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Lefschetz fibrations, open books, and symplectic fillings of contact 3-manifolds

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Abstract. Ever since Donaldson showed that every closed symplectic 4-manifold admits a Lefschetz pencil and Giroux proved that every closed contact 3-manifold admits an adapted open book decomposition, Lefschetz fibrations and open books have been used fruitfully to obtain significant results about the topology of symplectic 4-manifolds and contact 3-manifolds. In this expository article, we present the highlights of our contribution to the subject at hand based on joint work with several coauthors during the past twenty years.

1. Introduction

At the turn of the century, two groundbreaking results have surfaced which had a long-lasting impact on the study of global topology of symplectic 4-manifolds and contact 3-manifolds. These results respectively are Donaldson's existence theorem [19] about Lefschetz pencils on closed symplectic 4-manifolds and Giroux's correspondence [30] between open books and contact structures on closed 3-manifolds.

In the first half of this short expository article, we briefly review the results of Donaldson and Giroux. In the last half, we first present an analogous result on Stein domains of complex dimension two, with an eye towards some applications to the study of the topology of symplectic fillings of contact 3-manifolds. Then we demonstrate how Lefschetz fibrations and open books interact with the classical theory of complex surface singularities as well as trisections of arbitrary smooth 4-manifolds, which were relatively recently discovered by Gay and Kirby [25].

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2. Topological characterization of symplectic 4-manifolds

Suppose that X and Σ are compact, oriented, and smooth manifolds of dimensions four and two, respectively, possibly with *nonempty* boundaries.

Definition 2.1. A *Lefschetz fibration* $\pi\colon X \to \Sigma$ is a submersion except for finitely many points $\{p_1, \ldots, p_k\}$ in the interior of X, such that around each p_i and $\pi(p_i)$, there are orientation-preserving complex charts, on which π is of the form $\pi(z_1, z_2) = z_1^2 + z_2^2$.

The topology of Lefschetz fibrations is well understood with multiple points of view. We advise the reader to turn to the book [33] of Gompf and Stipsicz for an excellent introduction to the subject.

Lefschetz critical points can be viewed as complex analogs of Morse critical points, and they correspond to 2-handles. As a result, one obtains a handle decomposition of the 4-manifold X. Since a Lefschetz fibration is locally trivial in the complement of finitely many singular fibers, it can also be described combinatorially by means of its *monodromy*. Locally, the fiber of the map $(z_1, z_2) \to z_1^2 + z_2^2$ above $0 \neq t \in \mathbb{C}$ is smooth (topologically an annulus), while the fiber above the origin has a transverse double point (aka nodal singularity) and is obtained from the nearby fibers by collapsing an embedded simple closed curve called the *vanishing cycle*, as illustrated in Figure 1.

Let $\pi\colon X \to \Sigma$ be a Lefschetz fibration and let γ be a loop in Σ enclosing a single critical value, whose critical fiber has a single node. Then π restricts to surface fibration over γ, whose monodromy (a diffeomorphism of the fiber) is given by the right-handed Dehn twist about the vanishing cycle, as depicted in Figure 2.

For the purposes of this article, we assume that each singular fiber carries a *unique singularity* and there are *no homotopically trivial* vanishing cycles. Moreover, we restrict our attention to the following two cases.

First case, $\Sigma = S^2$, $\partial X = \emptyset$, and hence the fibers are closed surfaces. Suppose that $q_1, \ldots, q_k \in D^2 \subset S^2$ are the critical values of a genus g Lefschetz fibration $\pi\colon X \to S^2$. Let $q_0 \in D^2$ be a regular value and for each $1 \leq i \leq k$, let $\gamma_i \subset D^2$ be a loop based at q_0 enclosing a single critical value q_i as shown in Figure 3. By the discussion above, the monodromy of the fibration over each γ_i is a positive Dehn twist along the corresponding vanishing cycle.

Since the fibration π is trivial over the complement $S^2 \setminus D^2$, the product of positive Dehn twists along the vanishing cycles is isotopic to the identity. The upshot is that a Lefschetz fibration $\pi\colon X \to S^2$ is characterized by a positive Dehn twist factorization of the identity element in Map_g, the mapping class group of an oriented closed surface of genus g.

