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Uniqueness results for solutions of continuous and discrete PDE

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Abstract. We give an overview of some recent results on unique continuation property "at infinity" for solutions of elliptic and dispersive PDE and their discrete counterparts. The proofs of most of the results are given in previous works written with coauthors.

1. Introduction

Let L be a differential operator. We say that L has the (weak) unique continuation property if any solution u to the equation $Lu = 0$ in some domain Ω which vanishes on an open subset of Ω equals zero on Ω. For the case of a linear operator, we conclude that two solutions which coincide on an open subset should coincide on the whole domain. The unique continuation property holds for the class of holomorphic functions, this corresponds to the first-order differential operator $\bar{\partial}$, and, more interestingly, for a large class of second-order elliptic operators. The operator L has the strong unique continuation property if any solution u to the equation $Lu = 0$ in Ω that vanishes at some point $x \in \Omega$ to an infinite order is identically zero in Ω.

In this survey, we consider versions of the uniqueness property at infinity. Let $Lu = 0$ on $\mathbb{R}^d$, assuming some decay or growth restriction condition for u, we want to conclude that u is a trivial solution. The simplest example of such result is the classical Liouville theorem for harmonic functions. If a harmonic function on $\mathbb{R}^d$ is bounded, then it is constant. This theorem has a very short and elegant proof; see [30]. It also has numerous generalizations, which include the analogous statement for harmonic functions on $\mathbb{Z}^d$; see for example [21]. The first topic of this note is a surprising improvement of the Liouville theorem for discrete harmonic functions on $\mathbb{Z}^2$ obtained in [3]. We discuss some follow up questions and very deep related results on Anderson localization for the Anderson–Bernoulli model.

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In the next part of the note, we consider the stationary Schrödinger operator with a bounded potential, $Lu = -\Delta u + Vu$. We suggest an elementary analysis of the decay properties of solutions to the corresponding equation on the lattice $\mathbb{Z}^d$ and then describe a recent progress on the continuous question, known as the Landis conjecture. The result is proved in [28] and answers the question on the plane; the problem is open in higher dimensions.

Finally, we describe uniqueness results for the operator $Lu = \partial_t u + i(-\Delta + V)u$, obtained by Luis Escauriaza, Carlos Kenig, Gustavo Ponce, and Luis Vega in a remarkable series of articles [12–15], and discuss the semi-discrete operator, citing the results of [1, 16, 17, 22].

2. Uniqueness results for discrete harmonic functions

2.1. Harmonic functions on $\mathbb{Z}^d$

For each point $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, the $2d$ points $y = (y_1, \dots, y_d)$ such that $\sum_j |x_j - y_j| = 1$ are called the neighbors of x; we write $x \sim y$. Let $V \subset \mathbb{Z}^d$. We define the interior of V as the set of all $x \in V$ such that all neighbors of x also lie in V. Then a function $h : V \to \mathbb{R}$ is called harmonic in V if for any point x in the interior we have

$$h(x) = \frac{1}{2d} \sum_{y \sim x} h(y).$$

This definition easily extends to graphs with finite degrees of vertices. The systematic study of harmonic functions on $\mathbb{Z}^d$ started about a century ago with the classical works of Phillips and Wiener [31], and of Courant, Friedrichs, and Lewy, [5]. It is interesting to note that the first classical articles on the discrete potential theory already mentioned its connections to the probability and random walks. The motivation for these works was the approximation of continuous harmonic functions by discrete ones. One of the results, that can be obtained using such approximation, is the solvability of the Dirichlet problem for bounded domains in $\mathbb{R}^d$ with sufficiently smooth boundary. One might argue that motivation now is reversed; we think that the real world is discrete and study the discrete mathematical models in their own right.

2.2. Weak unique continuation

We start with some simple examples that show the absence of the weak unique continuation property for harmonic functions on $\mathbb{Z}^d$.

