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Smooth compactifications in derived non-commutative geometry

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Abstract. This is a short overview of the author's results related to the notion of a smooth categorical compactification. We cover the construction of a categorical smooth compactification of the derived categories of coherent sheaves, using the categorical resolution of Kuznetsov and Lunts. We also mention examples of homotopically finitely presented DG categories which do not admit a smooth compactification. This is closely related to Kontsevich's conjectures on the generalized versions of categorical Hodge-to-de Rham degeneration, which we disproved. Finally, we mention our new result on the DG categorical analogue of Wall's finiteness obstruction, which in particular gives a criterion for existence of a smooth compactification of a homotopically finite DG category.

1. Introduction

We give a short overview of some of our results concerning smooth compactifications of differential graded categories [8–10].

Suppose that $X \subset \bar{X}$ is a smooth compactification, i.e., X is open in $\bar{X}$ and $\bar{X}$ is smooth and proper over a base field k. Then the restriction functor

$$D^b_{\mathrm{coh}}(\bar{X}) \to D^b_{\mathrm{coh}}(X)$$

is a *localization*. Namely, the induced functor

$$D^b_{\mathrm{coh}}(\bar{X})/D^b_{\mathrm{coh},\bar{X}-X}(\bar{X}) \to D^b_{\mathrm{coh}}(X)$$

is an equivalence of categories.

This motivates a general categorical notion of a smooth compactification. There are notions of smoothness and properness for DG categories, which are defined in terms of the diagonal bimodule. By definition, a categorical smooth compactification of a pre-triangulated DG category $\mathcal{A}$ is given by a smooth and proper pre-triangulated

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DG category $\mathcal{C}$, with a functor $\Phi : \mathcal{C} \to \mathcal{A}$, such that Φ is a localization up to direct summands, with an additional assumption that $\ker(\Phi)$ is generated by a single object (see Definition 3.5). Here being a localization means that the induced functor $\overline{\Phi} : \mathcal{C}/\ker(\Phi) \to \mathcal{A}$ is fully faithful, and it is essentially surjective up to direct summands.

Existence of a categorical smooth compactification of a DG category $\mathcal{A}$ automatically implies that $\mathcal{A}$ is smooth. Moreover, $\mathcal{A}$ is actually homotopically finitely presented (hfp); see Definition 3.3.

The following result has been proved in [9].

Theorem 1.1 ([9, Theorem 1.8, part (1)]). *Let X be a separated scheme of finite type over a field* k *of characteristic zero. Then there exists a categorical smooth compactification of the form $D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$, where Y is smooth and proper.*

In [9], Theorem 1.1 was used to prove the homotopy finiteness for derived categories of coherent sheaves over a field of characteristic zero, confirming a conjecture of Kontsevich.

The construction of a smooth compactification in Theorem 1.1 uses the categorical resolution of singularities of Kuznetsov and Lunts [15], as well as Orlov's results on semi-orthogonal gluings of geometric DG categories [21].

The statement of Theorem 1.1 is conceptually very closely related with the following conjecture of Bondal and Orlov.

Conjecture 1.2 ([2]). *Let Y be a variety with rational singularities, and $f : X \to Y$ a resolution of singularities. Then the functor $\mathbf{R}f_* : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ is a localization.*

The methods of the proof of Theorem 1.1 allow to prove Conjecture 1.2 in a certain class of cases.

Theorem 1.3 ([9, Theorem 1.10]). *Suppose that Y has rational singularities, $Z \subset Y$ is a closed smooth subscheme, and $X = Bl_Z Y$ is smooth, so that $f : X \to Y$ is a resolution of singularities. Denote by $T = f^{-1}(Z)$ the exceptional divisor, by $p : T \to Z$ the induced morphism, and by $j : T \to X$ the embedding. Suppose that $\mathbf{R}f_* I_T^n = I_Z^n$ for $n \geq 1$. Then the functor $\mathbf{R}f_* : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ is a localization, and its kernel is generated by $j_*((p^* D^b_{\mathrm{coh}}(Z))^\perp)$.*

In particular, Theorem 1.3 applies in the case when Y is a cone over a projectively normal embedding of a smooth Fano variety, and Z is the origin.

The following question for general homotopically finite DG categories was formulated by Toën.

Question 1.4. Is it true that any homotopically finite DG category admits a categorical smooth compactification?

It turns out surprisingly that the answer is "no", and a counterexample has been obtained in [8]. Question 1.4 is closely related with two (unpublished) conjectures of Kontsevich on the generalized versions of Hodge-to-de Rham degeneration, which we disproved in [8] (these are Conjectures 5.3 and 5.4).

One can further ask "what are the necessary and sufficient conditions for an hfp DG category to have a categorical smooth compactification?". We have the following (new) result.

Theorem 1.5 ([10]). *Let $\mathcal{A}$ be an hfp pre-triangulated DG category. The following are equivalent.*

(1) *$\mathcal{A}$ admits a smooth categorical compactification.*

(2) *There exists a DG functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{C}$ is smooth and proper, such that*

$$[I_{\mathcal{A}}] \in \operatorname{Im}\left(K_0(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})\right).$$

Here $I_{\mathcal{A}}$ is the diagonal $\mathcal{A}$-$\mathcal{A}$-bimodule.

This theorem is closely related with a certain DG categorical analogue of Wall's finiteness obstruction theorem; see Section 6.

The paper is organized as follows.

In Section 2, we briefly recall some basic notions and statements about triangulated categories and DG categories.

In Section 3, we discuss the general notion of a categorical smooth compactification.

In Section 4, we formulate our result on smooth compactifications of derived categories of coherent sheaves, and briefly explain the idea of the proof.

Section 5 discusses the question of existence of smooth compactifications, and the closely related Conjectures 5.3 and 5.4.

Finally, in Section 6 we briefly mention our new results on the DG categorical analogue of Wall's finiteness obstruction theorem about finitely dominated spaces. This in particular gives a criterion for when a homotopically finite DG category has a smooth compactification.

