_{j=1}^{N}\left(\frac{\Gamma\left(-\frac{j}{2}\frac{d}{N-1}\right)}{\Gamma\left(1+\frac{j}{2}\frac{d}{N-1}\right)}\frac{\Gamma\left(1-w_1-\frac{j}{2}\frac{d}{N-1}\right)}{\Gamma\left(w_1+\frac{j}{2}\frac{d}{N-1}\right)}\cdots\right),$$

where the dots indicate similar factors obtained by replacing w_1 with w_2 and w_3. It is a classical fact that $\Gamma(x)$ is a meromorphic zero-free function of $x \in \mathbb{C}$ with poles at $0, -1, -2, \dots$. Hence, the zeros of $\mathcal{Z}_N(\Delta)$ can only come from the poles of the Gamma factors appearing in the denominators above. First, consider the case when $d \neq 0$. Since $N \geq 2$ and $2 > \Re d$, the factor $\Gamma(-\frac{1}{2}\frac{d}{N-1})$ has no poles in Ω. Similarly, since $\Re d > -1$, the factor $\Gamma(1+\frac{j}{2}\frac{d}{N-1})$ has no poles and since $\Re w_1 < 1$, the factor $\Gamma(w_1 + \frac{j}{2}\frac{d}{N-1})$ has no poles in Ω (using that, for $\boldsymbol{w} \in \mathbb{R}^3$, when $d < 0$, $w_1 + \frac{j}{2}\frac{d}{N-1}$

is minimal when $j = N$ and $N = 2$, i.e., the minimum is $w_1 + d = 2 - w_1 - w_2 > 0$) and likewise when w_1 is replaced by w_2 and w_3. Finally, when $d = 0$, we get

$$\mathcal{Z}_N(\Delta) = N!\pi^N \left(\frac{\Gamma(1 - w_1)}{\Gamma(w_1)} \cdots \right)^N$$

which is non-zero, since $\Re w_i > 0$ (and thus the denominator above has no poles).

This argument also reveals that the "first" negative poles of $\mathcal{Z}_N(\Delta)$ appear when $1 - x = 0$, for $x = w + td/2$ for $w \in \{w_0, w_1, w_2\}$ and $t = i/(N-1)$ for $i = 1, \ldots, N$, i.e., when $w + td/2 = 1$. In particular, if $w + td/2 > 1$ for the maximal value of t, i.e., for $t = N/(N-1)$, then $\mathcal{Z}_N(\Delta) < \infty$. This is precisely the condition for the finiteness of $\mathcal{Z}_N(\Delta)$ that came up in the beginning of Section 4.1 which is equivalent to the weight condition (4.2) for w real. The explicit formula (4.10) for $\mathcal{Z}_N(\Delta)$ then also gives

$$\mathcal{Z}_N(\Delta) \leq C^N.$$

4.3.1. Proving Theorem 4.1 by deforming Δ in the case when $m = 3$. We finally explain how to give an alternative proof of Theorem 4.1 in the case $m = 3$ using the zero-free property and the bound on $\mathcal{Z}_N(\Delta)$ established in the previous section, combined with the approach discussed in Section 2.4.3. In this case, the affine space $\mathcal{A}$ of all "admissible" $(s, \boldsymbol{w})$ is defined by the condition

$$d_L^{-1}\left(2 - \left(\sum_{i=1}^{m} w_i\right)\right) = s,$$

where, as before, d_L denotes the degree of the anti-canonical line bundle of the given log Fano variety (whose weight vector is denoted by $\boldsymbol{w}_0$ in Section 2.4.3). In particular, since we consider the case when $m = 3$, we get $s < 0$ by choosing a real weight vector $\boldsymbol{w}_1$ with components sufficiently close to 1 (which can be done as soon as $m > 2$) and, in particular, $\boldsymbol{w}_1 \in \Omega$ (where Ω is the domain in formula (4.11)). Since the components $p_1, \ldots, p_m$ of Δ are, trivially, non-singular and mutually non-intersecting, the implicit function theorem does apply. Hence, so does the approach in Section 2.4.3.

5. Speculations on the strong zero-free hypothesis, L-functions, and arithmetic geometry

In this last section, we discuss some intriguing relations between the strong zero-free hypothesis for the partition functions $\mathcal{Z}_N(\beta)$ on Fano manifolds introduced in Section 2.4.1 and the zero-free property of the representation-theoretic (automorphic) local zeta functions $L_p(s)$ appearing in the Langlands program [65]. Conjecturally, the latter zeta functions are related to arithmetic/motivic L-functions [66].

First, recall that given a reductive group G over a global field F together with automorphic representations π and ρ of G and its Langlands dual, respectively, one attaches a local L-function $L_p(s)$ to any place (prime) p of F. By definition, the places p of F correspond to multiplicative (normalized) absolute value $|\cdot|_p$ on F. In the case when $|\cdot|_p$ is non-Archimedean, the local L-function $L_p(s)$ is defined as the inverse of a characteristic polynomial attached to the induced representation of G_p and thus $L_p(s)$ is automatically zero-free. For Archimedean $|\cdot|_p$, the local L-function $L_p(s)$ may be defined as an appropriate product of Γ-functions and is thus also zero-free; see [59, Section 4] for the case $G = \mathrm{GL}(N, \mathbb{C})$ and the relation to the local Langlands correspondence. Conjecturally, any local automorphic L-function $L_p(s)$ is a product of the *standard L-functions* corresponding to the case when $G = \mathrm{GL}(N, F_p)$ and ρ is the standard representation of $\mathrm{GL}(N, \mathbb{C})$ [65] (generalizing the local versions of the classical Hecke L-functions, e.g. the Riemann zeta function when $N = 1$).