Example 2.1. First we consider $\mathbb{Z}^2$. It is easy to see that if h is a harmonic function on $\mathbb{Z}^2$ and $h(x) = 0$ when $x = (x_1, 0)$ and $x = (x_1, 1)$ for all $x_1 \in \mathbb{Z}$, then $h = 0$

on $\mathbb{Z}^2$. On the other hand, we construct a non-trivial harmonic function h on $\mathbb{Z}^2$ such that $h(x) = 0$ when $x = (x_1, x_2)$ with $x_1 + x_2 < 0$. We define $h(x_1, -x_1) = (-1)^{x_1}$ and notice then that one can choose freely the values $h(0, n)$ for $n = 1, 2, \ldots$ and all other values of h are then uniquely determined. We note also that this large region of zeros enforces a rigid structure to the values of the harmonic function nearby. On each next diagonal, the harmonic function $h(x_1, n - x_1) = (-1)^{x_1} p_n(x_1)$, where p_n is a polynomial of degree n.

The situation is even more counter-intuitive in higher dimensions.

Example 2.2. We consider the function h_0 on $\mathbb{Z}^2$ defined by

$$h_0(x) = \begin{cases} 0, & \text{when } x = (x_1, x_2),\ x_1 + x_2 \neq 0, \\ (-1)^{x_1}, & \text{when } x = (x_1, -x_1). \end{cases}$$

Then we extend h_0 to the function on $\mathbb{Z}^3 = \mathbb{Z}^2 \times \mathbb{Z}$ as

$$H(x_1, x_2, x_3) = c^{x_3} h_0(x_1, x_2),$$

where $c + c^{-1} = 6$. The resulting harmonic function H equals zero everywhere on $\mathbb{Z}^3$ except for the hyperplane $x_1 + x_2 = 0$.

These examples demonstrate that some of our continuous intuition does not work for discrete harmonic functions.

Nevertheless, there is a trace of the unique continuation property for discrete harmonic functions on $\mathbb{Z}^d$. We denote by Q_N^d the discrete cube $[-N, N]^d \cap \mathbb{Z}^d$.

Lemma 2.3 ([20]). *There exist $C = C(d) > 0$, $c = c(d) > 0$, and $\alpha = \alpha(d) \in (0, 1)$ such that for any discrete harmonic function U on Q_{4N}^d the following inequality holds:*

$$\max_{Q_{2N}^d} |U| \leq C\Big(\max_{Q_N^d} |U|^\alpha \max_{Q_{4N}^d} |U|^{1-\alpha} + e^{-cN} \max_{Q_{4N}^d} |U|\Big).$$

A similar result was also proven by Lippner and Mangoubi in [26] using a different method. We remark that the error term $e^{-cN} \max_{Q_{4N}^d} |U|$ cannot be omitted, as Example 2.1 shows, and that the decay of this term as N grows to infinity is sharp. In the continuous setting, the corresponding estimate (without the error term) is known as the three-ball inequality; see for example [24]. This estimate serves as a quantitative version of the weak unique continuation property.

The inequality of Lemma 2.3 was generalized in [3], where we showed that there exist C, c, α as above such that

$$\max_{Q_{2N}^d} |U| \leq C\Big(\max_{E} |U|^\alpha \max_{Q_{4N}^d} |U|^{1-\alpha} + e^{-cN} \max_{Q_{4N}^d} |U|\Big) \tag{2.1}$$

holds for any $E \subset Q_N^d$ with $|E| > |Q_N^d|/2$. The proof is based on the fact that discrete harmonic function is a restriction to the lattice of a real analytic function with controlled speed of convergence. On the other hand, it is known that the three-ball inequality and its generalizations concerning propagation of smallness from sets of positive measure hold for a large class of elliptic equations with non-analytic coefficients; see [27]. Recently, interesting three balls inequalities were obtained for solutions of the discrete magnetic Schrödinger equation on the lattice using new discrete Carleman estimates [19].