2. Some preliminaries on triangulated categories and DG categories

For a very nice introduction to DG categories and their derived categories, we refer to [12]. For triangulated categories, we refer to Neeman's book [19]. The notion of a DG enhancement of a triangulated category has been introduced in [3]. The notion of a DG quotient of DG categories has been introduced in [13] and an explicit construction has been given in [6]. For model structures on the categories of DG algebras and DG categories we refer to [22, 23].

Fix some base field k. For a quasi-projective scheme X over k, we have the category of finite rank vector bundles on X, or equivalently the category of locally free sheaves of finite rank. After adding to it the cokernels, we get the abelian category $\mathrm{Coh}(X)$ of coherent sheaves. More generally, the abelian category $\mathrm{Coh}(X)$ can be defined for any noetherian (or even locally coherent) scheme X. In this note, we deal only with separated schemes of finite type over k.

The objects of the derived category $D^b(\mathrm{Coh}(X)) = D^b_{\mathrm{coh}}(X)$ are bounded complexes of coherent sheaves. The morphisms are more complicated: they are obtained from the naive category of complexes by inverting the quasi-isomorphisms. A quasi-isomorphism is a morphism of complexes that induces an isomorphism in cohomology.

The derived category $D^b_{\mathrm{coh}}(X)$ is always triangulated. It has a full triangulated subcategory of perfect complexes $D_{\mathrm{perf}}(X) \subset D^b_{\mathrm{coh}}(X)$, which is formed by bounded complexes of locally free sheaves (that is, of vector bundles). More precisely, if X is not necessarily quasi-projective, an object $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$ is a perfect complex if it is locally quasi-isomorphic to a bounded complex of locally free sheaves.

A DG category $\mathcal{A}$ is given by the following data:

- a class of objects $\mathrm{Ob}(\mathcal{A})$;
- for any pair of objects $x, y \in \mathrm{Ob}(\mathcal{A})$, a complex of vector spaces $\mathcal{A}(x, y) = \mathrm{Hom}_{\mathcal{A}}(x, y)$;
- for any objects $x, y, z \in \mathrm{Ob}(\mathcal{A})$, a composition map $\mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z)$.

The composition maps are required to be morphisms of complexes: they are homogeneous of degree zero and satisfy the (super-)Leibniz rule. They are also required to be associative. For each object $x \in \mathrm{Ob}(\mathcal{A})$, it is required that there is a unit morphism 1_x of degree zero (and automatically $d(1_x) = 0$).

The homotopy category of a DG category $\mathcal{A}$ is a k-linear category $H^0(\mathcal{A})$ which has the same objects as $\mathcal{A}$, and the morphisms are given by

$$H^0(\mathcal{A})(x, y) = H^0\big(\mathcal{A}(x, y)\big).$$

It is also convenient to define similarly the k-linear category $Z^0(\mathcal{A})$ with the same objects as $\mathcal{A}$, and with the morphisms given by $Z^0(\mathcal{A})(x, y) = Z^0(\mathcal{A}(x, y))$.

For a small DG category $\mathcal{A}$, just as for DG algebras, there is a notion of a right DG $\mathcal{A}$-module: it is a DG functor $\mathcal{A}^{\mathrm{op}} \to \text{Mod-k}$, where Mod-k is the DG category of complexes of vector spaces. DG $\mathcal{A}$-modules form a DG category Mod-$\mathcal{A}$. The derived category $D(\mathcal{A})$ of right $\mathcal{A}$-modules is obtained from $H^0(\text{Mod-}\mathcal{A})$ by inverting quasi-isomorphisms. Equivalently, $D(\mathcal{A})$ is obtained from $Z^0(\text{Mod-}\mathcal{A})$ by inverting quasi-isomorphisms. Again, as for DG algebras, $D_{\mathrm{perf}}(\mathcal{A}) \subset D(\mathcal{A})$ is the full subcategory of compact objects.

The Yoneda embedding $\mathcal{A} \hookrightarrow \text{Mod-}\mathcal{A}$ induces a fully faithful functor $H^0(\mathcal{A}) \hookrightarrow D(\mathcal{A})$. If its image is a triangulated subcategory of $D(\mathcal{A})$, then we call the DG category $\mathcal{A}$ *pre-triangulated*. In this case, we have $D_{\text{perf}}(\mathcal{A}) \simeq H^0(\mathcal{A})^{\text{Kar}}$ – Karoubi completion.

The basic example is the following: for a separated scheme X of finite type over k we take the DG category $\mathfrak{D}^b_{\text{coh}}(X)$ of bounded below complexes of injective quasi-coherent sheaves with bounded coherent cohomology. Then $\mathfrak{D}^b_{\text{coh}}(X)$ is pre-triangulated and $H^0(\mathfrak{D}^b_{\text{coh}}(X))$ is equivalent to $D^b_{\text{coh}}(X)$. We denote by $\text{Perf}(X) \subset \mathfrak{D}^b_{\text{coh}}(X)$ the full DG subcategory of perfect complexes.

If $\mathcal{T}$ is a small triangulated category and $\mathcal{S} \subset \mathcal{T}$ is a full triangulated subcategory, then there is a notion of a quotient category $\mathcal{T}/\mathcal{S}$, due to Verdier [26,27]. The category $\mathcal{T}/\mathcal{S}$ is again triangulated, and we have an exact quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. The category $\mathcal{T}/\mathcal{S}$ is obtained from $\mathcal{T}$ by inverting the morphisms $f : x \to y$ such that $\text{Cone}(f) \in \mathcal{S}$.

The basic example is coming from geometry: let X be as above, $Z \subseteq X$ a closed subscheme, and $U = X - Z$. Denote by $D^b_{\text{coh},Z}(X) \subseteq D^b_{\text{coh}}(X)$ the full subcategory of complexes whose cohomology is supported on Z. Then we have an equivalence

$$D^b_{\text{coh}}(X)/D^b_{\text{coh},Z}(X) \simeq D^b_{\text{coh}}(U);$$

see [20, Lemma 2.2].