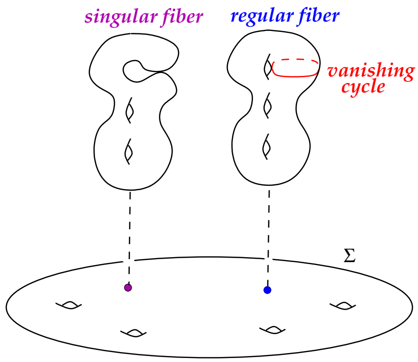


Figure 1. A nodal singularity.

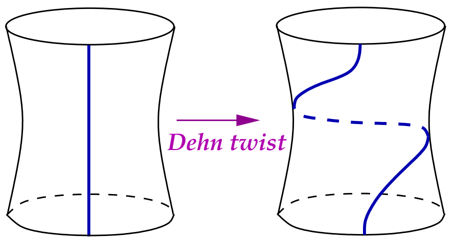


Figure 2. The right-handed (positive) Dehn twist.

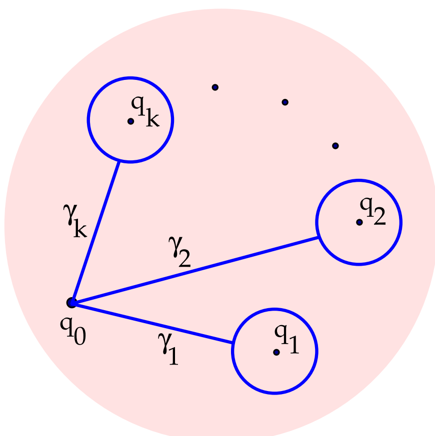


Figure 3. Loops in the base disk.

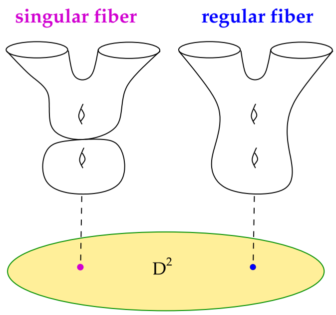


Figure 4. Fibers in a Lefschetz fibration.

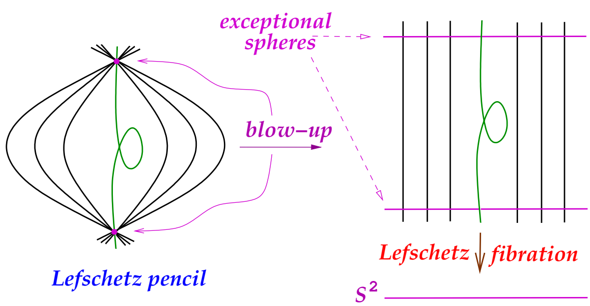


Figure 5. Blowing up the base-locus of a Lefschetz pencil.

Second case, $\Sigma = D^2$*, the fibers have nonempty boundary and hence* $\partial X \neq \emptyset$*.* In this case, the global monodromy over the boundary of the base disk D^2 is a product of positive Dehn twists in $\mathrm{Map}_{g,r}$ (the mapping class group of an oriented genus g surface with $r > 0$ boundary components), *with no other constraints* (see Figure 4). Moreover, ∂X inherits a natural *open book decomposition*, which we will discuss in details later in Section 3.

Definition 2.2. A *Lefschetz pencil* on a closed and oriented 4-manifold X is a map $\pi\colon X - \{b_1, \ldots, b_n\} \to S^2$, submersive except for a finite set $\{p_1, \ldots, p_k\}$, conforming to local models

(i) $(z_1, z_2) \to z_1/z_2$ near each b_i and

(ii) $(z_1, z_2) \to z_1^2 + z_2^2$ near each p_j.

By blowing up X at the base-locus $\{b_1, \ldots, b_n\}$, we obtain a Lefschetz fibration

$$X \# n\, \overline{\mathbb{CP}^2} \to S^2$$

with n disjoint sections, which are the exceptional spheres in the blow-up, as illustrated in Figure 5.

In the early twentieth century, Lefschetz showed that every *algebraic surface* (4-manifold arising as the zero-locus of a collection of homogeneous polynomials in $\mathbb{CP}^n$) admits "Lefschetz" pencils, which enabled him to study the topology of algebraic surfaces. This result was extended by Donaldson, to the case of the much larger class of symplectic 4-manifolds (i.e., those admitting closed non-degenerate 2-forms).