2.3. Discrete harmonic functions bounded on a large portion of $\mathbb{Z}^d$

Let U be a discrete harmonic function on $\mathbb{Z}^d$, we say that it is bounded by one on a ρ-portion of $\mathbb{Z}^d$ if

$$\left|\left\{x \in Q_N^d : \left|U(x)\right| \le 1\right\}\right| \ge \rho |Q_N^d|$$

for all N large enough. The inequality (2.1) shows that discrete harmonic functions behave similar to continuous ones and we expect a discrete harmonic function which is bounded on a large portion of $\mathbb{Z}^d$ to grow fast at infinity. More precisely, the following result holds.

Theorem 2.4 ([3]). *There exist $\varepsilon = \varepsilon(d) > 0$ and $b = b(d) > 0$ such that for any sufficiently large N and any discrete harmonic function U on Q_{2N}^d which satisfies* $\max_{Q_M^d} |U| \ge 2$ *and*

$$\left|\left\{x \in Q_k : \left|U(x)\right| \le 1\right\}\right| \ge (1-\varepsilon)|Q_K|$$

for every $K \in [M, 2N]$, where $M \le \sqrt{N}$, we have

$$\max_{Q_N^d} |U| \ge e^{bN}.$$

Example 2.2 shows that for $d \ge 3$ there are discrete harmonic functions bounded on $(1-\varepsilon)$ portion of $\mathbb{Z}^d$, which grow exponentially at infinity. We remark that the continuous intuition would predict for very small ε even faster growth at infinity.

A new uniqueness result for harmonic functions on $\mathbb{Z}^2$ found in [3] says that a discrete harmonic function which vanishes on a $(1-\varepsilon)$ portion of $\mathbb{Z}^2$ for sufficiently small ε is zero. The key observation, exploited in [3], is that near a tilted rectangle of zeros, the restrictions of a discrete harmonic function to diagonals have polynomial structure and thus either vanish or have a few zeros. This result follows from a more general statement.

Theorem 2.5 ([3]). *There exist $\varepsilon_0 > 0$ and $a(\varepsilon) > 0$ such that if U is a discrete harmonic function on Q_{2N}^2, N is sufficiently large, and U is bounded by one on*

$(1-\varepsilon)$ portion of Q^2_{2N}, $\varepsilon < \varepsilon_0$, then

$$\max_{Q^2_N} |U| \le e^{a(\varepsilon)N}.$$

Moreover, $a(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorems 2.4 and 2.5 imply that any discrete harmonic function that is bounded on a $(1-\varepsilon)$ portion of $\mathbb{Z}^2$ with ε small enough is constant.

Theorem 2.5 also implies that *there exist constants a and $\varepsilon < 1$ such that for any discrete harmonic function on Q^2_{2N}, for N large enough, we have*

$$\left|\left\{|U| > e^{-aN} \max_{Q^2_N} |U|\right\} \cap Q^2_{2N}\right| \ge \varepsilon N^2. \tag{2.2}$$

It would be interesting to obtain sharp generalizations of this result to harmonic functions on higher dimensional lattices. For example, a toy statement in $\mathbb{Z}^3$ is the following:

> *Suppose that U is a discrete harmonic function on Q^3_{2N} such that*
>
> $$\left|\{U \ne 0\}\right| \le cN^2,$$
>
> *where c is sufficiently small and N is sufficiently large. Then $U = 0$ on Q^3_N.*

The interest in the uniqueness theorems for discrete harmonic functions and more general solutions to the Schrödinger equation on lattices is partly due to its connections to the problem of the exponential decay of eigenfunctions of the Schrödinger operator with a random Bernoulli potential, known as the Anderson localization. This connection is discovered and exploited by Bourgain and Kenig in [2], where the continuous model is studied. Recently, Ding and Smart [10], combining the approach developed in [2] with ideas introduced in [3], obtained new results on localization near the edge for the Anderson–Bernoulli model on $\mathbb{Z}^2$. One of the tools developed in [10] is a probabilistic version of (2.2) for solutions of the equation $\Delta U + VU = \lambda U$ with random Bernoulli potential V. It is worth mentioning, that in dimension three the following deterministic statement holds (see [25]):