There is a notion of a DG quotient $\mathcal{A}/\mathcal{B}$ of a small DG category $\mathcal{A}$ by a full DG subcategory $\mathcal{B} \subseteq \mathcal{A}$, which was first defined by Keller [13], and then an explicit construction has been given by Drinfeld [6]. The main property of the DG quotient is its compatibility with the Verdier quotient of triangulated categories. Namely, if $\mathcal{A}$ is a pre-triangulated small DG category, and $\mathcal{B} \subset \mathcal{A}$ is a full pre-triangulated DG subcategory, then we have an equivalence $H^0(\mathcal{A}/\mathcal{B}) \simeq H^0(\mathcal{A})/H^0(\mathcal{B})$.

In particular, within the above notation we have a quasi-equivalence

$$\mathfrak{D}^b_{\text{coh}}(X)/\mathfrak{D}^b_{\text{coh},Z}(X) \simeq \mathfrak{D}^b_{\text{coh}}(U).$$

3. Categorical smooth compactifications

The following notions of smoothness and properness for DG categories are due to Kontsevich.

A DG category $\mathcal{A}$ is called *proper* (over k) if for any $x, y \in \mathcal{A}$ the complex $\mathcal{A}(x, y)$ has finite-dimensional total cohomology, and the triangulated category $D_{\text{perf}}(\mathcal{A})$ is generated by a single object. Here and below we say that a triangulated category T is generated by an object x if T is the smallest *idempotent complete* triangulated subcategory of T containing x. Equivalently, any (isomorphism class of an) object of T can be obtained from x using cones and direct summands.

A DG category $\mathcal{A}$ is called *smooth* (over k) if the diagonal $\mathcal{A}$-$\mathcal{A}$-bimodule $I_{\mathcal{A}}$ is perfect over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. Here $I_{\mathcal{A}}(x, y) = \mathcal{A}(x, y)$.

These properties are compatible with the corresponding properties of schemes. Namely, the following holds.

Proposition 3.1 ([21, Proposition 3.30] and [17, Proposition 3.13]). *If X is a separated scheme of finite type over* k, *then the DG category* $\mathrm{Perf}(X)$ *is smooth (resp. proper) if and only if X is smooth (resp. proper).*

Much more surprising is the following theorem of Lunts.

Theorem 3.2 ([17, Theorem 6.3]). *For any separated scheme X of finite type over a perfect field* k, *the DG category $\mathfrak{D}^b_{\mathrm{coh}}(X)$ is smooth.*

There is a notion of an hfp DG category. Before giving its formal definition, we mention that it is an analogue of the notion of a *finitely dominated* topological space. Recall that a (possibly infinite) CW complex X is called finitely dominated if there exists a finite CW complex Y and continuous maps $f : X \to Y$, $g : Y \to X$ such that $gf \sim \mathrm{id}_X$. Equivalently, the identity map id_X is homotopic to a map $r : X \to X$ such that the closure $\overline{r(X)}$ is compact.

Formal definition of hfp DG algebras and DG categories is the following.

Definition 3.3 ([25]). (1) A finite cell DG algebra B is a DG algebra which is isomorphic as a graded algebra to the free finitely generated associative algebra:

$$B^{gr} \cong \mathrm{k}\langle x_1, \dots, x_n \rangle,$$

and moreover we have

$$dx_i \in \mathrm{k}\langle x_1, \dots, x_{i-1} \rangle, \quad 1 \le i \le n. \tag{3.1}$$

(2) A DG algebra A is hfp if in the homotopy category $\mathrm{Ho}(dgalg_{\mathrm{k}})$ the object A is a retract of some finite cell DG algebra B.

(3) A DG category $\mathcal{A}$ is hfp if it is Morita equivalent to an hfp DG algebra.

Recall that in any category $\mathcal{C}$ an object X is a retract of Y iff there exists morphisms $f : X \to Y$, $g : Y \to X$ such that $gf = \mathrm{id}_X$.

Proposition 3.4 ([25]). *Let $\mathcal{A}$ be a small DG category over* k.

(1) *If $\mathcal{A}$ is hfp, then $\mathcal{A}$ is smooth.*

(2) *If $\mathcal{A}$ is smooth and proper, then $\mathcal{A}$ is hfp.*

Informally, an hfp DG category is a smooth DG category “given by a finite amount of data”. For example, the k-algebra of rational functions $\mathrm{k}(x)$ is smooth but not hfp.

An equivalent definition of an hfp DG category is the following. First, there is a notion of a finite cell DG category: as a k-linear graded category, it is a path category of a finite graded quiver with arrows $x_1, \dots, x_n$ such that the differential satisfied the condition analogous to (3.1). Now, a DG category $\mathcal{A}$ is hfp if it is a retract of a finite cell DG category in the Morita homotopy category of DG categories $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ (which is obtained by inverting Morita equivalences).

Recall that a usual smooth compactification of a smooth algebraic variety X is given by a smooth and proper variety $\bar{X}$ and an open embedding $X \hookrightarrow \bar{X}$. Denote by $Z = \bar{X} - X$ the infinity locus. As already mentioned in the previous section, we have an equivalence $D^b_{\mathrm{coh}}(X) \simeq D^b_{\mathrm{coh}}(\bar{X})/D^b_{\mathrm{coh},Z}(\bar{X})$. Hence, we also have a quasi-equivalence of DG categories

$$\mathfrak{D}^b_{\mathrm{coh}}(X) \simeq \mathfrak{D}^b_{\mathrm{coh}}(\bar{X})/\mathfrak{D}^b_{\mathrm{coh},Z}(\bar{X}).$$

This motivates a general definition of a categorical smooth compactification, which we already mentioned in the introduction.