Theorem 2.3 (Donaldson [19]). *Any closed symplectic* 4*-manifold admits a Lefschetz pencil.*

For a sketch of the proof of Theorem 2.3 (other than Donaldson's original papers [18, 19]), the interested reader may consult the lecture notes [6] of Auroux and Smith, which is a wide-ranging survey, touching on the uses of Donaldon's theory of Lefschetz pencils and their relatives in 4-dimensional topology and mirror symmetry.

Conversely, generalizing a similar result of Thurston [58] on surface bundles over surfaces, Gompf [33] showed that if $\pi : X \to \Sigma$ is a Lefschetz fibration for which the fiber represents a non-torsion homology class,[1] then X admits a symplectic structure with symplectic fibers. As a corollary, he showed that any closed 4-manifold which admits a Lefschetz pencil, is symplectic.

Combining the results of Donaldson and Gompf, we obtain a *topological characterization* of symplectic 4-manifolds which has lead to a renewed interest in Lefschetz pencils/fibrations and hundreds of papers have been devoted to the study of various aspects and generalizations of Lefschetz fibrations, over the past twenty years. Here is one of the earlier results.

Theorem 2.4 (Ozbagci and Stipsicz [47]). *There are infinitely many pairwise non-homeomorphic closed* 4*-manifolds, each of which admits a genus two Lefschetz fibration over* S^2 *but does not carry complex structure with either orientation.*[2]

The examples in Theorem 2.4 are obtained by fiber sums of genus two Lefschetz fibrations $S^2 \times T^2 \# 4\,\overline{\mathbb{CP}^2} \to S^2$ of Matsumoto [39], which also shows that fiber sums of *holomorphic* Lefschetz fibrations are *not necessarily holomorphic*.

3. Topological characterization of contact 3-manifolds

Definition 3.1. An *open book decomposition* of a closed and oriented 3-manifold Y is a pair (B, π) consisting of an oriented link $B \subset Y$, and a locally trivial fibration $\pi \colon Y - B \to S^1$ such that B has a trivial tubular neighborhood $B \times D^2$ in which π is

[1] This hypothesis is automatically satisfied if the fiber genus is not equal to one.
[2] This result was independently observed by Ivan Smith.

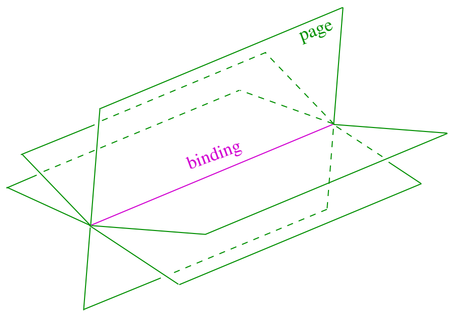


Figure 6. I am an open book!

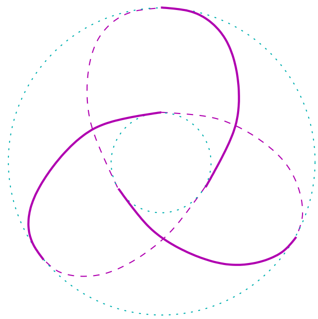


Figure 7. $(2, 3)$-torus knot (*the trefoil*).

given by the angular coordinate in the D^2-factor (see Figure 6). Here B is called the *binding* and the closure of each fiber of π, which is a Seifert surface for B, is called a *page*.

Example 3.2 (Milnor's fibration). Consider the polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(z_1, z_2) = z_1^p + z_2^q$, where $p, q \geq 2$ are relatively prime. Then $B = f^{-1}(0) \cap S^3$ is the (p, q)-torus knot in S^3 whose complement fibers over S^1:

$$\pi\colon S^3 - B \to S^1 := \frac{f(z_1, z_2)}{\left|f(z_1, z_2)\right|}.$$

Hence (B, π) is an open book for S^3 with connected binding. The torus knot for the case $p = 2$ and $q = 3$ is depicted in Figure 7.