> *There exists constant $p > 3/2$ such that for each $K > 0$, there is $C > 0$, such that if $\Delta U + VU = 0$ on Q^3_N, N is large enough, and $|V| \le K$, then*
>
> $$\left|\left\{|U| > e^{-CN}\left|U(0)\right|\right\}\right| \ge N^p.$$

This result is due to Li and Zhang, who generalized the Anderson localization near the edge of the spectrum to the Anderson–Bernoulli model on $\mathbb{Z}^3$ [25].

3. Landis conjecture on decay of solutions to Schrödinger equations

3.1. Decay at infinity

In this section, we consider bounded solutions to the stationary Schrödinger equation with bounded potential, $\Delta u + Vu = 0$, $|V| \le 1$. Landis conjectured that a solution to this equation cannot decay faster than exponential at infinity. An example of a function that decays exponentially is $u(x) = \exp(-(1+x^2)^{1/2})$.

We assume that there is a bounded solution to the Schrödinger equation with a bounded potential, and we are interested in the possible decay of the quantity $m_u(R) = \sup_{|x|>R} |u(x)|$. A local version of the Landis conjecture, which appeared in [2] in connection to the Anderson–Bernoulli model, is about the possible decay of the quantity $\mu_u(R) = \inf_{|x|=R} \sup_{B(x,1)} |u(x)|$.

For solutions of the continuous Schrödinger equation, the Landis conjecture was disproved by Meshkov, [29]. He gave an example of a complex valued function $u(x)$ which decays as $C \exp -c|x|^{4/3}$ and satisfies the inequality $|\Delta u| \le |u|$ everywhere. The proof is based on a Carleman inequality. Bourgain and Kenig proved the following local version of the estimate.

Theorem. *Let $\Delta U + Vu = 0$, let $u(0) = 1$, and let u and V be bounded on $\mathbb{R}^d$. Then*

$$\mu_u(R) \ge c \exp(-CR^{4/3} \log R).$$

The proof also exploits a Carleman-type inequality. The remaining question is whether the original Landis conjecture holds for the class of real-valued potentials. For this case one may consider only real-valued solutions. This question is open in dimension $d \ge 3$.

3.2. Discrete equation

First, we consider the corresponding equation on the lattice $\mathbb{Z}^d$, here there is no difference between the real-valued and complex-valued cases, to the best of my knowledge.

Suppose that $\Delta U + VU = 0$, $U : \mathbb{Z}^d \to \mathbb{R}$, $|V| \le C_0$, and $U \ne 0$, where

$$\Delta U(x) = \sum_{y \sim x} \big(U(y) - U(x)\big).$$

We also refer the reader to [1] for the discussion of this problem. Let

$$m_U(N) = \sup_{x \notin Q_N^d} \big|U(x)\big|.$$

We consider any $x \in Q_{N+1}^d \setminus Q_N^d$. Then there is one of its neighbors y such that $y \in Q_{N+2}^d \setminus Q_{N+1}^d$ and all neighbors of y except x are not in Q_{N+1}^d. Then the equation $\Delta U(y) + V(y)U(y) = 0$ can be written as

$$U(x) = U(y) + \sum_{z\sim y, z\neq x} \big(U(y) - U(z)\big) - V(y)U(y).$$

This implies that $m_U(N) \leq (2^{d+1} + 1 + C_0)m_U(N+1)$. Thus $m_U(N)$ does not decay faster than e^{-CN} as $N \to \infty$, where $C = C(d, C_0)$.