5.1. The "minimal" partition function on $\mathbb{P}^n_{\mathbb{C}}$ as a standard local L-function

In the standard case, it was shown in [51] (generalizing Tate's thesis [78] to $N > 1$) that $L_p(s)$ may – for any given admissible irreducible representation π – be realized as a "zeta integral":

$$L_p(s) = \int_{\mathrm{GL}(N,F_p)} \left|\det(g)\right|_p^s \mu_p(g) \tag{5.1}$$

for a distinguished measure μ_p on $\mathrm{GL}(F_p, N)$, depending on π, which is absolutely continuous with respect to Haar measure. As a consequence, for such particular measures $\mu_p(g)$ the zeta integral above is zero-free (since $L_p(s)$ is).

To see the relation to the partition functions $\mathcal{Z}_N(\beta)$ for Fano manifolds, first note that we may, in the zeta integral above, replace the group $\mathrm{GL}(F_p, N)$ with the algebra $\mathrm{Mat}(F_p, N)$ of $N \times N$ matrices A with coefficients in F_p (since μ_P puts no mass on the complement of $\mathrm{GL}(F_p, N)$ in $M(F_p, N)$). Then, after a suitable shift, $s \to s + \lambda$, the measure μ_p is of the form

$$\mu_p = f_\pi \Phi dA,$$

where dA is the additive Haar measure on $\mathrm{Mat}(F_p, N)$, the function f_π is an appropriate matrix element of π, and Φ is a suitable Schwartz–Bruhat function on $\mathrm{Mat}(F_p, N)$. In the "unramified case", f_π is the spherical function attached to π and Φ is its own Fourier transform [51, Proposition 6.12]. In case when p is non-Archimedean, this means that Φ is the characteristic function of $M(O_p, N)$, where O_p denotes the ring of integers of F_p, while in the Archimedean case, Φ is the Gaussian (see [54] for the case $F_p = \mathbb{C}$). Now, when p is taken to be the standard (squared) Archimedean absolute value on $\mathbb{C} (= F_p)$, with π the trivial representation, we get

$$\mathcal{Z}_N(\beta) = c_n \big(\Gamma(s + n + 1)\big)^{-(n+1)} L_p(s), \quad s = \beta(n+1), \tag{5.2}$$

where $\mathcal{Z}_N(\beta)$ denotes the partition function for the standard Kähler–Einstein metric on the Fano manifold $\mathbb{P}^n_{\mathbb{C}}$ with N the minimal one (i.e., $N = n + 1$) considered in Example 2.9. Indeed, this follows directly from combining formula (5.1) (for $f_\pi = 1$) with formula (A.5) for $\mathcal{Z}_N(\beta)$ in the appendix. Note that the first factor in the right-hand side above is non-vanishing when $\Re\beta > -1$ and thus the zero-free property of $\mathcal{Z}_N(\beta)$ in the strip $\Re\beta > -1$ can be attributed to the zero-free property of the corresponding local L-function $L_p(s)$.

5.2. Zeta integrals associated to Calabi–Yau subvarieties of Mat(N_k, $\mathbb{C}$)

It would be interesting to compute $\mathcal{Z}_{N_k}(\beta)$ in more examples to check if it can be expressed as products (and quotients) of Γ-function and related to local Archimedean L-functions as above. For example, if a reductive group G acts holomorphically on X (e.g. if X is a flag variety), one might be able to exploit that the section $\det S^{(k)}$ over X^{N_k} is invariant under the diagonal action of G on X^{N_k}, up to multiplication by the determinant of the induced G-action on $H^0(X, -kK_X)$.

For a general Fano manifold X and N_k, it seems, however, unlikely that $\mathcal{Z}_{N_k}(\beta)$ can be related to an automorphic local L-function. Anyhow, as next explained the integral $\mathcal{Z}_{N_k}(\beta)$ can be expressed in terms of an integral over a Calabi–Yau subvariety of Mat$(N_k, \mathbb{C})$, which has some intriguing structural similarities with the zeta integral for the standard L-function $L_p(s)$ in formula (5.1). We start by lifting the integral $\mathcal{Z}_{N_k}(\beta)$ to an integral where the projective variety X is replaced by the affine variety Y_k of dimension $n+1$ obtained by blowing down of the zero-section in the total space of the line bundle $-kK_X \to X$. To this end, first note that the standard $\mathbb{C}^*$-action on $-kK_X$ induces a $\mathbb{C}^*$-action on the affine variety Y_k with a unique fixed point y_0, i.e., Y_k can be viewed as an affine cone over X:

$$X \simeq \big(Y_k - \{y_0\}\big)/\mathbb{C}^*.$$

On the affine variety Y_k, there is a unique $\mathbb{C}^*$-equivariant holomorphic top form Ω (modulo a multiplicative constant). The Kähler–Einstein metric ω_{KE} on X corresponds to a conical Calabi–Yau metric ω_{CY} on Y_k, i.e., a Ricci-flat Kähler metric with a conical singularity at y_0 [49]. Denote by r the distance to the fixed point y_0 in Y_k with respect to the Calabi–Yau metric ω_{CY}. We may then express