For any given open book, one can choose a vector field which is transverse to the pages and meridional near the binding. Then the isotopy class of the first return map on a fixed page is called the *monodromy* of the open book. The topology of an open book is determined by the topology of its page and its monodromy.

Suppose that $\pi : X \to D^2$ is a Lefschetz fibration such that the regular fiber F has nonempty boundary ∂F. Then ∂X is the union of two pieces:

- the horizontal boundary, $\partial F \times D^2$ (see Figure 8) and
- the vertical boundary, $\pi^{-1}(\partial D^2)$ (see Figure 9),

glued together along the tori $\partial F \times \partial D^2$. It follows that ∂X inherits a natural open book, whose page is the fiber F and whose monodromy coincides with the monodromy of the Lefschetz fibration $\pi : X \to D^2$.

A differential 1-form α on a 3-manifold Y is called a *contact form* if $\alpha \wedge d\alpha$ is a volume form. A 2-dimensional distribution ξ in TY is called a contact structure if it can be given as the kernel of a contact form α. The pair (Y, ξ) is called a *contact 3-manifold*.

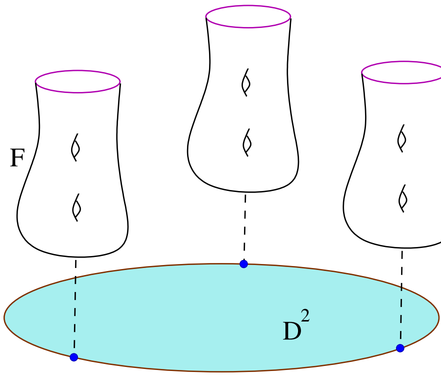


Figure 8. The vertical boundary: $\pi^{-1}(\partial D^2)$.

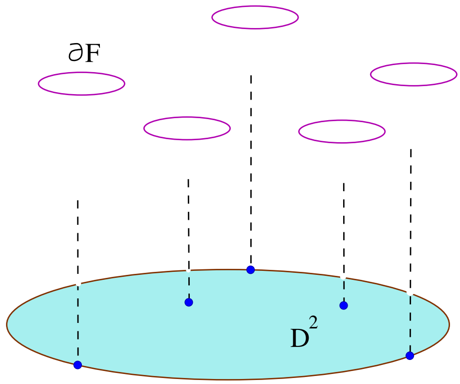


Figure 9. The horizontal boundary: $\partial F \times D^2$.

There are no local invariants of contact structures by *Darboux's theorem*, which says that any point in a contact 3-manifold has a neighborhood isomorphic to a neighborhood of the origin in the standard contact structure $\xi = \ker(dz + x dy)$ in $\mathbb{R}^3$, which is depicted in Figure 10.

We advise the reader to turn to the book [28] of Geiges, for a thorough introduction to contact topology in general dimensions and to the book [49] of Stipsicz and the author for a rapid course in dimension 3.

A classical theorem of Alexander [5] says that every closed oriented 3-manifold admits an open book decomposition and Martinet [38] showed that every closed oriented 3-manifold carries a contact structure. In 1975, Thurston and Winkelnkemper [59] presented an alternate proof of Martinet's theorem by constructing contact forms on closed 3-manifolds using open books.

Definition 3.3. A contact structure ξ on a 3-manifold Y is said to be supported by an open book (B, π) if ξ can be given by a contact form α such that $\alpha(B) > 0$ and $d\alpha > 0$ on every page.

In view of Definition 3.3, the result of Thurston and Winkelnkemper can be rephrased as follows: every open book on a closed oriented 3-manifold supports a contact structure.

The converse (i.e., every contact structure on a closed oriented 3-manifold is supported by an open book) was proven by Giroux. In fact, he proved the following theorem, which is known as *Giroux's correspondence*.