On the other hand, simple example shows that

$$\mu_U(N,k) = \inf_{x\in Q_N^d \setminus Q_{N-1}^d} \max_{|y-x|\leq k} \big|U(y)\big|$$

may be equal to zero for a non-trivial function U and bounded V; see [1]. Let us describe this example on $\mathbb{Z}^2$. We consider a function U which is zero on a tilted square

$$\tilde{Q}_N^2 = \big\{x = (x_1, x_2) \in \mathbb{Z}^2 : |x_1 + x_2| \leq 2N,\ |x_1 - x_2| \leq 2N\big\}$$

and takes non-zero values everywhere else. On the four diagonals $x_1 \pm x_2 = \pm 2N$, we define $U(x_1, x_2) = (-1)^{x_1}$, so that the function is harmonic at each point of $\tilde{Q}_N^2$. Then the values are arbitrary such that, for any $x \sim y$, we have $|U(x)| \leq (1+\varepsilon)|U(y)|$. Then we define $V(x) = -(\Delta U(x))/U(x)$ when $x \notin \tilde{Q}_N^2$. We see that $|V| \leq 8 + 4\varepsilon$. The example shows that there is no local version of the Landis conjecture when the potential is bounded but large enough. It would be interesting to obtain a local version for the case of the small potential.

3.3. Landis conjecture for real-valued potentials on the plane

The question of the estimates for the $m_u(R)$ and $\mu_u(R)$ for real-valued solutions of the Schrödinger equations in $\mathbb{R}^2$ is considered in [7–9,23], where local estimates were obtained under some assumptions on the potential. The decay estimate of the solution for the case of a periodic (in all but one variables) potential in $\mathbb{R}^2$ and $\mathbb{R}^3$ is discussed in [11].

The global and local versions of the result for solution of the Schrödinger equation with general bounded potential on $\mathbb{R}^2$ were recently obtained in [28]. It turns out that the Landis conjecture holds for this case (up to a logarithmic factor). More precisely, the following theorem holds.

Theorem 3.1 ([28]). *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function which satisfies $|\Delta u| \leq |u|$. Then*

(i) *if $|u(x)| \leq \exp(-C|x|(\log|x|)^{1/2})$ and C is large enough, then $u = 0$;*

(ii) *if $\inf_{|x|=R} \sup_{B_1(x)} |u(x)| \leq \exp(-CR(\log R)^{3/2})$, then $u = 0$.*

There are three main steps in the proof. First, one constructs a family of separated D_j disks of equal radii r such that $\operatorname{dist}(D_j, \{u = 0\}) \geq 10r$ and each connected component Ω_k of $\{u \neq 0\} \setminus \bigcup_j D_j$ has the small first Laplace eigenvalue. Then, constructing an auxiliary solution of the equation $\Delta f + Vf = 0$ in Ω_k with boundary values $f = 1$ on $\partial\Omega_k$, one considers the ration $v = u/f$. This reduces the problem to the following one:

Let $v : \mathbb{R}^2 \setminus \bigcup_j D_j \to \mathbb{R}$ *be a solution to the equation* $\operatorname{div}(f^2 \nabla v) = 0$ *and let* v *not change sign in each set* $10D_j \setminus D_j$*. Then if* v *decays as* $\exp(-C|x|(\log|x|)^{1/2})$ *with large* C*, then* $v = 0$.

The second step uses quasiconformal mappings to replace the general elliptic equation in divergence form by the Laplace equation; the factor $\log|x|^{1/2}$ in the exponent appears on this step. This step uses the specifics of dimension two. Finally, the above statement is proved for harmonic functions defined on $\mathbb{R}^2 \setminus \bigcup \tilde{D}_j$. The version of the last step for harmonic functions in higher dimensions is also discussed in [28].

4. Uncertainty principle and uniqueness for Schrödinger evolutions

4.1. Hardy's uncertainty principle

The Hardy uncertainty principle says that if $f \in L^2(\mathbb{R})$, $|f(x)| \leq Ce^{-a|x|^2}$, $|\widehat{f}(\xi)| \leq Ce^{-b|\xi|^2}$, and $ab > 1/4$, then $f = 0$. If $ab = 1/4$, then $f(x) = ce^{-a|x|^2}$. Its dynamical interpretation was found in [4, 12], where it is shown that the principle is equivalent to the following statement.