$$\mathcal{Z}_{N_k}(\beta) = c_n\Big(\Gamma\big((n+1)\beta + n + 1\big)\Big)^{-N_k}\tilde{\mathcal{Z}}_{N_k}(\beta),$$
$$\tilde{\mathcal{Z}}_{N_k}(\beta) := \int_{Y_k^{N_k}} |\det \Psi^{(k)}|^{2\beta/k}(e^{-r^2}\Omega \wedge \bar{\Omega})^{\otimes N_k},$$

where $\Psi^{(k)}$ is the holomorphic function on $Y_k^{N_k}$ corresponding to the section $\det S^{(k)}$ of $-kK_{X^{N_k}}$ and c_n is a (computable) positive constant c_n. This is shown essentially as in the proof of Proposition A.3 in the appendix. Next, assume that k is sufficiently large to ensure that $-kK_X$ is very ample. Then one obtains a holomorphic $(\mathbb{C}^*)^{N_k}$-equivariant embedding

$$Y_k^{N_k} \to \mathrm{Mat}(N_k, \mathbb{C}), \quad (y_1, \dots, y_{N_k}) \mapsto \big(\mathbf{\Psi}^{(k)}(y_1), \dots, \mathbf{\Psi}^{(k)}(y_{N_k})\big),$$

where $\mathbf{\Psi}^{(k)}(y)$ denotes the N_k-tuple of holomorphic functions $\psi_1^{(k)}, \dots, \psi_{N_k}^{(k)}$ on Y_k corresponding to the fixed bases in $H^0(X, -kK_X)$. In geometric terms, the embedding above is just the embedding induced from the Kodaira embedding of X in the projectivization of $H^0(X, -kK_X)^*$. Denoting by $\mathcal{Y}_k$ the image of $Y_k^{N_k}$ in $\mathrm{Mat}(N_k, \mathbb{C})$, we can thus express $\widetilde{\mathcal{Z}}_{N_k}(\beta)$ as a matrix integral:

$$\widetilde{\mathcal{Z}}_{N_k}(\beta) := \int_{\mathcal{Y}_k \Subset \mathrm{Mat}(N_k, \mathbb{C})} |\det A|^{2\beta/k} e^{-r^2} \Omega \wedge \bar{\Omega},$$

where now r denotes the distance to the origin in $\mathrm{Mat}(N_k, \mathbb{C})$ with respect to the Calabi–Yau metric on the subvariety $\mathcal{Y}_k$ and Ω denotes the equivariant holomorphic top form on $\mathcal{Y}_k$ (which can be viewed as a Poincaré-type residue of the standard holomorphic top form on $\mathrm{Mat}(N_k, \mathbb{C})$ along $\mathcal{Y}_k$). This matrix integral is reminiscent of the integral expression (5.1) for the local L-functions $L_p(s)$, if μ_p is taken to be the measure on $\mathrm{Mat}(N_k, \mathbb{C})$ induced by pairing of $\Omega \wedge \bar{\Omega}$ with the subvariety $\mathcal{Y}_k$, weighted by the Gaussian-type factor e^{-r^2} (and $s := \beta/k$). In view of this structural similarity, it is tempting to speculate on a very strong zero-free hypothesis, saying that, in general, the "lifted" partition function $\widetilde{\mathcal{Z}}_{N_k}(\beta)$ is zero-free on all of $\mathbb{C}$, when viewed as a meromorphic function.

Remark 5.1. The same considerations apply when X is a Fano orbifold if K_X is replaced by the orbifold canonical line bundle (coinciding with $-K_X + \Delta$ as $\mathbb{Q}$-line bundle). Then the natural projection from $Y_k - \{y_0\}$ to X is a submersion over the complement of the branching divisor Δ and the orbifold Kähler–Einstein metric on X corresponds to a bona fide Calabi–Yau metric on $Y_k - \{y_0\}$ [49].

One further piece of evidence for the very strong form of the zero-free hypothesis (complementing the "minimal" case on $\mathbb{P}^n$ appearing in Proposition A.3) is provided by the case when $X = \mathbb{P}^1$ and $k = 1$, i.e., $N_k = 3$ (which is the case next to minimal dimension, $N_k = n + 1$). Then identifying $-K_X$ with $2\mathcal{O}(1)$ and $\det S^{(1)}$ with the Vandermonde determinant $D^{(3)}$ on $\mathbb{C}^3$ (as in Lemma 4.3) and using that the Kähler–Einstein metric is explicitly given by the Fubini–Study metric (formula (4.7)), $\mathcal{Z}_{N_k}(\beta)$ may be expressed as

$$\mathcal{Z}_{N_k}(\beta) = \int_{\mathbb{C}^3} \prod_{i<j\leq 3} |z_i - z_j|^{2\beta} \prod_{i<j\leq 3} \big(1 + |z_i|^2\big)^{-(2\beta+2)},$$

integrating with respect to Lebesgue measure. Applying the formula in [22, Theorem 1] (to $\sigma_i = \nu_i = \beta + 1$), which originally appeared in the CFT, thus yields

$$\mathcal{Z}_{N_k}(\beta) = \pi^3\big(\Gamma(2\beta+2)\big)^{-3}\Gamma(3\beta+2)\Gamma(\beta+1)^3. \tag{5.3}$$

This means that the meromorphic function $\widetilde{\mathcal{Z}}_{N_k}(\beta)$ is a product of four Gamma functions and thus zero-free on all of $\mathbb{C}$. The elegant proof in [22] leverages the diagonal action of $\mathrm{GL}(N_k, \mathbb{C})$ on X^{N_k} alluded to above (following the corresponding real case considered in [21] in the context of automorphic triple products).