Definition 3.5. A smooth categorical compactification of a DG category $\mathcal{A}$ is a DG functor $F : \mathcal{C} \to \mathcal{A}$, where the DG category $\mathcal{C}$ is smooth and proper, the extension of scalars functor $F : \mathrm{Perf}(\mathcal{C}) \to \mathrm{Perf}(\mathcal{A})$ is a localization (up to direct summands), and its kernel is generated by a single object.

We have the following implication, which is quite easy to prove.

Proposition 3.6 ([9, Corollary 2.9]). *If a DG category $\mathcal{A}$ has a smooth categorical compactification, then it is hfp.*

4. Smooth compactifications of derived categories of coherent sheaves

We have the following general result.

Theorem 4.1 ([9, Theorem 1.8]). *For any separated scheme X of finite type over a field* k *of characteristic zero, there exists a smooth projective variety Y and a quasi-equivalence $\mathfrak{D}^b_{\mathrm{coh}}(Y)/\mathcal{S} \simeq \mathfrak{D}^b_{\mathrm{coh}}(X)$, where $\mathcal{S} \subset \mathfrak{D}^b_{\mathrm{coh}}(Y)$ is a pre-triangulated subcategory generated by a single object. In particular, the DG category $\mathfrak{D}^b_{\mathrm{coh}}(Y)$ is hfp.*

This confirms a conjecture of Kontsevich on the homotopy finiteness of the DG category $\mathfrak{D}^b_{\mathrm{coh}}(X)$.

Remark. A similar result is expected to hold over any perfect field. In our proof, we cannot get rid of the characteristic zero assumption: we use the categorical resolution of singularities of Kuznetsov and Lunts, which in turn uses the classical Hironaka's theorem.

We now explain the idea of the proof of Theorem 4.1. It is based on the following constructions.

The first one is the categorical resolution of singularities due to Kuznetsov and Lunts [15]. Let us restrict to proper schemes. For any proper scheme X over k, they construct a smooth and proper DG category $\mathcal{C}$ together with a fully faithful functor $\mathrm{Perf}(X) \hookrightarrow \mathcal{C}$. Moreover, this DG category $\mathcal{C}$ has a semi-orthogonal decomposition into derived categories of some smooth and proper varieties:

$$\mathcal{C} = \langle \mathfrak{D}^b_{\mathrm{coh}}(Y_1), \ldots, \mathfrak{D}^b_{\mathrm{coh}}(Y_m) \rangle.$$

More precisely, one chooses a resolution $Z \to X_{\mathrm{red}}$ by a sequence of blow-ups with smooth centers. Then the varieties $Y_1, \ldots, Y_m$ are exactly the centers of the blow-ups and the resolution Z (each of these varieties can appear in the list several times).

Another general construction due to Orlov [21] allows to embed such a semi-orthogonal gluing of $\mathfrak{D}^b_{\mathrm{coh}}(Y_i)$ into a single derived category $\mathfrak{D}^b_{\mathrm{coh}}(Y)$ (here Y_i and Y are smooth and proper). Taking such embedding $\mathcal{C} \hookrightarrow \mathfrak{D}^b_{\mathrm{coh}}(Y)$ (where $\mathcal{C}$ is as above), we obtain the fully faithful composition functor $\mathrm{Perf}(X) \hookrightarrow \mathcal{C} \hookrightarrow \mathfrak{D}^b_{\mathrm{coh}}(Y)$. Passing to large categories (i.e., categories of ind-objects), we can take a right adjoint to this embedding, which restricts to a functor $\Phi : \mathfrak{D}^b_{\mathrm{coh}}(Y) \to \mathfrak{D}^b_{\mathrm{coh}}(X)$. It turns out (but it is not easy to prove) that this functor is actually a desired localization functor promised by Theorem 4.1.

Remark. Strictly speaking, in [9] it is proved that the functor $\Phi : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ is a localization under some assumptions on the choices of integer parameters in the construction of the category $\mathcal{C}$ in [15]. We do not discuss these details in the present note.

The construction of the categorical resolution from [15] uses two general methods to "partially resolve" the category $\mathrm{Perf}(X)$. The first one allows to deal with nilpotents in the structure sheaf $\mathcal{O}_X$. Namely, assuming that the reduced part $X_{\mathrm{red}} \subset X$ is smooth, one can find a categorical resolution by a certain ringed space $(X, \mathcal{A}_X)$, where $\mathcal{A}_X$ is a sheaf of associative algebras (and non-commutative unless $X = X_{\mathrm{red}}$). This ringed space is equipped with a morphism $(X, \mathcal{A}_X) \xrightarrow{\rho_X} X$, and the pullback functor

$$\rho_X^* : D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(X, \mathcal{A}_X)$$

is fully faithful. It is not hard to show that the pushforward functor

$$\rho_{X*} : D^b_{\mathrm{coh}}(X, \mathcal{A}_X) \to D^b_{\mathrm{coh}}(X)$$

is a localization.

Remark. The ringed space $(X, \mathcal{A}_X)$ is given by a certain generalization of algebras considered by Auslander [1]. Namely, for a finite-dimensional algebra A of finite representation type, Auslander constructs an algebra $B = \mathrm{End}_A(\bigoplus_i M_i)$, where M_i are representatives of all isomorphism classes of indecomposable finite-dimensional A-modules. In fact, for $X = \mathrm{Spec}\, \mathrm{k}[x]/x^n$ we have

$$(X, \mathcal{A}_X) = \left(\mathrm{pt}, \mathrm{End}_{\mathrm{k}[x]/x^n}\left(\bigoplus_{i=1}^{n} \mathrm{k}[x]/x^i\right)\right).$$