Theorem 3.4 (Giroux [30]). *On a closed oriented* 3*-manifold, there is a one-to-one correspondence between the set of isotopy classes of contact structures and open books up to positive stabilization.*

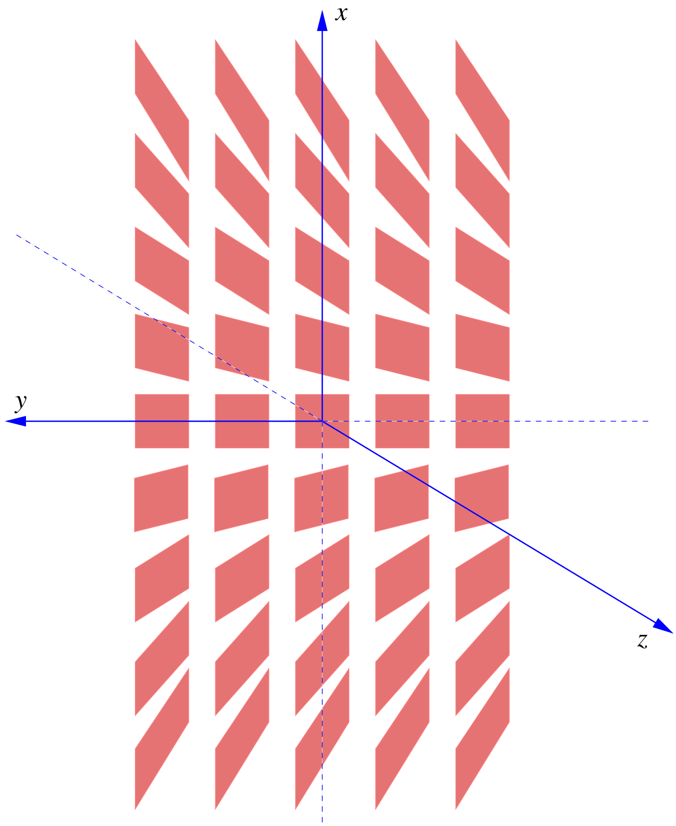


Figure 10. The standard contact structure $\xi = \ker(dz + x dy)$ in $\mathbb{R}^3$.

For a detailed sketch of the proof of Theorem 3.4, we refer to Etnyre's lecture notes [21].

4. Topological characterization of Stein domains of complex dimension two

Definition 4.1. A *Stein manifold* is an affine complex manifold, i.e., a complex manifold that admits a proper holomorphic embedding into some $\mathbb{C}^N$.

Suppose that $\phi\colon X \to \mathbb{R}$ is a smooth function on a complex manifold (X, J). Let ω_ϕ denote the 2-form $-d(d\phi \circ J)$. Then the map $\phi\colon X \to \mathbb{R}$ is called *J-convex* (aka *strictly plurisubharmonic*) if $\omega_\phi(u, Ju) > 0$ for all nonzero vectors $u \in TX$. It follows that ω_ϕ is an exact symplectic form on X.

Grauert's characterization. A complex manifold (X, J) is Stein if and only if it admits a proper J-convex function $\phi\colon X \to [0, \infty)$.

We advise the reader to turn to the book [17] of Eliashberg and Cieliebak, for a meticulous treatment of Stein (and Weinstein) manifolds. For the purposes of this article, we now restrict our attention to *Stein surfaces* (of complex dimension two), for which the reader may consult [32] for an elaborate discussion.

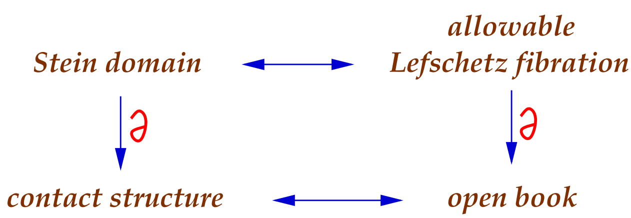
Suppose that (X, J) is a Stein surface. For any proper J-convex *Morse* function $\phi\colon X \to [0, \infty)$, each regular level set Y of ϕ is a contact 3-manifold, where the contact structure is given by the kernel of $\alpha_\phi = -d\phi \circ J$ or, equivalently, by the *complex tangencies* $TY \cap JTY$. For any regular value c of ϕ, the sublevel set $W = \phi^{-1}([0, c])$ is called a *Stein domain*. We also say that the compact 4-manifold (W, J) is a *Stein filling* of its contact boundary $(\partial W, \ker \alpha_\phi)$.

By the work of Eliashberg [20] and Gompf [32] a handle decomposition of a Stein domain (W, J) is well understood: it consists of a 0-handle, some 1-handles, and some 2-handles attached along Legendrian knots (those tangent to the contact planes) with framing -1 relative to the contact planes.