The general case on $X = \mathbb{P}^1$, when $N_k > 3$, appears to be open. However, a similar formula does hold for any N_k when X is replaced by its *real* points, i.e., when $\mathbb{P}^1_{\mathbb{C}}$ is replaced by $\mathbb{P}^1_{\mathbb{R}}$. Then the role of $\mathcal{Z}_{N_k}(\beta)$ is played by

$$\begin{aligned}\mathcal{Z}_{N_k}(\beta)_{\mathbb{R}} &:= \int_{(\mathbb{P}^1_{\mathbb{R}})^{N_k}} \|\det S^{(k)}\|^{\beta/k}\, dV^{\otimes N_k} \\ &= \int_{(S^1)^{N_k}} \prod_{i<j\leq N_k} |z_i - z_j|^{\frac{2\beta}{N_k-1}}\, d\theta^{\otimes N_k}, \quad N_k = 2k+1,\end{aligned}$$

where $\|\cdot\|$ denotes the Fubini–Study metric and dV denotes the corresponding volume form on $(\mathbb{P}^1_{\mathbb{R}})$. In the second equality above, we have exploited that the integrand is invariant under the diagonal action of $\mathrm{SU}(2)$ to replace the real points $\mathbb{P}^1_{\mathbb{R}}$ of $\mathbb{P}^1_{\mathbb{C}}$ with the unit-circle S^1 in $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$. The latter integral over $(S^1)^{N_k}$ coincides with the partition function for the 2D Coulomb gas confined to $S^1 \subset \mathbb{C}$ at inverse temperature $2\beta/(N_k-1)$ (known as the circular ensemble). Applying [44, formula (1.12)] (originally conjectured by Dyson and established by Gunson and Wilson) thus yields

$$\mathcal{Z}_{N_k}(\beta)_{\mathbb{R}} = (2\pi)^{N_k}\Gamma\Big(1+\beta\frac{1}{N_k-1}\Big)^{-N_k}\Gamma\Big(1+\beta\frac{N_k}{N_k-1}\Big), \quad N_k = 2k+1.$$

This formula reveals that the real analog $\mathcal{Z}_{N_k}(\beta)_{\mathbb{R}}$ of the partition function on $\mathbb{P}^1_{\mathbb{C}}$ does satisfy the strong zero-free hypothesis. This real analog may, from the point of view of localization, be obtained by replacing the squared absolute value $|\cdot|^2_{\mathbb{C}}$ corresponding to the complex Archimedean place of the global field $\mathbb{Q}$ with the absolute value $|\cdot|_{\mathbb{R}}$ corresponding to the real Archimedean place of $\mathbb{Q}$. The extension to non-Archimedean places is discussed in Section 5.4. But first we start by a brief detour on arithmetic aspects of the partition function.

5.3. Invariants of arithmetical Fano varieties

Let $\mathcal{X}$ be an arithmetic variety of dimension $n+1$ (i.e., a projective scheme flat over $\mathbb{Z}$, $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$) such that the corresponding n-dimensional complex variety X (i.e., the complexification of the generic fiber $X_{\mathbb{Q}}$ of $\mathcal{X}$) is Fano. Assume that $\mathcal{X}$

is endowed with a relatively nef line bundle $\mathcal{L}$ such that the induced line bundle on X equals $-K_X$. Then $(\mathcal{X}, \mathcal{L})$ induces a section $\det S^{(k)}$ of $-kK_{X^{N_k}} \to X^{N_k}$ which is uniquely determined up to multiplication by ± 1. Indeed, $(\mathcal{X}, \mathcal{L})$ induces a lattice $H^0(\mathcal{X}, k\mathcal{L})$ of integral sections in $H^0(X, -kK_X)$ and $\det S^{(k)}$ may be defined as in formula (1.3) with respect to any basis in $H^0(\mathcal{X}, k\mathcal{L})$ (any two such bases are related by a matrix with integral coefficients, which thus has determinant equal to ± 1). As a consequence, the corresponding partition function $\mathcal{Z}_{N_k}(\beta)$ only depends on $(\mathcal{X}, \mathcal{L})$ and the choice of a metric $\|\cdot\|$ on $-K_X$ (and is independent of the metric at $\beta = -1$). In fact, the explicit expression for $\mathcal{Z}_{N_k}(\beta)$ appearing in Proposition A.3 – related to a local L-function in formula (5.2) – was computed with respect to the standard integral model $(\mathcal{X}, \mathcal{L})$ for $(\mathbb{P}^n, \mathcal{O}(1))$ (where $H^0(\mathcal{X}, k\mathcal{L})$ is the lattice spanned by the sections defined by multinomials). In the light of the speculations in the previous section, this appears to fit well with the arithmetical side of the Langlands program.

In particular, taking $\beta = -1$ yields an invariant $\mathcal{Z}_{N_k}$ of $(\mathcal{X}, \mathcal{L})$ (which is finite iff X is Gibbs stable at level k). The following conjecture relates the arithmetic invariants $\mathcal{Z}_{N_k}$ to the arithmetic intersection numbers introduced by Gillet–Soulé in the context of Arakelov geometry (see the book [77]).