Another more interesting construction involved in the categorical resolution is the "categorical blow-up". Without going into details, this is a certain categorical modification of the usual blow-up. Given any noetherian scheme X and a closed subscheme S, consider the blow-up $f : Y \to X$, i.e., $Y = \mathrm{Proj}_X(\bigoplus_{n\geq 0} I_S^n)$. Then under some assumptions on S (always achievable by replacing S with its sufficiently large infinitesimal neighborhood), one can define a certain semi-orthogonal gluing of $\mathfrak{D}^b_{\mathrm{coh}}(Y)$ and $\mathfrak{D}^b_{\mathrm{coh}}(S)$, denoted by $\mathcal{D}_{\mathrm{coh}}(Y, S)$, with a functor

$$\pi_* : \mathcal{D}_{\mathrm{coh}}(Y, S) \to \mathfrak{D}^b_{\mathrm{coh}}(X).$$

It is proved in [9] that under some additional assumptions on S (again they always hold after infinitesimally enlarging S) this functor π_* is a localization. This is the most difficult part of the proof of Theorem 4.1. Note that if we use $\mathfrak{D}^b_{\mathrm{coh}}(Y)$ instead of $\mathfrak{D}_{\mathrm{coh}}(Y, S)$, then

(1) the pushforward functor $\mathbf{R}f_* : \mathfrak{D}^b_{\mathrm{coh}}(Y) \to \mathfrak{D}^b_{\mathrm{coh}}(X)$ is usually not a localization, and a necessary condition is that $\mathbf{R}f_*(\mathcal{O}_Y) = \mathcal{O}_X$;

(2) if we assume that this condition is satisfied, we are not able in general to prove that $\mathbf{R}f_*$ is a localization (this is a generalization of Conjecture 1.2). So even in this case we use $\mathfrak{D}_{\mathrm{coh}}(Y, S)$ instead of $\mathfrak{D}^b_{\mathrm{coh}}(Y)$.

Using these localization statements as building blocks, the proof of Theorem 4.1 is obtained by induction of the number of blow-ups of smooth centers in the resolution process of X_{red}.

5. Existence of smooth compactifications

In this section, we assume that the characteristic of the base field k is zero.

Recall the question of Toën, mentioned in the introduction.

Question 5.1. Let $\mathcal{A}$ be a homotopically finite DG category. Does it admit a smooth compactification?

Quite surprisingly, the paper [8] gives a negative answer. We briefly explain the idea of a counterexample, and its close relation with generalized versions of the non-commutative Hodge-to-de Rham degeneration.

Recall that the classical Hodge theory implies that for any smooth and proper algebraic variety X over a field k of characteristic zero the spectral sequence

$$E_1^{p,q} = H^q_{Zar}(X, \Omega^p_X) \Rightarrow H^{p+q}_{DR}(X)$$

degenerates. Here the limit of the spectral sequence is the algebraic de Rham cohomology.

The following categorical generalization was conjectured by Kontsevich and Soibelman [14], and proved by Kaledin [11].

Theorem 5.2 ([11, Theorem 5.4]). *Let A be a smooth and proper DG algebra over a field of characteristic zero. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism $HP_\bullet(A) \cong HH_\bullet(A)((u))$.*

The following conjectures were formulated by Kontsevich (unpublished).

Conjecture 5.3 (Kontsevich). *Let A be a smooth DG algebra over a field of characteristic zero. Then the composition*

$$K_0(A \otimes A^{\mathrm{op}}) \xrightarrow{\mathrm{ch}} \big(HH_\bullet(A) \otimes HH_\bullet(A^{\mathrm{op}})\big)_0 \xrightarrow{\mathrm{id}\otimes\delta^-} \big(HH_\bullet(A) \otimes HC^-_\bullet(A^{\mathrm{op}})\big)_1$$

vanishes on the class $[A]$ of the diagonal bimodule.

Here $\delta^- : HH_\bullet(A^{\mathrm{op}}) \to HC^-_\bullet(A^{\mathrm{op}})[-1]$ denotes the boundary map in the long exact sequence

$$\cdots \to HC^-_{n+1}(A^{\mathrm{op}}) \to HC^-_{n-1}(A^{\mathrm{op}}) \to HH_{n-1}(A^{\mathrm{op}}) \xrightarrow{\delta^-} HC^-_n(A^{\mathrm{op}}) \to \cdots;$$

see for example [7, Section 3].

Conjecture 5.4 (Kontsevich). *Let B be a proper DG algebra over a field* k *of characteristic zero. Then the composition map*

$$\big(HH_\bullet(B) \otimes HC_\bullet(B^{\mathrm{op}})\big)[1] \xrightarrow{\mathrm{id}\otimes\delta^+} HH_\bullet(B) \otimes HH_\bullet(B^{\mathrm{op}}) \to \mathrm{k} \qquad (5.1)$$

is zero.

Here $\delta^+ : HC_\bullet(B^{\mathrm{op}})[1] \to HH_\bullet(B^{\mathrm{op}})$ denotes the boundary map in a similar long exact sequence

$$\cdots \to HH_{n+1}(B^{\mathrm{op}}) \to HC_{n+1}(B^{\mathrm{op}}) \to HC_{n-1}(B^{\mathrm{op}}) \xrightarrow{\delta^+} HH_n(B^{\mathrm{op}}) \to \cdots;$$

see [16, Section 2.2]. The second map in (5.1) is given by the composition

$$HH_\bullet(B) \otimes HH_\bullet(B^{\mathrm{op}}) \cong HH_\bullet(B \otimes B^{\mathrm{op}}) \to HH_\bullet\big(\mathrm{End}_{\mathrm{k}}(B)\big) \cong HH_\bullet(\mathrm{k}) = \mathrm{k},$$

where we used the Künneth isomorphism for HH, the diagonal bimodule structure on B, and the (derived) Morita equivalence between $\mathrm{End}_{\mathrm{k}}(B)$ and k.

Both Conjectures 5.3 and 5.4 had a strong motivation. Namely, in the case of smooth DG algebras, the following holds.