The following theorem, whose proof is based on the handle decomposition above, is somewhat analogous to Donaldson's theorem on the existence of Lefschetz pencils on closed symplectic manifolds.

Theorem 4.2 (Akbulut and Ozbagci [1] and Loi and Piergallini [36]). *A Stein domain admits an allowable*[3] *Lefschetz fibration over D^2 and, conversely, any allowable Lefschetz fibration over D^2 admits a Stein structure.*

Moreover, by modifying the proof of Akbulut and the author, Plamenevskaya [52] showed that the contact structure induced on the boundary of the Stein domain is supported by the open book inherited by the Lefschetz fibration. As a result we have the diagram



which gives a *criterion for Stein fillability: a contact 3-manifold is Stein fillable if and only if it admits a supporting open book whose monodromy can be factorized into positive Dehn twists.*[4]

[3] The vanishing cycles are homologically non-trivial.

[4] This was independently proved by Giroux.

Definition 4.3. A compact symplectic 4-manifold (X, ω) is a (strong) *symplectic filling* of a contact 3-manifold (Y, ξ) if $\partial X = Y$ (as oriented manifolds), ω is exact near the boundary, and its primitive α can be chosen so that $\ker(\alpha|_Y) = \xi$. A symplectic filling is called *minimal* if it does not contain any symplectically embedded sphere of self-intersection -1.

An active line of research in symplectic/contact topology is to classify *all Stein fillings* or more generally *all minimal symplectic fillings* of a given contact 3-manifold, up to diffeomorphism. It is clear by definition that every Stein filling is a minimal symplectic filling. The converse, however, is *not true* as shown by Ghiggini [29], using the celebrated Ozsváth–Szabó contact invariants [50].

The classification of Stein or more generally minimal symplectic fillings of a given contact 3-manifold is difficult in general. Nevertheless, this problem has been solved for many contact 3-manifolds, each of which has finitely many fillings. See the author's survey article [46] for the state of affairs until 2015.

The existence of a contact 3-manifold which admits infinitely many distinct Stein fillings was discovered by Stipsicz and the author. Let Y_g denote the closed 3-manifold, which is the total space of the open book whose page is a genus g surface with connected boundary and whose monodromy is the square of the boundary Dehn twist. Let ξ_g denote the contact structure on Y_g supported by this open book.

Theorem 4.4 (Ozbagci and Stipsicz [48]). *For each odd integer $g \geq 3$, the contact 3-manifold (Y_g, ξ_g) admits infinitely many pairwise non-homeomorphic Stein fillings.*

Outline of proof. A positive word in Map_g, for $g \geq 3$ (generalizing Matsumoto's genus two word [39]), was discovered independently by Cadavid [12] and Korkmaz [34]. For g odd, the word is $(c_0 c_1 c_2 \cdots c_g a^2 b^2)^2 = 1$, where, by an abuse of notation, each letter represents the right-handed Dehn twist along the curve decorated with the same letter, depicted in Figure 11. For each odd integer $g \geq 3$, there is a Lefschetz fibration over S^2, which corresponds to the aforementioned word. First we take (twisted) fiber sums of two copies of this Lefschetz fibration over S^2 and then remove a regular neighborhood of the union of a section and a regular fiber to get Stein fillings of the common contact boundary. The Stein fillings are distinguished by the torsion in their first homology groups, coming from the twistings in the fiber sums.

Remark 4.5. For a fixed odd integer $g \geq 3$, all the Stein fillings mentioned in Theorem 4.4 have the same Euler characteristic and the signature. In contrast, Baykur and Van Horn-Morris [8] showed that there are vast families of contact 3-manifolds each member of which admits infinitely many Stein fillings with arbitrarily large Euler characteristics and arbitrarily small signatures.

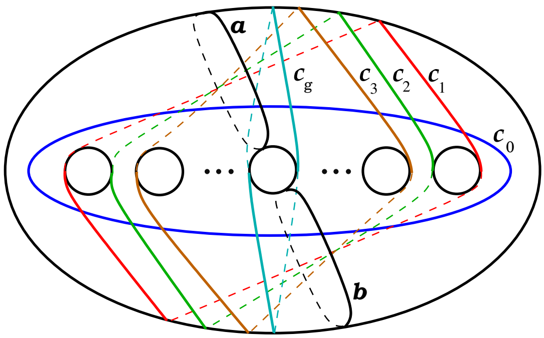


Figure 11. Curves on a genus g surface, for odd g.