Conjecture 5.2. *Let $(\mathcal{X}, \mathcal{L})$ be an arithmetic variety as above and assume that the corresponding Fano manifold X admits a unique Kähler–Einstein metric, whose volume form is denoted by dV_{KE}, normalized to have unit total volume. Then, as $k \to \infty$, $\frac{(n+1)!}{k^n} \log \mathcal{Z}_{N_k}$ converges towards the $(n+1)$-fold arithmetic self-intersection number of the line bundle $\mathcal{L}$, metrized by dV_{KE}.*

In fact, using the arithmetic Hilbert–Samuel theorem in [83, Theorem 1.4] (generalizing the relative ample case in [50]), this conjecture is equivalent to the convergence of the partition function appearing in Theorem 2.4, defined with respect to any basis of $H^0(X, kK_X)$ which is orthonormal with respect to the Hermitian product induced by a Kähler metric on X. Thus, by Theorem 2.6, in order to establish the conjecture it would, for example, be enough to show that the lifted partition function $\widetilde{\mathcal{Z}}_{N_k}(\beta)$ may be expressed as a product of $O(N_k)$ shifted Γ-functions all of whose poles are located in the region where $\Re\beta < -1 - \varepsilon$ for some $\varepsilon > 0$.

Remark 5.3. Other (polarized) arithmetic varieties on arithmetic varieties $\mathcal{X}$, endowed with a relatively ample line bundle $\mathcal{L}$, are introduced in [23, 84] (which are finite precisely when $(X, k\mathcal{L})$ is Chow stable) and related to constant scalar curvature metrics in [74].

The analog of Conjecture 5.2 does hold when $-K_X$ is replaced by K_X (assumed ample) and $\log \mathcal{Z}_{N_k}$ is replaced by the arithmetic invariant $-\log \mathcal{Z}_{N_k}$ (as follows from combining the convergence of $\mathcal{Z}_{N_k}(1)$ in Theorem 2.1 with the arithmetic Hilbert–Samuel theorem).

5.4. Extension to non-Archimedean places

In view of the connections to local L-functions, L_p at the (complex) Archimedean place p, exhibited in Section 5.1, one may wonder if the probabilistic setup can be extended to non-Archimedean places p. The case of the trivial place is discussed in (5.1), in connection to Gibbs stability. What follows are some speculations on the case of non-trivial non-Archimedean places p, inspired by the adelic geometric setup in [29], where geometric Igusa local zeta functions are studied (see Section A.2).

Let X be a non-singular variety defined over $\mathbb{Q}$ and first consider the case when $K_{X(\mathbb{Q})}$ is ample. Given a non-trivial non-Archimedean place p (i.e., a prime number), denote by $X(\mathbb{Q}_p)$ the projective variety over the corresponding p-adic local field $\mathbb{Q}_p$ (the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$), which comes with the structure of a $\mathbb{Q}_p$-analytic manifold. By general principles, any continuous metric on $K_{X(\mathbb{Q}_p)}$ induces a measure on $X(\mathbb{Q}_p)$, which is absolutely continuous with respect to the local Haar measures [29, Section 2.1]. In particular, a section s_k of $kK_{X(\mathbb{Q}_p)}$ induces a measure on $X(\mathbb{Q}_p)$, whose local density may be symbolically expressed as $|s_k|_p^{1/k}$.[4] Hence, replacing the squared Archimedean absolute value appearing in formula (1.2) with $|\cdot|_p$, one arrives at a symmetric probability measure $\mu_p^{(N_k)}$ on $X(\mathbb{Q}_p)^{N_k}$. This construction thus yields a canonical random point process on $X(\mathbb{Q}_p)$. Accordingly, it seems natural to ask if the convergence in Theorem 1.1 can be extended to this non-Archimedean setup, if dV_{KE} is replaced by an appropriate measure $dV_{KE,p}$ on $X(\mathbb{Q}_p)$. In analogy with the Archimedean setup, the measure $dV_{KE,p}$ should be characterized as the unique minimizer of a free-energy type functional F_1 on the space of probability measure μ on $X(\mathbb{Q}_p)$ of the form

$$F_1(\mu) = E(\mu) + \mathrm{Ent}(\mu), \tag{5.4}$$

where $\mathrm{Ent}(\mu)$ denotes the entropy of the measure μ relative to a fixed measure on $X(\mathbb{Q}_p)$, absolutely continuous with respect to the local Haar measure and $E(\mu)$ is a non-Archimedean analog of the energy discussed in Section 2.2. In particular, $dV_{KE,p}$ is then absolutely continuous with respect to the local Haar measure.

Ideally, one might hope that the collection of metrics on $-K_{X(\mathbb{Q}_p)}$ defined by $dV_{KE,p}$, as p ranges over all primes p, is induced by some model $(\mathcal{X}, \mathcal{L})$ for $(X, K_{X(\mathbb{Q})})$ over $\mathbb{Z}$, away from primes p with bad reduction (cf. [29, Section 2.2.3]). This would, loosely speaking, yield a probabilistic construction of a "canonical" integral model attached to $X(\mathbb{Q})$. This is in line with the analogy between the Kähler–Einstein condition of a metric on $X(\mathbb{C})$ (i.e., at $p = \infty$) and the minimality condition of an integral model for $X(\mathbb{Q})$ put forth in [72] and further studied in [74].