Proposition 5.5 ([8, Proposition 4.1]). *Let B be a smooth DG algebra and F : $\mathrm{Perf}(A) \to \mathrm{Perf}(B)$ a localization functor, where A is a smooth and proper DG algebra. Then Conjecture 5.3 holds for B.*

This is easy to prove (of course, assuming Kaledin's theorem (Theorem 5.2)). Namely, it almost immediately follows from the commutative diagram

$$\begin{array}{ccc} HH_\bullet(A) \otimes HH_\bullet(A^{\mathrm{op}}) & \xrightarrow{\mathrm{id}\otimes\delta^-} & HH_\bullet(A) \otimes HC_\bullet^-(A^{\mathrm{op}})[-1] \\ \downarrow & & \downarrow \\ HH_\bullet(B) \otimes HH_\bullet(B^{\mathrm{op}}) & \xrightarrow{\mathrm{id}\otimes\delta^-} & HH_\bullet(B) \otimes HC_\bullet^-(B^{\mathrm{op}})[1], \end{array}$$

and from the degeneration of the Hochschild-to-cyclic spectral sequence for A. We have the following corollary.

Corollary 5.6 ([8, Corollary 4.2]). *Let X be a separated scheme of finite type over* k, *and $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$ – a generator. Then Conjecture 5.3 holds for the smooth DG algebra $A = \mathbf{R}\mathrm{End}(\mathcal{G})$.*

Indeed, this follows from Proposition 5.5 and from Theorem 1.1 (in fact, a weakened version of Theorem 1.1 is sufficient; see [8, Remark 4.3]).

Similar (dual) statements hold for proper DG algebras.

Proposition 5.7 ([8, Proposition 5.1]). *Let B be a proper DG algebra and $\mathrm{Perf}(B) \hookrightarrow \mathrm{Perf}(A)$ a fully-faithful functor, where A is a smooth and proper DG algebra. Then Conjecture 5.4 holds for B.*

Corollary 5.8 ([8, Corollary 5.2]). *Let X be a separated scheme of finite type over* k, *and $Z \subset X$ a closed proper subscheme. For any object $\mathcal{F} \in \mathrm{Perf}_Z(X)$, Conjecture 5.4 holds for the proper DG algebra $B = \mathbf{R}\mathrm{End}(\mathcal{F})$.*

However, we disproved both Conjectures 5.3 and 5.4. The counterexamples are provided by [8, Theorems 4.5 and 5.4]. The counterexample to Conjecture 5.3 is in fact hfp, hence by Proposition 3.6 it gives a negative answer to Question 1.4.

We briefly describe the counterexample to Conjecture 5.4. Recall that given DG algebras A and B, together with an A-B-bimodule M, we can form a gluing $C = \left(\begin{smallmatrix} B & 0 \\ M & A \end{smallmatrix}\right)$. This is a DG algebra which equals $A \oplus B \oplus M$ as a complex of vector spaces, and the multiplication is given by

$$(a, b, m) \cdot (a', b', m') = (aa', bb', am' + mb').$$

Let us take $A = \mathrm{k}[x]/x^6$ and $B = \mathrm{k}[y]/y^3$, where $\deg(x) = 0$, $\deg(y) = 1$, and $dx = 0$, $dy = 0$. Then one can show that there exists a DG A-B-bimodule M such that $H^\bullet(M) = \mathrm{k}[0]$ and the DG algebra $C = \left(\begin{smallmatrix} B & 0 \\ M & A \end{smallmatrix}\right)$ is a counterexample to Conjecture 5.4. Namely, the DG algebra C is proper (but not smooth), and its cohomology $H^\bullet(C)$ is 10-dimensional. Further, we have the elements $x \in H^0(C)$, $y \in H^1(C)$. Using natural maps $H^n(C) \to HH_{-n}(C) \to HC_{-n}(C)$ and similarly for C^{op}, we can consider x and y as classes in Hochschild and cyclic homology, respectively: $x \in HH_0(C)$, $y \in HC_{-1}(C^{\mathrm{op}})$. Now, a bimodule M is constructed in such a way that $\langle x, \delta^+(y)\rangle \neq 0$, disproving conjecture 5.4. For details see [8, Theorem 5.4].

6. Wall's finiteness obstruction for DG categories

Here we mention some new results, to appear in [10]. In particular, we formulate a criterion for a homotopically finite DG category to have a smooth compactification.

As we already mentioned, the notion of an hfp DG category is analogous to the notion of a finitely dominated CW complex.

In 1959, Milnor [18] asked if every finitely dominated CW complex X is homotopy equivalent to a finite CW complex. This was already known in the case when each connected component of X is simply connected, but it was considered to be a difficult problem in general.

For simplicity, let us assume that X is connected. In 1965, C. T. C. Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (an element of the reduced Grothendieck group of $\mathbb{Z}[\pi_1(X)]$) for any finitely dominated space X. Recall that for an associative unital ring A the group $K_0(A)$ is generated by isomorphism classes of finitely generated projective (right) A-modules $[P]$, subject to relations $[P \oplus Q] = [P] + [Q]$. If a ring A is equipped with a unital homomorphism $A \to \mathbb{Z}$ (i.e., A is augmented), its reduced Grothendieck group $\widetilde{K_0}(A)$ is defined to be the kernel $\ker(K_0(A) \to K_0(\mathbb{Z}) = \mathbb{Z})$. In fact, we have a decomposition $K_0(A) \cong \mathbb{Z} \oplus \widetilde{K_0}(A)$. Note that for any group G the group ring $\mathbb{Z}[G]$ is naturally augmented. Wall proved the following result.

Theorem 6.1 ([28, Theorem F]). *A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.*

Probably the simplest description (and different from the original one) of the class $w(X)$ is the following. Recall that for a DG ring B the group $K_0(B)$ is defined to be the Grothendieck group $K_0(D_{\mathrm{perf}}(B))$. Here for a small triangulated category T the group $K_0(T)$ is generated by the isomorphism classes of objects $[X]$, $X \in T$, subject to relations $[Y] = [X] + [Z]$ for an exact triangle

$$X \to Y \to Z \to X[1]$$

in T. If the DG ring B is concentrated in degree zero, i.e., $B = H^0(B)$, then the two definitions of $K_0(B)$ agree. Moreover, if B is (cohomologically) non-positively graded, then $K_0(B) \cong K_0(H^0(B))$; see [4, Theorem 5.3.1 and Proposition 6.2.1].