5. Canonical contact structures on the links of isolated complex surface singularities

A fruitful source of Stein fillable contact 3-manifolds is given by the links of isolated complex surface singularities. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complex surface singularity. Then for a sufficiently small sphere $S_\varepsilon^{2N-1} \subset \mathbb{C}^N$ centered at the origin, $Y = X \cap S_\varepsilon^{2N-1}$ is a closed, oriented, and smooth 3-dimensional manifold, which is called *the link of the singularity*.

If J denotes the complex structure on X, then the plane field given by the complex tangencies $\xi := TY \cap JTY$ is a contact structure on Y—called the canonical (aka *Milnor fillable*) contact structure on the singularity link. The contact 3-manifold (Y, ξ) is called the *contact singularity link*. Note that ξ is determined uniquely, up to isomorphism, by a theorem of Caubel, Némethi, and Popescu-Pampu [14].

We advise the reader to turn to the comprehensive lecture notes [54] of Popescu-Pampu for an introduction to complex singularity theory and its relation to contact topology.

The *minimal resolution* of an isolated complex surface singularity provides a Stein filling of its contact singularity link (Y, ξ), by the work of Bogomolov and de Oliveira [11]. Moreover, if the singularity is smoothable, the general fiber X of a smoothing is called a *Milnor fiber*, which is a compact smooth 4-manifold such that $\partial X = Y$. Furthermore, X has a natural Stein structure so that it provides a Stein (hence minimal symplectic) filling of (Y, ξ). Therefore, a natural question arises as follows (see, for example, [41]): Does there exist a contact singularity link which admits Stein (or minimal symplectic) fillings other than the Milnor fibers (and the minimal resolution)?

The answer is negative for simple and simple elliptic singularities as shown by Ohta and Ono [43–45]. The answer is negative for cyclic quotient singularities as shown by the culmination of the work of several people: McDuff [40], Christophersen

[16], Stevens [56], Lisca [35], and Némethi and Popescu-Pampu [42]. The answer is negative for non-cyclic quotient singularities as well by the work of Stevens [57], Bhupal and Ono [9], and H. Park, J. Park, Shin, and Urzúa [51].

The first examples where the answer is affirmative were discovered by Akhmedov and the author.

Theorem 5.1 (Akhmedov and Ozbagci [3]). *There exists an infinite family of Seifert fibered contact singularity links such that each member of this family admits infinitely many exotic*[5] *Stein fillings. Moreover, none of these Stein fillings are homeomorphic to Milnor fibers.*

The exotic fillings mentioned in Theorem 5.1 are not simply connected. The first examples of infinitely many exotic simply-connected Stein fillings were discovered by Akhmedov, Etnyre, Mark, and Smith [2].

Moreover, Plamenevskaya and Starkston [53] recently showed that many *rational singularities* admit simply-connected Stein fillings that are not diffeomorphic to any Milnor fibers.

Theorem 5.2 (Akhmedov and Ozbagci [4]). *For any finitely presented group G, there exists a contact singularity link which admits infinitely many exotic Stein fillings such that the fundamental group of each filling is G.*

Some key ingredients in the proofs of Theorem 5.1 and Theorem 5.2 are Luttinger surgery [37], symplectic sum [31], Fintushel–Stern knot surgery [24], and the Seiberg–Witten invariants [61].

We now turn our attention to Lefschetz fibrations on minimal symplectic fillings of lens spaces. Let ξ denote the canonical contact structure on the lens space $L(p,q)$, which is the link of a *cyclic quotient surface singularity*. The minimal symplectic fillings of $(L(p,q),\xi)$ have been classified by Lisca [35], generalizing the classification by McDuff [40] for $(L(p,1),\xi)$.

Theorem 5.3 (Bhupal and Ozbagci [10]). *There is an algorithm to describe any minimal symplectic filling of $(L(p,q),\xi)$ as an explicit genus-zero allowable Lefschetz fibration over D^2. Moreover, any minimal symplectic filling of $(L(p,q),\xi)$ is obtained by a sequence of rational blowdowns*[6] *starting from the minimal resolution of the corresponding cyclic quotient singularity.*

Theorem 5.3 was recently extended to the case of non-cyclic quotient singularities by H. Choi and J. Park [15].