[4] One can also consider a field extension F_p of $\mathbb{Q}_p$ and get a measure on the corresponding analytic manifolds $X(F_p)$, as in [56], but here $F_p = \mathbb{Q}_p$, for simplicity.

Remark 5.4. Embedding $X(\mathbb{Q}_p)$ in its Berkovich analytification X_p^{an} and pushing forward a measure μ on $X(\mathbb{Q}_p)$ to X_p^{an}, the functional on $C^0(X_p^{an})$ defined as the Legendre–Fenchel transform of the functional $E(\mu)$ in formula (5.4) should, in analogy to the Archimedean setup [4, 17], be given by the primitive of the non-Archimedean Monge–Ampère operator introduced in [28, 62]. The primitive in question is called the "energy functional" in [24]. In the case of a trivial non-Archimedean absolute value, such an energy $E(\mu)$ appears in [25, formula (6.1)] and plays an important role in the non-Archimedean approach to K-stability.

Similar considerations apply in the Fano case. In particular, to a given metric on $-K_{X(\mathbb{Q}_p)}$ one can associate a lifted partition function $\widetilde{\mathcal{Z}}_{N_k,p}(\beta)$. By general principles [29, Section 4.1], this defines a meromorphic function on $\mathbb{C}$ which in the light of Section 5.1 plays the role of the local L-functions L_p in the Langlands program. More precisely, in order to render $\widetilde{\mathcal{Z}}_{N_k,p}(\beta)$ as canonical as possible, the metric on $-K_{X(\mathbb{Q}_p)}$ should be taken to be defined by a "canonical" integral model $(\mathcal{X}, \mathcal{L})$ for $(X(\mathbb{Q}), -K_{(\mathbb{Q})})$ and $\det S^{(k)}$ should be defined with respect to any basis in $H^0(\mathcal{X}, \mathcal{L})$ (as in Section 5.3). Finally, one could then attempt to define a global L-type function as a Euler product of $\widetilde{\mathcal{Z}}_{N_k,p}(\beta)$ over all p, generalizing the Riemann zeta function.

A. Log canonical thresholds and Archimedean zeta functions

In this appendix, we recall the basic notions of lct's, α-invariants, and their connections to Archimedean zeta functions, which are as essentially well known. We conclude with a proof of the formula appearing in Example 2.9.

A.1. Log canonical thresholds

Let X be a compact complex manifold.

A.1.1. The lct of a divisor on X. By definition, an $\mathbb{R}$-divisor D is a finite formal sum of irreducible analytic subvarieties $D_i \subset X$ of complex codimension one:

$$D = \sum_{i=1}^{m} c_i D_i, \quad c_i \in \mathbb{R}.$$

The *log canonical threshold* $\mathrm{lct}_X(D)$ of an $\mathbb{R}$-divisor D has various algebro-geometric formulations (using discrepancies, valuations, multiplier ideal sheaves, etc.) [60], but for the purposes of the present paper, it will be enough to recall its analytic definition as an integrability threshold. First, consider the case when the coefficients D are in $\mathbb{Z}_+$. This equivalently means that there exists a holomorphic line bundle $L_D \to X$ and a holomorphic section s_D such that D is cut-out by s_D, including multiplicities, i.e., s_D vanishes to order c_i along the irreducible varieties D_i. The lct may then be defined as the following integrability index:

$$\mathrm{lct}_X(D) := \sup_{\gamma>0}\left\{\gamma : \int_X \|s_D\|^{-2\gamma}\, dV < \infty\right\}, \tag{A.1}$$

in terms of any Hermitian metric $\|\cdot\|$ on L and volume form dV on X. This definition first extends to the case when $c_i \in \mathbb{Z}$, if s_D is viewed as a meromorphic section, so that the negative coefficients correspond to the poles of s_D, and then to $c_i \in \mathbb{Q}$ by viewing s_D as a multi-valued holomorphic section and noting that $\|s\|$ is still a well-defined function on X (taking values in $[0, \infty]$). Finally, the definition extends, by continuity, to any $\mathbb{R}$-divisor D or, alternatively, by noting that the function $\|s_D\|$ is still well defined (and can be viewed as the norm on an $\mathbb{R}$-line bundle, i.e., a formal sum of the line bundles L_{D_i}).

A.1.2. The lct of a divisor on (X, Δ). More generally, if Δ is a given $\mathbb{Q}$-divisor of X, then the lct of D relative to the *log pair* (X, Δ) [30] may be analytically defined as

$$\mathrm{lct}_{(X,\Delta)}(D) := \sup_{\gamma>0}\left\{\gamma : \int_X \|s\|^{-2\gamma}\, dV_\Delta < \infty\right\},$$

where dV_Δ is a measure on X with singularities encoded by Δ, i.e., locally dV_Δ may be expressed as

$$dV_\Delta = \|s_\Delta\|^{-2} dV_X$$

for some bona fide volume form dV_X on X and metric $\|\cdot\|$ on the $\mathbb{Q}$-line bundle with multivalued holomorphic section s_Δ corresponding to Δ. More generally, as in the previous section, Δ may be taken to be an $\mathbb{R}$-divisor on X.