Theorem. *Let* $u(t,x)$ *be a solution to the free Schrödinger equation*

$$\partial_t u = i\Delta u(t,x).$$

Suppose that $u \in C^1([0,T], W^{2,2}(\mathbb{R}^d))$ *and*

$$\left|u(0,x)\right| \leq Ce^{-\alpha|x|^2} \quad \textit{and} \quad \left|u(T,x)\right| \leq Ce^{-\beta|x|^2},$$

where $\alpha, \beta > 0$*. Then the following hold.*

(i) *If* $\alpha\beta > (16T^2)^{-1}$*, then* $u(t,x) = 0$.

(ii) *If* $\alpha\beta = (16T^2)^{-1}$*, then* $u(t,x) = ce^{-(\alpha + i/(4T))|x|^2}$.

A real-variable proof of this result is given by Cowling, Escauriaza, Kenig, Ponce, and Vega in [6]. The last theorem was generalized to a large class of Schrödinger evolutions of the form $\partial_t u = i(\Delta u + Vu)$ in the series of articles [12–14].

4.2. Uniqueness results for discrete Schrödinger evolutions

Let Δ be again the discrete Laplacian on $\mathbb{Z}^d$. We consider the equation

$$\partial_t U(t,n) = i\big(\Delta_U(t,n) + V(t,n)U(t,n)\big),$$

where V is a bounded potential. We are interested in uniqueness results which says that if a solution to the discrete Schrödinger equation decays fast on $\mathbb{Z}^d$ at two distinct times, then it is trivial. First, we consider the free evolution with $V = 0$. In dimension $d = 1$, there is a solution $U_0(t,n) = i^{-n}e^{-2it}J_n(1-2t)$, where J_n is the Bessel function. This solution has optimal decay at $t = 0$ and $t = 1$. The role of the Gaussian is now played by the Bessel function. We get the following result for the free evolution:

Let $U(t,n)$ be a solution to $\partial_t U(t,n) = i\Delta U(t,n)$ on $[0,1]\times\mathbb{Z}$. Suppose that

$$\big|U(0,n)\big| + \big|U(1,n)\big| \le \frac{C}{\sqrt{|n|}}\left(\frac{e}{2|n|}\right)^{|n|}, \quad n \in \mathbb{Z}\setminus\{0\}.$$

Then $U(t,n) = C i^{-n}e^{-2it}J_n(1-2t)$.

This result was generalized to general bounded potentials in [22] (in dimension $d = 1$) and [1] (in arbitrary dimension). The result is as follows.

Theorem 4.1. *Let $U(t,n) \in C^1([0,1] : \ell^2(\mathbb{Z}^d))$ be a solution to*

$$\partial_t U(t,n) = i\big(\Delta U(t,n) + V(t,n)U(t,n)\big),$$

on $[0,1]\times\mathbb{Z}^d$. Suppose that $\|V\|_\infty \le 1$. There exists constant γ such that if

$$\big|U(0,n)\big| + \big|U(1,n)\big| \le C\exp\big(-\gamma|n|\log|n|\big), \quad n \in \mathbb{Z}^d\setminus\{0\},$$

then $U = 0$.

This result is not precise; we expect the same decay bounds as for the case of the free Schrödinger equation. One of the interesting applications of the uniqueness theorem with general potential which may depend on time is to the nonlinear Schrödinger equation. For this case, we have the same decay result as for the free equation. Let $U : [0,1]\times\mathbb{Z} \to \mathbb{R}$ be a solution to the equation

$$\partial_t U = i\big(\Delta U + c|U|^2 U\big).$$

Suppose that

$$\big|U(0,n)\big| + \big|U(1,n)\big| \le \left(\frac{c}{|n|}\right)^{|n|}, \quad n \in \mathbb{Z}\setminus\{0\},$$

where $c < e/2$. Then $U = 0$. We refer the reader to a recent survey [18] for detailed discussions of the uniqueness results for discrete and continuous Schrödinger evolutions.

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