[5] Homeomorphic but pairwise not diffeomorphic.

[6] Rational blow-down is a surgery operation discovered by Fintushel and Stern [23], where a negative definite linear plumbing submanifold is replaced by a rational 4-ball.

Remark 5.4. Since $(L(p,q),\xi)$ is known to be planar [55], i.e., it admits a planar open book that supports ξ, it also follows by a theorem of Wendl [60], that each minimal symplectic filling of $(L(p,q),\xi)$ is *deformation equivalent* to a genus-zero allowable Lefschetz fibration over D^2, although we have not relied on Wendl's theorem in our proof of Theorem 5.3.

6. Lefschetz fibrations and trisections

A handlebody is a compact manifold admitting a handle decomposition with a single 0-handle and some 1-handles. A *trisection* of a closed 4-manifold X is a decomposition of X into three 4D-handlebodies, whose pairwise intersections are 3D-handlebodies and whose triple intersection is a closed embedded surface.

A trisection of a 4-manifold is analogous to a Heegaard splitting of a closed 3-manifold, which is a decomposition into two 3D-handlebodies whose intersection is an embedded surface. Moreover, trisections can be presented by *trisection diagrams*, similar to the Heegaard diagrams. We refer to Gay's lecture notes [27] for a gentle introduction to trisections of 4-manifolds.

Theorem 6.1 (Gay and Kirby [25]). *Every closed oriented* 4*-manifold admits a trisection.*

Based on a splitting of an arbitrary closed 4-manifold into two *achiral*[7] Lefschetz fibrations over D^2 due to Etnyre and Fuller [22] and a gluing technique for *relative* trisections for 4-manifolds with boundary, Castro and the author [13] obtained an alternate proof of Theorem 6.1 using *Lefschetz fibrations* and *contact geometry*, instead of *Cerf theory* as utilized by Gay and Kirby. The following result is an application of this alternate proof.

Theorem 6.2 (Castro and Ozbagci [13]). *Suppose that X is a closed, oriented* 4*-manifold which admits a Lefschetz fibration over S^2 with a section of square* -1*. Then, an explicit trisection of X can be described by a corresponding trisection diagram, which is determined by the vanishing cycles of the Lefschetz fibration.*

We would like to point out that Gay [26] also constructed a trisection of any 4-manifold which admits a Lefschetz pencil, turning one type of decomposition into another, but without describing an *explicit* trisection diagram.

Remark 6.3. Baykur and Saeki [7] obtained yet another proof of Theorem 6.1, setting up a correspondence between *broken* Lefschetz fibrations and trisections, using

[7] Possibly including nodes with opposite orientation.

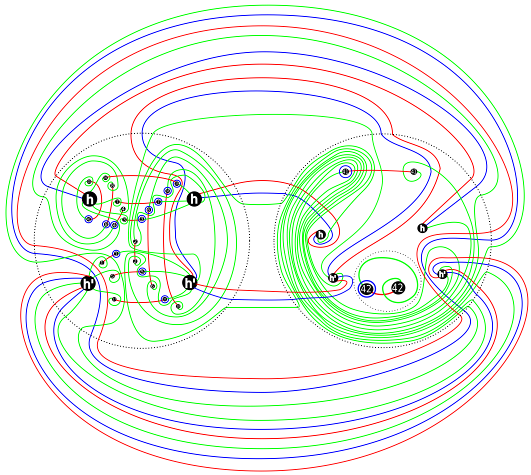


Figure 12. A trisection diagram for the Horikawa surface $H'(1)$.

a method which is very different from ours. They also proved a stronger version of Theorem 6.2.

Example 6.4 ([13]). The Horikawa surface $H'(1)$, a simply-connected complex surface of general type, admits a genus two Lefschetz fibration over S^2 with a section of square -1. The trisection diagram obtained by applying Theorem 6.2 is depicted in Figure 12. Notice that $H'(1)$ is an *exotic copy* of $5\,\mathbb{CP}^2 \,\#\, 29\,\overline{\mathbb{CP}^2}$.

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