A.1.3. The lct of a line bundle L and the α-invariant. The log canonical threshold $\mathrm{lct}_X(L)$ of a line bundle $L \to X$ is now defined by

$$\mathrm{lct}_X(L) := \inf_{D\sim L} \mathrm{lct}_X(D),$$

where D ranges over the divisors attached to all the many-valued holomorphic section s of L. By [35], this coincides with Tian's α-invariant of L:

$$\alpha(L) := \sup_{\gamma>0}\left\{\gamma : \exists C \int_X e^{-\gamma(\phi-\phi_0)}\, dV \leq C \ \forall \phi \in \mathcal{H}(L)\right\}, \tag{A.2}$$

where $\mathcal{H}(L)$ denotes the space of all metrics on L with positive curvature and ϕ_0 denotes a fixed smooth reference metric on L using additive notation for metrics so that $\phi - \phi_0$ defines a function on X. More generally, the log canonical threshold $\mathrm{lct}_{(X,\Delta)}(L)$ of a line bundle $L \to X$ with respect to a log pair (X, Δ) [30] is defined by

$$\mathrm{lct}_{(X,\Delta)}(L) := \inf_{D\sim L} \mathrm{lct}_{(X,\Delta)}(D).$$

This coincides with the α-invariant defined with respect to the log pair (X, Δ) obtained by replacing dV in formula (A.2) with $dV_{(X,\Delta)}$, as shown in the appendix of [4].

A.2. Archimedean zeta functions

Let μ_0 be a measure on $\mathbb{C}^n$ with compact support and $\psi \in L^1(\mu_0)$. Then we may define the integrability threshold $\mathrm{lct}_{\mu_0}(\psi)$ as in formula (A.1), by replacing $\log \|s\|^2$ with ψ and dV by μ_0. The integral

$$Z(\beta) = \int_{\mathbb{C}^n} e^{2\beta\psi}\mu_0,$$

defines a holomorphic function on the strip $\{\Re\beta > -\mathrm{lct}_{\mu_0}(\psi)\}$ in $\mathbb{C}$ (using that, in this strip, $e^{\beta\psi} \in L^1(\mu_0)$ and that the integrand is holomorphic in β). In the case when $\psi = \log|f|^2$ for f holomorphic, or more precisely,

$$Z(\beta) = \int_{\mathbb{C}^n} |f|^{2\beta}\Phi\, dx, \tag{A.3}$$

for a Schwartz function Φ, the holomorphic function $Z(\beta)$ on the strip $\{\Re\beta > -\mathrm{lct}_{\mu_0}(\psi)\}$ extends to a meromorphic function in $\mathbb{C}$, whose poles are located at the negative real axes.

Remark A.1. This follows from classical results of Atiyah and Bernstein, extended by Igusa to a more general setting of zeta function attached to polynomials defined over local fields [53]. Briefly, meromorphic functions $Z(\beta)$ of the form (A.3) can be defined more generally by replacing $\mathbb{C}$ and its standard Archimedean absolute value $|\cdot|$ with any local field F, endowed with an absolute value $|\cdot|_F$. Such functions $Z(\beta)$ are usually called *Igusa local zeta function* [53] and thus $Z(\beta)$ in formula (A.3) is called an Igusa Archimedean zeta function or simply an *Archimedean zeta function* in the literature on algebraic and arithmetic geometry. In the case when f is a polynomial with integer coefficients and F is the p-adic field, $F = \mathbb{Q}_p$, the meromorphic function $Z(\beta)$ encodes the number of solutions of the equation $f(x_1, \dots, x_n) = 0$, modulo powers of p, when Φ is taken as the characteristic function of the n-fold product of the ring $\mathbb{Z}_p$ of integers of $\mathbb{Q}_p$,

Similarly, given a holomorphic section s of a line bundle $L \to X$ over a compact complex manifold, a metric $\|\cdot\|$ on L and a singular volume form dV_Δ associated to a log pair (X, Δ)

$$\mathcal{Z}(\beta) := \int_X \|s\|^{2\beta}\, dV_{(X,\Delta)} \tag{A.4}$$

defines a holomorphic function in the strip $\{\Re\beta > -\mathrm{lct}_{(X,\Delta)}(D)\}$ in $\mathbb{C}$, where D denotes the divisor cut out by the section s. More precisely, the function $\mathcal{Z}(\beta)$ extends to a meromorphic function on $\mathbb{C}$, whose poles are located on the negative real axes (using a partition of unity to reduce to the case of $X = \mathbb{C}^n$). The first negative pole is precisely $-\operatorname{lct}_{(X,\Delta)}(D)$.

Remark A.2. Functions of the form (A.4) have previously appeared in a general adelic setup [29] (containing both the Archimedean and the p-adic setup), motivated by number theory and arithmetic geometry on log Fano varieties.

In the present probabilistic setup on Fano manifolds, discussed in Section 2.4.1, the manifold is of the form X^{N_k}, the section is the many-valued holomorphic section $(\det S^{(k)})^{1/k}$ of $-K_{X^{N_k}}$, and the measure is of the form $dV_X^{\otimes N_k}$ (and similarly in the case of log Fano pairs). We conclude by proving the explicit formula for $\mathcal{Z}(\beta)$ stated in Example 2.9.