Now choose a base point $x_0 \in X$. Consider the DG ring $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space. By the result of Brav–Dyckerhoff [5, Proposition 5.1], the DG ring $C_\bullet(\Omega_{x_0} X)$ is smooth over $\mathbb{Z}$ (and moreover it is hfp). It follows that the augmentation module $\mathbb{Z}$ is perfect: $\mathbb{Z} \in \mathrm{Perf}(C_\bullet(\Omega_{x_0} X))$. Any perfect module defines a class in K_0, hence we have a well-defined class

$$\widetilde{w}(X) := [\mathbb{Z}] \in K_0\big(C_\bullet(\Omega_{x_0} X)\big) \cong K_0\big(\mathbb{Z}\big[\pi_1(X, x_0)\big]\big),$$

since $H_0(\Omega_{x_0} X) \cong \mathbb{Z}[\pi_1(X, x_0)]$. The class $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X, x_0)])$ is simply the projection of $\widetilde{w}(X)$.

Remark. The class $\widetilde{w}(X) \in K_0(\mathbb{Z}[\pi_1(X, x_0)])$ contains essentially the same information as $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X, x_0)])$. Namely, under the identification

$$K_0\big(\mathbb{Z}\big[\pi_1(X, x_0)\big]\big) \cong \mathbb{Z} \oplus \widetilde{K_0}\big(\mathbb{Z}\big[\pi_1(X, x_0)\big]\big)$$

the class $\widetilde{w}(X)$ is given by $(\chi(X), w(X))$, where $\chi(X)$ is the Euler characteristic.

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0} X))$ is an integer multiple of the class $[C_\bullet(\Omega_{x_0} X)]$.

Now fix some base field k of arbitrary characteristic. For a small DG category $\mathcal{A}$, we put $K_0(\mathcal{A}) := K_0(D_{\mathrm{perf}}(\mathcal{A}))$. Recall that we denote by $I_{\mathcal{A}}$ the diagonal $\mathcal{A}$-$\mathcal{A}$-bimodule.

Theorem 6.2 ([10]). *For a small DG category $\mathcal{A}$, the following are equivalent:*

(i) *$\mathcal{A}$ is Morita equivalent to a finite cell DG category;*

(ii) *$\mathcal{A}$ is hfp, and moreover $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{\mathrm{op}}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}))$;*

(iii) *$\mathcal{A}$ is Morita equivalent to a DG quotient $\mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S}$ is a subcategory generated by a single object.*

Remark. To explain the analogy between Theorem 6.2 and Wall's theorem, let us consider the following three categories with a class of morphisms called weak equivalences (the most important part of the model structure):

(1) the category Top of topological spaces, with a class of weak homotopy equivalences;

(2) the category $Z^0(\text{Mod-}\mathcal{A})$ (see Section 2 for this notation) of right DG modules over a fixed DG category $\mathcal{A}$, with a class of quasi-isomorphisms;

(3) the category dgcat_k of small DG categories over a field k, with a class of Morita equivalences.

For each of these categories, one has the class of "finite cell" objects, namely: finite CW complexes in Top, semi-free finitely generated $\mathcal{A}$-modules in $Z^0(\text{Mod-}\mathcal{A})$, and finite cell DG categories in dgcat_k. Then we have the classes of hfp objects: these are homotopy retracts of finite cell objects. Thus, the hfp objects are as follows: finitely dominated spaces in Top, perfect $\mathcal{A}$-modules in $Z^0(\text{Mod-}\mathcal{A})$, hfp DG categories in dgcat_k.

Now, Wall's theorem (more precisely, an analogue of Theorem 6.1 for not necessarily connected spaces) gives a K-theoretic criterion for a finitely dominated space to have a homotopy type of a finite CW complex.

Further, Thomason's classification of dense subcategories of triangulated categories [24, Theorem 2.1] gives a K-theoretic criterion for a perfect $\mathcal{A}$-module M to be quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module. This happens if and only if the class $[M] \in K_0(\mathcal{A})$ is contained in the subgroup generated by the classes of representable $\mathcal{A}$-modules.

From this point of view, our theorem (Theorem 6.2) is an analogue of the results of Wall and Thomason for DG categories, plus also an alternative characterization of finite cell DG categories (equivalence (i)⇔(iii)). The following table summarizes the above discussion.

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k
Weak homotopy equivalences	Quasi-isomorphisms	Morita equivalences
Finite CW complexes	Semi-free finitely generated $\mathcal{A}$-modules	Finite cell DG categories
Finitely dominated spaces	Perfect $\mathcal{A}$-modules	hfp DG categories
Wall's finiteness obstruction theorem	Thomason's classification of dense subcategories	Theorem 6.2

There are different ways to formulate a "relative" version of Theorem 6.2. We choose the following "minimalistic" formulation.

Theorem 6.3 ([10]). *Let $\mathcal{A}$ and $\mathcal{B}$ be hfp, pre-triangulated, Karoubi complete DG categories, and $\mathcal{B} \neq 0$. The following are equivalent.*

(i) *The class* $[I_{\mathcal{A}}] \in K_0(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ *is contained in the subgroup generated by the images* $\mathrm{Im}(K_0(\mathcal{B} \otimes \mathcal{B}^{\mathrm{op}}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}))$ *under various pairs of quasi-functors* $(\mathcal{B} \to \mathcal{A}, \mathcal{B}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}})$.