Proposition A.3. *In the setup of Example 2.9, the following formula holds:*

$$\mathcal{Z}(\beta) = c_n \frac{\prod_{j=1}^{n} \Gamma\big(\beta(n+1)+j\big)}{\big(\Gamma\big(\beta(n+1)+n+1\big)\big)^n}.$$

In particular, the maximal holomorphicity strip of $\mathcal{Z}(\beta)$ is given by $\Omega = \{\Re(\beta) > -1/(n+1)\} \Subset \mathbb{C}$ and $\mathcal{Z}(\beta)$ is zero-free in Ω. More precisely, the zeros of $\mathcal{Z}(\beta)$ are located at $\beta = -1 + j/(n+1)$, where $j = 0, 1, 2, \ldots$.

Proof. In this "minimal" case, a basis $s_1, \ldots, s_{N_k}$ in the complex vector space

$$H^0(X, -kK_X) = H^0\big(\mathbb{P}^n, \mathcal{O}(1)\big)$$

is obtained from the homogeneous coordinates $Z_0, \ldots, Z_n$ on $\mathbb{P}^n$. Denote by $\boldsymbol{Z} := (Z_0, \ldots, Z_n)$ the corresponding vector in $\mathbb{C}^{n+1}$. We will represent an element in $(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_N) \in (\mathbb{C}^{n+1})^N$ with an $(n+1) \times N$-matrix, denoted by $[\boldsymbol{Z}]$. Then the corresponding Slater determinant $\det S^{(k)}$ may be identified with the homogeneous polynomial $\det[\boldsymbol{Z}]$ on $\mathbb{C}^{(n+1)^2}$, defined by the determinant of the matrix $[\boldsymbol{Z}]$. Using the $\mathrm{SU}(n+1)$-symmetry of the Fubini–Study metric on $\mathcal{O}(1) \to \mathbb{P}^n$, we may then first lift the integral $Z(\beta)$ on $(\mathbb{P}^n)^{n+1}$ to the product of unit-spheres S in $\mathbb{C}^{n+1}$:

$$\mathcal{Z}(\beta) = c_n \int_{S^{(n+1)}} \big|\det[\boldsymbol{Z}]\big|^{2s} \, d\sigma^{\otimes N}, \quad s := \beta/k,$$

where $d\sigma$ denotes the standard $\mathrm{SU}(n+1)$-invariant measure on S. Next, exploiting that $\det[\boldsymbol{Z}]$ is homogeneous of degree 1 in each column gives

$$\int_{S^{(n+1)}} \big|\det[\boldsymbol{Z}]\big|^{2s} \, d\sigma^{\otimes N} = c_n \frac{\int_{\mathbb{C}^{(n+1)^2}} \big|\det[\boldsymbol{Z}]\big|^{2s} e^{-|\boldsymbol{Z}|^2} \, d\lambda}{\big(\int_0^\infty (r^2)^s e^{-r^2} r^{2(n+1)-1} \, dr\big)^{n+1}}.$$

Hence, making the change of variables $t = r^2$ in the denominator (and rewriting $r^{2(n+1)-1}dr = r^{2(n+1)}r^{-2}d(r^2)/2$) reveals that

$$\mathcal{Z}(\beta) = c_n \frac{\int_{\mathbb{C}^{(n+1)^2}} \left|\det[\boldsymbol{Z}]\right|^{2s} e^{-|\boldsymbol{Z}|^2}\, d\lambda}{\left(\Gamma(s+n+1)\right)^{(n+1)}}, \quad \Gamma(a) := \int_0^\infty t^a e^{-t} \frac{dt}{t}. \tag{A.5}$$

Finally, the proof is concluded by invoking the following formula in [53, Theorem 6.3.1]:

$$Z(s) := \int_{\mathbb{C}^{(n+1)^2}} \left|\det[\boldsymbol{Z}]\right|^{2s} e^{-|\boldsymbol{Z}|^2}\, d\lambda = c_n \prod_{j=1}^{n+1} \Gamma(s+j). \tag{A.6}$$

■

Remark A.4. The proof of formula (A.6) in [53] exploits that the polynomial $f := \det[\boldsymbol{Z}]$ on $\mathbb{C}^{(n+1)^2}$ has the property that

$$P(\partial) f^{s+1} = b(s) f^s \tag{A.7}$$

with

$$b(s) = \prod_{j=1}^{n+1} (s+j),$$

when $P(z) = f(z)$. This leads to the functional relation $b(s)Z(s) = Z(s+1)$, that can then be compared with the classical functional relation for $\Gamma(s)$ to deduce formula (A.6). Recall that in general, given a polynomial $f(z)$ on $\mathbb{C}^m$, the monic polynomial $b(s)$ on $\mathbb{C}$ with minimal degree for which there exists a polynomial $P(z)$ satisfying formula (A.7) is called the *Bernstein–Sato polynomial* attached to f [53]. In general, it is very hard to compute $b(s)$ explicitly (and thus to also find $P(z)$) but the present case, $f(z) = \det[\boldsymbol{Z}]$, fits into Sato's theory of prehomogenuous vector spaces. This is explained in [53]. Alternatively, formula (A.6) follows from the Iwasawa decomposition of $\mathrm{GL}(N, \mathbb{C})$ (as in [54, Section 2]). It would be interesting to see if similar considerations could be applied to $X = \mathbb{P}^n$ when N_k is not assumed to be minimal, i.e., when $N_k > n+1$. However, even the case when $n = 1$ appears to be open (apart from the case when $N_k = 3$ appearing in formula (5.3), where a symmetry argument can be exploited).

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