(ii) *We have a Morita equivalence* $\mathcal{A} \simeq \mathcal{C}/\mathcal{S}$, *where* $\mathcal{C} = \langle \mathcal{B}, \dots, \mathcal{B} \rangle$ *is a (smooth) semi-orthogonal gluing of a finite number of copies of* $\mathcal{B}$, *and* $\mathcal{S} \subset \mathcal{C}$ *is generated by a single object.*

Using this relative version of Wall's finiteness obstruction for DG categories, we prove the following criterion for existence of a categorical smooth compactification.

Theorem 6.4 ([10]). *Let* $\mathcal{A}$ *be an hfp pre-triangulated DG category. The following are equivalent.*

(1) $\mathcal{A}$ *admits a smooth categorical compactification.*

(2) *There exists a DG functor* $\mathcal{C} \to \mathcal{A}$, *where* $\mathcal{C}$ *is smooth and proper, such that*

$$[I_{\mathcal{A}}] \in \mathrm{Im}\left(K_0(\mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})\right).$$

For example, this allows to show existence of a smooth compactification of the derived category of coherent D-modules on a separated scheme of finite type X over a field of characteristic zero (although it is not clear how to construct such compactification explicitly).

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References

[1] M. Auslander, Representation theory of Artin algebras. I, II. *Comm. Algebra* **1** (1974), 177–268; ibid. 1 (1974), 269–310 Zbl 0285.16029 MR 349747

[2] A. Bondal and D. Orlov, Derived categories of coherent sheaves. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 47–56, Higher Ed. Press, Beijing, 2002 Zbl 0996.18007 MR 1957019

[3] A. I. Bondal and M. M. Kapranov, Framed triangulated categories. *Mat. Sb.* **181** (1990), no. 5, 669–683 MR 1055981

[4] M. V. Bondarko, Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory* **6** (2010), no. 3, 387–504 Zbl 1303.18019 MR 2746283

[5] C. Brav and T. Dyckerhoff, Relative Calabi–Yau structures. *Compos. Math.* **155** (2019), no. 2, 372–412 Zbl 1436.18009 MR 3911626

[6] V. Drinfeld, DG quotients of DG categories. *J. Algebra* **272** (2004), no. 2, 643–691 Zbl 1064.18009 MR 2028075

[7] A. I. Efimov, Generalized non-commutative degeneration conjecture. *Proc. Steklov Inst. Math.* **290** (2015), no. 1, 1–10 Zbl 1338.14006 MR 3488776

[8] A. I. Efimov, Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration. *Invent. Math.* **222** (2020), no. 2, 667–694 Zbl 07269005 MR 4160877

[9] A. I. Efimov, Homotopy finiteness of some DG categories from algebraic geometry. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 9, 2879–2942 Zbl 1478.14010 MR 4127943

[10] A. I. Efimov, Wall's finiteness obstruction for DG categories. In preparation

[11] D. Kaledin, Spectral sequences for cyclic homology. In *Algebra, Geometry, and Physics in the 21st Century*, pp. 99–129, Progr. Math. 324, Birkhäuser/Springer, Cham, 2017 Zbl 1390.14011 MR 3702384

[12] B. Keller, Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)* **27** (1994), no. 1, 63–102 Zbl 0799.18007 MR 1258406

[13] B. Keller, On the cyclic homology of exact categories. *J. Pure Appl. Algebra* **136** (1999), no. 1, 1–56 Zbl 0923.19004 MR 1667558

[14] M. Kontsevich and Y. Soibelman, Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In *Homological Mirror Symmetry*, pp. 153–219, Lecture Notes in Phys. 757, Springer, Berlin, 2009 Zbl 1202.81120 MR 2596638

[15] A. Kuznetsov and V. A. Lunts, Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN* **2015** (2015), no. 13, 4536–4625 Zbl 1338.14020 MR 3439086

[16] J.-L. Loday, *Cyclic Homology*. Grundlehren Math. Wiss. 301, Springer, Berlin, 1992 Zbl 0780.18009 MR 1217970

[17] V. A. Lunts, Categorical resolution of singularities. *J. Algebra* **323** (2010), no. 10, 2977–3003 Zbl 1202.18006 MR 2609187

[18] J. Milnor, On spaces having the homotopy type of a CW-complex. *Trans. Amer. Math. Soc.* **90** (1959), 272–280 Zbl 0084.39002 MR 100267

[19] A. Neeman, *Triangulated Categories*. Ann. of Math. Stud. 148, Princeton University Press, Princeton, NJ, 2001 Zbl 0974.18008 MR 1812507

[20] D. Orlov, Formal completions and idempotent completions of triangulated categories of singularities. *Adv. Math.* **226** (2011), no. 1, 206–217 Zbl 1216.18012 MR 2735755

[21] D. Orlov, Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.* **302** (2016), 59–105 Zbl 1368.14031 MR 3545926

[22] G. N. Tabuada, Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris* **340** (2005), no. 1, 15–19 Zbl 1060.18010 MR 2112034

[23] G. N. Tabuada, *Théorie homotopique des DG-catégories*. Thèse de doctorat, Université Paris-Diderot - Paris 7, 2007

[24] R. W. Thomason, The classification of triangulated subcategories. *Compos. Math.* **105** (1997), no. 1, 1–27 Zbl 0873.18003 MR 1436741

[25] B. Toën and M. Vaquié, Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)* **40** (2007), no. 3, 387–444 Zbl 1140.18005 MR 2493386

[26] J.-L. Verdier, Catégories dérivées: quelques résultats (état 0). In *Cohomologie étale*, pp. 262–311, Lecture Notes in Math. 569, Springer, Berlin, 1977 Zbl 0407.18008 MR 3727440

[27] J.-L. Verdier, Des catégories dérivées des catégories abéliennes. *Astérisque* **239** (1996), xii+253 pp. Zbl 0882.18010 MR 1453167

[28] C. T. C. Wall, Finiteness conditions for CW-complexes. *Ann. of Math. (2)* **81** (1965), 56–69 Zbl 0152.21902 MR 171284

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